Problem Set 1, Algebraic Stacks, due Nov 4, 2020
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1. Write a full proof of the Yoneda lemma (Theorem A.2.2 in Olsson). Conclude that the functor

$$\text{Schemes} \to \text{Func}(\text{Schemes}, \text{Set}), X \mapsto h_X$$

is fully faithful.

2. Consider the subscheme $C \subset \mathbb{P}^2 \times \mathbb{A}^1_t \setminus \{0\}$ defined by the equation

$$Y^2 Z = X^3 - tZ,$$

where $[X, Y, Z]$ are the homogeneous coordinates on $\mathbb{P}^2$. Show that the family of elliptic curves $f : C \to \mathbb{A}^1_t \setminus \{0\}$ (with section $[0 : 1 : 0]$) is not isomorphic to the trivial family $C_1 \times \mathbb{A}^1_t \setminus \{0\}$. Do this in two different ways:

(1) Base change to $\mathbb{C}$ and compute the monodromy $H^1(E_1, \mathbb{Z}) \to H^1(E_1, \mathbb{Z})$.
(2) Follow the steps in Exercise 4.D of Olsson's book.

The above is an example of an *isotrivial family* of elliptic curves (i.e. a family that becomes trivial after an étale base change). The elliptic curve $y^2 = x^3 - 1$ is special in the sense that it has an order 6 automorphism. Can you find similarly an isotrivial family of elliptic curves over the pictured disc $\mathbb{A}^1 \setminus 0$ with fiber isomorphic to the elliptic curve $y^2 = x(x^2 - 1)$ automorphism such that the family has $\mathbb{Z}_4$-monodromy? Here $y^2 = x(x^2 - 1)$ is an elliptic curve with an order 4 automorphism given by $x \mapsto -x, y \mapsto iy$. What is the degenerate fiber in your construction?

3. Let $g \geq 2$. A family of curves of genus g over a base T is a smooth proper morphism $\pi : C \to T$ such that every geometric fiber is a connected curve of genus g. A morphism $(C' \to T') \to (C \to T)$ of such families is a commutative diagram

$$\begin{array}{ccc} C' & \longrightarrow & C \\ \downarrow & & \downarrow \\ T' & \longrightarrow & T \end{array}$$

which is fibered. Let $\mathcal{M}'_g : \text{Sch} \to \text{Set}$ be the functor that sends a scheme T to the set of *isomorphism classes* of curves of genus g over T, and send a morphism to the pullback map by that morphism.

Show that $\mathcal{M}'_g$ is not representable by a scheme. (Hint: You can imitate what was done in Problem 2. All you need is a curve with a non-trivial automorphism.)

4. Olsson, Exercise 1.D.

5. (Optional) If you feel you need more practice, do also Olsson, Exercise 1.B, 1.C, 1.E.