A REMARK ON THE BANANA CALABI-YAU THREEFOLD

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Abstract. Bryan computed the Donaldson-Thomas invariants of the Banana-Calabi-Yau threefold $X \to \mathbb{P}^1$ in fiber classes. We explain in this note how this determines the 2-dimensional generalized Donaldson-Thomas invariants counting semi-stable sheaves supported on fibers of $X \to \mathbb{P}^1$ in terms of the modular form $12\eta(\tau)^2/\eta(4\tau)^6\eta(2\tau)$ and a multiple cover rule.

1. Introduction

Let $\pi_R : R \to \mathbb{P}^1$ be a generic rational elliptic surface with fixed section $B \subset R$. The Banana Calabi-Yau threefold is the fibered product

$$X = \mathrm{Bl}_\Delta(R \times_{\mathbb{P}^1} R).$$

Let $\pi_i : X \to R$ for $i = 1, 2$ be the projections to the factors. Consider the fibration

$$\pi := \pi_R \circ \pi_1 = \pi_R \circ \pi_2 : X \to \mathbb{P}^1.$$

The smooth fibers of π are the self-product of the corresponding fibers of π_R, hence the self product of an elliptic curve. The singular fibers of π correspond to the 12 singular fibers of π_R, and are of Banana type [2]. We see that π is a 1-parameter family of abelian surfaces with 12 banana fibers. There exists a lattice polarization of the fibers of π by

$$L = \mathbb{Z}\omega_1 \oplus \mathbb{Z}\omega_2 \oplus \mathbb{Z}\omega_3 \subset \mathrm{Pic}(X)$$

where

$$\omega_1 = \pi_1^*(B), \quad \omega_2 = \pi_2^*(B), \quad \omega_3 = [\widetilde{\Delta}]$$

with $\widetilde{\Delta}$ the proper transform of the diagonal. The intersection form of these divisor classes restricted to a generic fiber is given by the matrix

$$\begin{pmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{pmatrix}$$

Dually, let

$$H_2(X, \mathbb{Z})_\pi \subset H_2(X, \mathbb{Z})$$

be the sublattice generated effective classes contracted under π. One has that

$$H_2(X, \mathbb{Z})_\pi = \mathbb{Z}[C_1] \oplus \mathbb{Z}[C_2] \oplus \mathbb{Z}[C_3]$$

where C_i are the banana curves (of any singular fiber). There is a natural isomorphism

$$H_2(X, \mathbb{Z})_\pi \cong L^*$$

and the C_i are the dual basis of the ω_i. The induced intersection form on $H_2(X, \mathbb{Z})$ has matrix

$$\frac{1}{2}\left(\begin{pmatrix} -1 & 1 & 1 \\ 1 & -1 & 1 \\ 1 & 1 & -1 \end{pmatrix} \right). \tag{1.1}$$

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We define thus a quadratic form on $H_2(X,\mathbb{Z})_\pi$ by letting $\beta = d_1C_1 + d_2C_2 + d_3C_3$ have norm

$$||\beta|| = 2d_1d_2 + 2d_1d_3 + 2d_2d_3 - d_1^2 - d_2^2 - d_3^2$$

Let $\mathsf{Hilb}_{n,\beta}(X)$ be the Hilbert scheme of 1-dimensional subschemes $Z \subset X$ with $[Z] = \beta$ and $\chi(\mathcal{O}_Z) = n$. Consider the Donaldson-Thomas invariants of the Calabi-Yau threefold X:

$$\mathsf{DT}_{n,\beta} := \int_{[\mathsf{Hilb}_{n,\beta}(X)]^{\mathrm{vir}}} 1.$$

The following beautiful result was proven by Bryan in [2]:

Theorem 1.1 ([2]). *We have the equality of formal power series*

$$\sum_{\beta \in H_2(X,\mathbb{Z})_\pi} \sum_{n\in\mathbb{Z}} \mathsf{DT}_{n,\beta}(-p)^n t^\beta = \prod_{(\beta,n)>0} (1 - p^n t^\beta)^{-12c(||\beta||,n)}$$

where

- t^β *is the basis element in the group ring* $\mathbb{C}[H_2(X,\mathbb{Z})_\pi]$, *completed along the cone of effective curves,*
- $(\beta, n) > 0$ *stands for* $\beta > 0$ *(i.e. effective) or* $\beta = 0$ *and* $n > 0$,
- *the coefficients* $c(d,n)$ *are given by*

$$\sum_{d,n} c(d,n)p^n q^d = \frac{-p}{q(1-p)^2} \prod_{n\geq 1} \frac{(1-q^{2n})(1-pq^{2n-1})(1-p^{-1}q^{2n-1})}{(1-q^{4n})^2(1-pq^{4n})^2(1-p^{-1}q^{4n})^2}$$

For $r \geq 0$, $\beta \in H_2(X,\mathbb{Z})_\pi$, and $n \in \mathbb{Z}$ let

$$\mathsf{DT}(r,\beta,n)$$

be the generalized Donaldson-Thomas invariant, definded by Joyce-Song [3], counting 2-dimensional Gieseker semistable sheaves $\mathcal{E}$ (with respect to some polarization) with Chern character

$$\mathsf{ch}_1(\mathcal{E}) = r[F], \quad \mathsf{ch}_2(\mathcal{E}) = \beta, \quad \mathsf{ch}_3(\mathcal{E}) = n,$$

where $F \subset X$ stands for any fiber of π. The condition on the Chern character hence implies that $\mathcal{E}$ is supported on fibers of π.

The goal of this note is to explain that Theorem 1.1 determines these invariants. Consider

$$\Lambda := \mathbb{Z} \oplus H_2(X,\mathbb{Z})_\pi \oplus \mathbb{Z}.$$

We define the Mukai pairing on Λ by

$$(r,\beta,n)\cdot(r',\beta',n') = \beta\cdot\beta' - rn' - r'n$$

where $\beta\cdot\beta'$ stands for the pairing dual to the pairing on L, that is which is given by the matrix (1.1). Hence for $v = (r,\beta,n)$ we have

$$2v\cdot v = ||\beta|| - 4rn.$$

Clearly, Λ is just the Mukai lattice of the abelian surface given by a generic fiber of π.

We call $v = (r,\beta,n)$ effective if

- $r \geq 0$, or
- $r = 0, \beta > 0$, or
- $r = 0, \beta = 0, n > 0$.

Define the coefficients $a(n)$ by

$$\begin{aligned}\sum_n a(n)q^n &= \frac{\eta(\tau)^2}{\eta(4\tau)^6\eta(2\tau)} \\ &= \frac{1}{q}\prod_{m\geq 1}\frac{(1-q^m)^2}{(1-q^{4m})^6(1-q^{2m})} \\ &= \frac{1}{q} - 2 + 8q^3 - 12q^4 + 39q^7 - 56q^8 + 152q^{11} - 208q^{12} + O(q^{15})\end{aligned}$$

Theorem 1.2. *For any effective $v = (r, \beta, n) \in \Lambda$ we have*

$$\mathsf{DT}(v) = 12\sum_{\substack{k|v\\ k\geq 1}}\frac{1}{k^2}a\left(\frac{2v\cdot v}{k^2}\right).$$

Another way to state the formula is as follows: Define invariants $\mathsf{dt}(v)$ by subtracting formally multiple cover contributions, that is inductively by the equality:

$$\mathsf{DT}(v) = \sum_{\substack{k|v\\ k\geq 1}}\frac{1}{k^2}\mathsf{dt}(v/k). \tag{1.2}$$

Then Theorem 1.2 is equivalent to:

Corollary 1.3. $\mathsf{dt}(v) = a(2v\cdot v)$.

That the invariant $\mathsf{dt}(v)$ only depends on the square $v \cdot v$ is pretty remarkable.

The invariants $\mathsf{DT}(v)$ have been computed for K3 fibrations also in three other instances:

(i) For the product $S \times \mathbb{C}$ where S is a K3 surface in [4].
(ii) For K3 fibered Calabi-Yau threefolds $\pi : X \to C$ with nodal singular fibers (i.e. K3 surfaces with ADE singularieties) in [1].
(iii) For the Enriques Calabi-Yau threefold $\pi : X \to \mathbb{P}^1$ which is an isotrivial K3 fibration with 4 double Enriques fibers, in [6].

In all three instances the invariants $\mathsf{DT}(r, \beta, n)$, after subtracting multiple cover contributions formally as in (1.2), do only depend on the square of (r, β, n). Moreover, they have modular behaviour. Hence it is natural expect to find similar multiple cover and modular behaviour for fiber class DT invariants of all K3/Abelian surface fibered Calabi-Yau threefolds.

Example 1.4. Before going to the proof, we give some examples of Theorem 1.2.

The simplest is:

$$\mathsf{DT}(0, 0, 1) = -24 = -e(X).$$

Next we consider the genus zero Gopakumar-Vafa invariants of X. These are given by

$$\mathsf{DT}(0, \beta, 1) = 12a(||\beta||).$$

This matches the results proven in [5] by the basic formula

$$\varphi_{-2,1}(p, q) = \sum_{d\geq 0}\sum_{r\in\mathbb{Z}} a(4d - r^2)p^r q^d$$

where the left hand side is the classical weight -2 index 1 Jacobi form

$$\varphi_{-2,1}(p, q) = (p^{1/2} - p^{-1/2})^2\prod_{m\geq 1}\frac{(1-pq^m)^2(1-p^{-1}q^m)^2}{(1-q^m)^4}.$$

For the stable sheaves on fibers of π of rank 1 we get

$$\sum_{n\geq 0}\mathsf{DT}(1,0,-n)q^n = -24\frac{\eta(q^4)^2}{\eta(q^2)\eta(q)^6}$$

For example, the equality $\mathsf{DT}(1,0,0) = -24$ computes minus the topological Euler number of the associated Jacobian fibration.

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2. Proof of Theorem 1.2

The idea of the proof is to apply Toda's wall-crossing argument of [7]. This works perfectly for the first two wall-crossings but fails in the last step. In any case, one obtains the following structural result: Define the generating series of DT invariants

$$\mathsf{DT}_\pi(p,t) = \sum_{\beta\in H_2(X,\mathbb{Z})_\pi}\sum_{n\in\mathbb{Z}}\mathsf{DT}_{n,\beta}(-p)^n t^\beta.$$

Proposition 2.1. *There exists a power series $\mathsf{DT}_{rest}(t)$ (depending only on t) such that*

$$\mathsf{DT}_\pi(X) = \prod_{\substack{r\geq 0\\ \beta\geq 0\\ n>0\\ r=0 \text{ if } \beta=0}}\exp(-n\mathsf{DT}(r,\beta,n)p^n t^\beta)\prod_{\substack{r>0\\ \beta>0\\ n>0}}\exp(-n\mathsf{DT}(r,\beta,n)p^{-n}t^\beta)\mathsf{DT}_{rest}(t)$$

Proof. The term $r=\beta=0$ and $n>0$ is simply the DT/PT wall-crossing. Next we apply the wall-crossing between PT invariants and the L-invariants of Toda, see [7, Eqn (13)], which accounts for the $r=0$ terms. Then we apply the wall-crossing formula in the category $\mathcal{A}_\omega\subset\mathcal{D}$ according to the notation of [7, Step 2 in 1.4]: This yields the remaining terms. The semistable sheaves after this wall-crossing must have vanishing ch_3, so the generating series of the invariants only depend on t. This completes the proof. Note that there is a final wall-crossing step in [7] which is done in the category $\mathcal{A}_\omega(1/2)$: This can not be done here since it involves infinitely many walls. The issue is that for K3 surfaces $\mathsf{DT}(r,0,0)$ (where we use the Chern character) becomes zero for $r\gg 0$ while in our case it does not. □

Taking log and using (1.2) this can be rewritten as

$$\log\mathsf{DT}_\pi(X) = \sum_{k\geq 1}\frac{1}{k}f_0(p^k) + \sum_{\beta>0}\sum_{k\geq 1}\frac{1}{k}t^{k\beta}f_\beta(p^k) + (\text{series in } t \text{ only})$$

where

$$f_0(p) = \sum_{n\geq 1} -n\mathsf{dt}(0,0,n)p^n$$

and for $\beta>0$ we let

$$f_\beta(p) = \sum_{n\geq 1}-n\mathsf{dt}(0,\beta,n)p^n + \sum_{r>0,n>0}-n\mathsf{dt}(r,\beta,n)(p^n+p^{-n}).$$

On the other hand, taking the log of Theorem 1.1 we get

$$\log\mathsf{DT}_\pi(X) = \sum_{\beta\geq 0}\sum_{k\geq 1}\frac{1}{k}t^{k\beta}\sum_{\substack{n\in\mathbb{Z}\\ n>0 \text{ if } \beta=0}} 12c(||\beta||,n)p^{kn}.$$

One has the basic identity

$$\sum_{d,n} c(d,n) q^d p^n = -f(q) \left[\frac{p}{(1-p)^2} + \sum_{k,n\geq 1} n(p^n + p^{-n}) q^{4kn} - \sum_{m\geq 1} \sum_{\substack{d|m \\ m/d \text{ odd}}} d q^m \right]$$

where

$$f(q) = \frac{\eta(\tau)^2}{\eta(4\tau)^6 \eta(2\tau)} = \frac{1}{q} - 2 + \ldots .$$

Comparing we find:

$$\sum_{n\geq 1} -n\mathsf{dt}(0,0,n) p^n = 12 \sum_{n>0} c(0,n) p^n = 24 \frac{p}{(1-p)^2}$$

so $\mathsf{dt}(0,0,n) = -24$. For $\beta > 0$ we obtain modulo the p^0-term:

$$\sum_{n\geq 1} -n\mathsf{dt}(0,\beta,n) p^n + \sum_{r>0,n>0} -n\mathsf{dt}(r,\beta,n)(p^n + p^{-n})$$
$$= \text{Coefficient}_{q^{||\beta||}} \left(-f(q) \left[\frac{p}{(1-p)^2} + \sum_{k,n\geq 1} n(p^n+p^{-n}) q^{4kn} \right] \right) + (\text{constant})$$

Using that $\mathsf{dt}(0,\beta,n) = \mathsf{dt}(0,\beta,n+\gcd(\beta))$ we get

$$\mathsf{dt}(0,\beta,n) = \mathsf{dt}(0,\beta,1) = \text{Coefficient}_{q^{||\beta||}}(f(q))$$

and for $r > 0$, $n \neq 0$ and $\beta > 0$

$$\mathsf{dt}(r,\beta,n) = a(||\beta|| - 4rn).$$

This proves the result for $r = 0$ fully, and for $r > 0$ if $\beta > 0$ and $n \neq 0$.

Now by the same argument as in [7] it follows that $\mathsf{DT}(r,\beta,n)$ is independent of the polarization. This implies that the DT invariant is invariant under tensoring the sheaves by a line bundles, that is: $\mathsf{dt}(r,\beta,n) = \mathsf{dt}(r,\beta + r\gamma, n + \beta\cdot\gamma + \frac{1}{2}r\gamma^2)$ for all $\gamma \in H_2(X,\mathbb{Z})_\pi$. Hence if $r > 0$ we can compute $\mathsf{dt}(r,\beta,n)$ by assuming $\beta > 0$ and $n > 0$. □

References

[1] V. Bouchard, T. Creutzig, D.-E. Diaconescu, C. Doran, C. Quigley, A. Sheshmani, *Vertical D4-D2-D0 bound states on K3 fibrations and modularity*, Comm. Math. Phys.**350**(2017), no.3, 1069–1121.

[2] J. Bryan, *The Donaldson-Thomas partition function of the banana manifold*, Algebr. Geom. **8** (2021), no. 2, 133–170.

[3] D. Joyce, Y. Song, *A theory of generalized Donaldson-Thomas invariants*, Mem. Amer. Math. Soc. **217** (2012), no. 1020, iv+199 pp.

[4] D. Maulik, R. P. Thomas, *Sheaf counting on local K3 surfaces*, Pure Appl. Math. Q. **14** (2018), no. 3-4, 419–441.

[5] N. Morishige, *Genus 0 Gopakumar-Vafa invariants of the banana manifold*, Q. J. Math. **73** (2022), no. 1, 175–212.

[6] G. Oberdieck, *Curve counting on the Enriques surface and the Klemm-Mariño formula*, arXiv:2305.11115

[7] Y. Toda, *Stable pairs on local K3 surfaces*, J. Differential Geom. **92** (2012), no. 2, 285–371

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