Practice problem for the exam in Algebraic Stacks
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1.

(a) Give a proof that $B\mathbb{G}_m$ is an algebraic stack.
(b) Is $B\mathbb{G}_m$ an algebraic space? If not, why?
(c) Describe a group presentation for $B\mathbb{G}_m$, and use it to compute the Picard group of $B\mathbb{G}_m$. Is it trivial?
(d) Write down (with proof) an automorphism $B\mathbb{G}_m \to B\mathbb{G}_m$ that is not isomorphic to the identity. Can you write down an endomorphism, that is not invertible?
(e) Compute the global sections of $\mathcal{O}_{B\mathbb{G}_m}$. If you found any other, do the same for the line bundles in (c). Do any line bundles on $B\mathbb{G}_m$ have higher sheaf cohomology?

2. Repeat problem 1 but for the stack $\mathcal{X} = [\mathbb{A}^1/\mathbb{G}_m]$. Can you still find automorphisms as in (d)?

3. Give an example (with proof) of an algebraic space that is not a scheme. Can you choose your algebraic space to be quasi-separated?

4. Let E be an elliptic curve over $\mathbb{C}$. Consider the stack $\mathcal{X} = [E/\mathbb{Z}_2]$ where $\mathbb{Z}_2$ acts by ± 1.

(a) Describe the $\mathbb{C}$-points of $\mathcal{X}$ and its stabilizers groups.
(b) Is $\mathcal{X}$ a Deligne-Mumford stack?
(c) Show that $\mathcal{X}$ admits a coarse moduli space $\pi : \mathcal{X} \to Y$. Is it also a good moduli space (e.g. can you compute $\pi_* \mathcal{O}_{\mathcal{X}}$)?
(d) Describe $\mathcal{X}$ as the root stack over some smooth curve C.

5. Let $\mathbb{G}_m$ act on $\mathbb{C}^2$ by $\lambda \cdot (x, y) = (\lambda x, y)$. Consider the induced action on $\mathbb{P}^1$ and equip $\mathcal{O}_{\mathbb{P}^1}(-1)$ with the natural $\mathbb{G}_m$-linearization. This induces a natural $\mathbb{G}_m$-linearization on $\mathcal{O}_{\mathbb{P}^1}(n)$. Compute $H^0([\mathbb{P}^1/\mathbb{G}_m], \mathcal{O}(n))$.
(Remark: For an a more advanced example consider Olsson 10.G)

6. Let $\varphi : G \to H$ be a group homomorphism of smooth affine $\mathbb{C}$-schemes. Describe the natural induced map $f : BG \to BH$.
(a) If φ is surjective, show that f is a gerbe (in fact a $K = \mathrm{Ker}(\varphi)$ gerbe if you know what that means). When is the gerbe trivial?
(b) If φ is not surjective, is f still a gerbe?
(c) Let H act on a variety X. The H-action on X induces a natural G-action on X (by acting through φ). If φ is surjective, show that $[X/H] \to [X/G]$ is a gerbe. When is it trivial?

Remark: This is very abstract, but for a start you may consider the group homomorphism $\mathbb{Z}_4 \to \mathbb{Z}_2$ or $\mathbb{Z}_2 \times \mathbb{Z}_2 \to \mathbb{Z}_2$, and we let $\mathbb{Z}_2$ act on E as in

Problem 4. or by translation by a 2-torsion point. For an other example, consider $\mathrm{SL}_2(\mathbb{Z}) \to PSL_2(\mathbb{Z})$ an the natural action on the upper half plane by Möbius transformations (in the analytic topology).

7. Let $C = \mathrm{Spec}\mathbb{C}[x,y]/(x^2 - y^3)$ and consider the $\mathbb{G}_m$ action on C given by $\lambda \cdot (x,y) = (\lambda^3 x, \lambda^2 y)$.
a. Is the quotient stack $\mathcal{X} = [C/\mathbb{G}_m]$ smooth?
b. Is the natural morphism $C \to \mathcal{X}$ smooth?
c. Can you find a finite morphism $f : \mathcal{Y} \to \mathcal{X}$ such that $\mathcal{Y}$ is smooth and f induces an equivalence $\mathcal{Y}(\mathbb{C}) \to \mathcal{X}(\mathbb{C})$?
d. Does $\mathcal{X}$ admit a good moduli space? If yes which one?

8. Write down the definition of the moduli stack of genus g curves $\mathcal{M}_g$.
Check that it is a stack in the étale topology.
How would you go about checking that it is a Deligne-Mumford stack? (no details needed)