Problem Set 2, Algebraic Stacks, due Nov 11, 2020
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1. Check (with proof) which of the following functors $p : F \to C$ define fibered categories. Which of them are fibered in sets, which are fibered in groupoids? See also Problem 2.

(a) Let C be a category and consider $p : C/X \to C$ sending $f : T \to X$ to T. Here C/X is the localized category, see [Olsson, 3.A] or the lecture for the definition.

(b) Let C be a category and let $\mathrm{Arr}(C)$ be the category of arrows in C, i.e. the objects of $\mathrm{Arr}(C)$ are arbitrary morphisms $f : T \to X$ in C, and a morphism in $\mathrm{Arr}(C)$ from $(f : T \to X)$ to $(g : U \to Y)$ is a commutative diagram

$$\begin{array}{ccc} T & \longrightarrow & U \\ \downarrow & & \downarrow \\ X & \longrightarrow & Y. \end{array}$$

The functor $p : \mathrm{Arr}(C) \to C$ sends $T \to X$ to X and the morphism given by the commutative diagram above to the morphism $X \to Y$.

(c) Let Set be the category and sets and let Top be the category of topological spaces (with morphisms the continuous maps). Let $p :$ Top $\to$ Set be the functor that sends a topological space (U, τ) to its underlying set U, and a continuous morphism to itself.

(d) Let Set $\to$ Top that sends a set U to the pair $(U, \tau_{\mathrm{discr}})$, where τ_{discr} is the discrete topology.

(e) Let G be the topological group and define BG to be the category whose objects are principle G-bundles $P \to T$, and whose morphisms from $P \to T$ to $P' \to T'$ are commutative diagrams

$$\begin{array}{ccc} P & \xrightarrow{h} & P' \\ \downarrow & & \downarrow \\ T & \longrightarrow & T' \end{array}$$

such that h is G-equivariant and the diagram is cartesion. Let $p : BG \to$ Top be the map that sends $P \to T$ to T.

Remark. This example can be generalized to the case where the group G acts on a topological space X, and we obtain the quotient stack $[X/G]$. We will also see the algebraic analog of the above construction later on.

(f) $p : \mathcal{M}_g \to$ Schemes, the moduli space of smooth curves of genus g.

(g) Let G be a finite group and let $\mathrm{Rep}_{G,\mathbb{C}}$ be the category of finite-dimensional representations of G (over the base $\mathbb{C}$). Consider the forgetful map $\mathrm{Rep}_{G,\mathbb{C}} \to \mathrm{Vec}_{\mathbb{C}}$ to the category of $\mathbb{C}$-vector spaces.

2. (Olsson, 3.D) Show that a fibered category $p : F \to C$ is a category fibered in groupoids if and only if every morphism in F is cartesian.

3. Olsson, Exercise 2.M (This requires you to read up a bit on Olsson about cohomology on sites, see Section 2.3)

4. (Optional) Olsson Exercise 2.C.