

RATIONAL CURVES IN HOLOMORPHIC SYMPLECTIC VARIETIES AND GROMOV–WITTEN INVARIANTS

GEORG OBERDIECK, JUNLIANG SHEN, AND QIZHENG YIN

ABSTRACT. We use Gromov–Witten theory to study rational curves in holomorphic symplectic varieties. We present a numerical criterion for the existence of uniruled divisors swept out by rational curves in the primitive curve class of a very general holomorphic symplectic variety of $K3^{[n]}$ type. We also classify all rational curves in the primitive curve class of the Fano variety of lines in a very general cubic 4-fold. Our proofs rely on Gromov–Witten calculations by the first author, and in the Fano case on a geometric construction of Voisin. In the Fano case a second proof via classical geometry is sketched.

CONTENTS

0. Introduction	1
1. Moduli spaces of stable maps	5
2. Gromov–Witten calculations	9
3. Rational curves in the Fano varieties of lines	15
Appendix A. Sketch of a classical proof of Theorem 0.2	21
References	23

0. INTRODUCTION

0.1. **Overview.** Rational curves in $K3$ surfaces have been investigated for decades from various angles. In contrast, not much is known about the geometry of rational curves in the higher-dimensional analogs of $K3$ surfaces—holomorphic symplectic varieties.¹ In this paper, we use *Gromov–Witten theory* (intersection theory of the moduli space of stable maps) together with classical methods to study these rational curves.

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¹A nonsingular projective variety X is holomorphic symplectic if it is simply connected and $H^0(X, \Omega_X^2)$ is generated by a nowhere degenerate holomorphic 2-form.

0.2. Rational curves. Let (X, H) be a very general polarized holomorphic symplectic variety of dimension $2n$, and let $\beta \in H_2(X, \mathbb{Z})$ be the primitive curve class. The moduli space $\overline{M}_{0,m}(X, \beta)$ of genus 0 and m -pointed stable maps to X in class β is pure of expected dimension $2n - 2 + m$; see Proposition 1.1. Consider the decomposition

$$\overline{M}_{0,1}(X, \beta) = M^0 \cup M^1 \cup \dots \cup M^{n-1} \quad (1)$$

such that the general fibers of the restricted evaluation map

$$\text{ev} : M^i \rightarrow \text{ev}(M^i) \subset X$$

are of dimension i . The image of M^0 under ev is precisely the union of all uniruled divisors swept out by rational curves in class β . More generally, the image $\text{ev}(M^i)$ is the codimension $i + 1$ locus of points on X through which passes an i -dimensional family of rational curves in class β .

In [21, Conjecture 4.3], Mongardi and Pacienza conjectured that for all i

$$M^i \neq \emptyset,$$

which would imply the existence of algebraically coisotropic subvarieties in X in the sense of Voisin [27].

In Theorems 0.1 and 0.2 below, we provide counterexamples to this conjecture which illustrate “pathologies” of rational curves in higher-dimensional holomorphic symplectic varieties. Two typical examples are as follows.

- (i) There exist a very general pair (X, H) of $K3^{[8]}$ type with $M^0 = \emptyset$. In other words, on (X, H) there exists no uniruled divisor swept out by rational curves in the primitive class β .
- (ii) For the Fano variety of lines in a very general cubic 4-fold, we have $M^1 = \emptyset$.

Here a variety is of $K3^{[n]}$ type if it is deformation equivalent to the Hilbert scheme of n points on a $K3$ surface.

0.3. Uniruled divisors. On a holomorphic symplectic variety X , let

$$(-, -) : H_2(X, \mathbb{Z}) \times H_2(X, \mathbb{Z}) \rightarrow \mathbb{Q}$$

denote the unique \mathbb{Q} -valued extension of the Beauville–Bogomolov form on $H^2(X, \mathbb{Z})$. If X is of $K3^{[n]}$ type and $n \geq 2$, there is an isomorphism of abelian groups

$$r : H_2(X, \mathbb{Z})/H^2(X, \mathbb{Z}) \rightarrow \mathbb{Z}/(2n - 2)\mathbb{Z},$$

unique up to multiplication by ± 1 , such that $r(\alpha) = 1$ for some $\alpha \in H_2(X, \mathbb{Z})$ with $(\alpha, \alpha) = \frac{1}{2-2n}$. Given a class $\beta \in H_2(X, \mathbb{Z})$, we define its residue set by

$$\pm[\beta] = \{\pm r(\beta)\} \subset \mathbb{Z}/(2n - 2)\mathbb{Z}.$$

In case $n = 1$, we set $\pm[\beta] = 0$.

The following theorem provides a complete numerical criterion for the existence of uniruled divisors swept out by rational curves in the primitive class of a very general variety of $K3^{[n]}$ type.

Theorem 0.1. *Let X be a holomorphic symplectic variety of $K3^{[n]}$ type, and let $\beta \in H_2(X, \mathbb{Z})$ be a primitive curve class. If*

$$(\beta, \beta) = -2 + \sum_{i=1}^{n-1} 2d_i - \frac{1}{2n-2} \left(\sum_{i=1}^{n-1} r_i \right)^2,$$

$$\pm[\beta] = \pm \left[\sum_{i=1}^{n-1} r_i \right]$$

for some $d_i, r_i \in \mathbb{Z}$ satisfying $2d_i - \frac{r_i^2}{2} \geq 0$, then there exists a uniruled divisor on X swept out by rational curves in class β . The converse holds if β is irreducible.

For a very general pair (X, β) with X of $K3^{[n]}$ type and β the primitive curve class, Theorem 0.1 implies that

- (i) $M^0 \neq \emptyset$ when $n \leq 7$, and
- (ii) for every $n \geq 8$, there exists (X, β) such that $M^0 = \emptyset$.

The first instance of case (ii) is given by a very general pair (X, β) of $K3^{[8]}$ type with $(\beta, \beta) = \frac{3}{14}$ and $\pm[\beta] = \pm[5]$.²

0.4. Fano varieties of lines. Let $Y \subset \mathbb{P}^5$ be a nonsingular cubic 4-fold. By Beauville and Donagi [4], the Fano variety of lines in Y

$$F = \{l \in \text{Gr}(2, 6) : l \subset Y\}$$

is a holomorphic symplectic 4-fold. These varieties form a 20-dimensional family of polarized holomorphic symplectic varieties of $K3^{[2]}$ type.

In [26], Voisin constructed a rational self-map

$$\varphi : F \dashrightarrow F \tag{2}$$

sending a general line l to its residual line with respect to the unique plane $\mathbb{P}^2 \subset \mathbb{P}^5$ tangent to Y along l . When Y is very general, the exceptional divisor associated to the resolution of φ

$$\begin{array}{ccc} D = \mathbb{P}(\mathcal{N}_{S/F}) & \xrightarrow{\phi} & F \\ \downarrow p & & \\ S & & \end{array} \tag{3}$$

²Such a pair (X, β) can be obtained by deforming $(\text{Hilb}^8(S), \beta')$, where S is a $K3$ surface of genus 2 with polarization H and $\beta' = H + 5A$ with A the exceptional curve class; see Section 2 for the notation.

is a \mathbb{P}^1 -bundle over a nonsingular surface $S \subset F$; see Amerik [1]. The image of each fiber

$$\phi(p^{-1}(s)) \subset F, \quad s \in S$$

is a rational curve lying in the primitive curve class in $H_2(F, \mathbb{Z})$.

The following theorem shows that every rational curve in the primitive curve class is of this form in a unique way.

Theorem 0.2. *Let F be the Fano variety of lines in a very general cubic 4-fold. Then for every rational curve $C \subset F$ in the primitive curve class, there is a unique $s \in S$ such that $C = \phi(p^{-1}(s))$.*

By a result of Huybrechts [15, Section 6], the surface S is connected.³ In particular, the moduli space of rational curves in the primitive curve class of a very general F is irreducible. This implies $M^1 = \emptyset$ in the decomposition (1) and the following.

Corollary 0.3. *For a very general F , there is a unique irreducible uniruled divisor swept out by rational curves in the primitive curve class.*

The moduli space of rational curves in the primitive curve class of a very general $K3$ surface always has more than one irreducible component. Corollary 0.3 indicates a difference between rational curves in $K3$ surfaces and in higher-dimensional holomorphic symplectic varieties.

0.5. Idea of proofs. We briefly explain how Gromov–Witten theory [10] controls rational curves in the primitive class β of a very general polarized holomorphic symplectic variety (X, H) of $K3^{[n]}$ type.

Since the evaluation map ev is generically finite on the component M^0 but contracts positive dimensional fibers on all other components in the decomposition (1), the (non)emptiness of M^0 is detected by the pushforward

$$\text{ev}_*[\overline{M}_{0,1}(X, \beta)] \in H^2(X, \mathbb{Q}). \quad (4)$$

For the Fano variety of lines $X = F$, a key observation is that the emptiness of M^1 can be further detected by the Gromov–Witten correspondence

$$\text{ev}_{12*}[\overline{M}_{0,2}(X, \beta)] \in H^{4n}(X \times X, \mathbb{Q}). \quad (5)$$

The class (5) has contributions from all of the components in (1), and contains strictly more information than the 1-pointed class (4).

Since $\overline{M}_{0,m}(X, \beta)$ is pure of the expected dimension, its fundamental class coincides with the (reduced) virtual fundamental class [5, 18],

$$[\overline{M}_{0,m}(X, \beta)] = [\overline{M}_{0,m}(X, \beta)]^{\text{vir}}.$$

³ In an earlier version of this paper the connectedness of S was claimed. We thank Daniel Huybrechts for pointing out a gap in our earlier argument and presenting a new proof. The connectedness of S is not needed in the proof of Theorem 0.2.

Hence the classes (4) and (5) are determined by the Gromov–Witten invariants of X . By deformation invariance, the Gromov–Witten invariants can be calculated on a special model given by the Hilbert scheme of points of an elliptic $K3$ surface; see [23] and Section 2.

Our proofs of Theorems 0.1 and 0.2 are intersection-theoretic. In Appendix A, we also sketch an alternative proof of Theorem 0.2 using a series of classification results in classical projective geometry.⁴

0.6. Conventions. We work over the complex numbers. A statement holds for a *very general* polarized projective variety (X, H) if it holds away from a countable union of proper Zariski-closed subsets in the corresponding component of the moduli space.

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1. MODULI SPACES OF STABLE MAPS

We discuss properties of the moduli spaces of stable maps to holomorphic symplectic varieties, and introduce tools from Gromov–Witten theory.

1.1. Dimensions. Let X be a holomorphic symplectic variety of dimension $2n$, and let $\beta \in H_2(X, \mathbb{Z})$ be an *irreducible* curve class. We show that the moduli space $\overline{M}_{0,1}(X, \beta)$ of genus 0 pointed stable maps to X in class β is pure of the expected dimension.

Let M be an irreducible component of $\overline{M}_{0,1}(X, \beta)$. We know *a priori*

$$\dim M \geq \int_{\beta} c_1(X) + \dim X - 1 = 2n - 1.$$

⁴The proof in Appendix A was found only after a first version of this article appeared online. While Theorem 0.2 can be proven classically, the quantitative information obtained from Gromov–Witten theory was essential for us to find the statement.

Consider the restriction of the evaluation map to M ,

$$\text{ev} : M \rightarrow Z = \text{ev}(M) \subset X. \quad (6)$$

Proposition 1.1. *If a general fiber of (6) is of dimension $r - 1$, then*

- (i) $\dim Z = 2n - r$, so that $\dim M = 2n - 1$;
- (ii) $r \leq n$;
- (iii) a general fiber of the MRC fibration⁵ $Z \dashrightarrow B$ is of dimension r .

Proof. Since the curve class β is irreducible, the family of rational curves $M \rightarrow T \subset \overline{M}_{0,0}(X, \beta)$ viewed as in X is unsplit in the sense of [16, IV, Definition 2.1]. Given a general point $x \in Z$, let $T_x \subset T$ be the Zariski-closed subset parametrizing maps passing through x . Consider the universal family $\mathcal{C}_x \rightarrow T_x$ and the restricted evaluation map

$$\text{ev} : \mathcal{C}_x \rightarrow V_x = \text{ev}(\mathcal{C}_x) \subset Z.$$

By [16, IV, Proposition 2.5], we have

$$\dim T = \dim Z + \dim V_x - 2.$$

Hence $\dim V_x = \dim M - \dim Z + 1 = r$. In other words, rational curves through a general point of Z cover a Zariski-closed subset of dimension r .

A general fiber of the MRC fibration $Z \dashrightarrow B$ is thus of dimension $\geq r$. By an argument of Mumford (see [27, Lemma 1.1]), this implies $\dim Z \leq 2n - r$ and $r \leq n$. On the other hand, since $\dim M \geq 2n - 1$, we have

$$\dim Z = \dim M - (r - 1) \geq 2n - r.$$

Hence there is equality $\dim Z = 2n - r$, and the dimension of a general fiber of $Z \dashrightarrow B$ is exactly r . \square

Proposition 1.1 shows that $\overline{M}_{0,1}(X, \beta)$ is pure of the expected dimension $2n - 1$ and justifies the decomposition (1). Similar arguments have also appeared in [2, Theorem 4.4] and [3, Proposition 4.10].

1.2. Gromov–Witten theory. Let X be a holomorphic symplectic variety of dimension $2n$, and let $\beta \in H_2(X, \mathbb{Z})$ be an arbitrary curve class. By Li–Tian [18] and Behrend–Fantechi [5], the moduli space of stable maps $\overline{M}_{0,m}(X, \beta)$ carries a (reduced⁶) virtual fundamental class

$$[\overline{M}_{0,m}(X, \beta)]^{\text{vir}} \in H_{2\text{vdim}}(\overline{M}_{0,m}(X, \beta), \mathbb{Q}).$$

⁵We refer to [11] for the definition and properties of the maximal rationally connected (MRC) fibration.

⁶Since X is holomorphic symplectic, the (standard) virtual fundamental class on the moduli space vanishes. The theory is nontrivial only after reduction; see [20, Section 2.2] and [23, Section 0.2]. The virtual fundamental class is always assumed to be reduced in this paper.

It has the following basic properties.

- (a) *Virtual dimension.* The virtual fundamental class is of dimension

$$\text{vdim} = 2n - 2 + m. \quad (7)$$

- (b) *Expected dimension.* If $\overline{M}_{0,m}(X, \beta)$ is pure of the expected dimension (7), then the virtual and the ordinary fundamental classes agree:

$$[\overline{M}_{0,m}(X, \beta)]^{\text{vir}} = [\overline{M}_{0,m}(X, \beta)].$$

- (c) *Deformation invariance.* Let $\pi : \mathcal{X} \rightarrow B$ be a family of holomorphic symplectic varieties, and let $\beta \in H^0(B, R\pi_*^{4n-2}\mathbb{Z})$ be a class which restricts to a curve class in $H_2(X_b, \mathbb{Z})$ on each fiber.⁷ Then there exists a class on the moduli space of relative stable maps

$$[\overline{M}_{0,m}(\mathcal{X}/B, \beta)]^{\text{vir}} \in H_{2(\text{vdim}+\dim B)}(\overline{M}_{0,m}(\mathcal{X}/B, \beta), \mathbb{Q})$$

such that for every fiber $X_b \hookrightarrow \mathcal{X}$, the inclusion $\iota_b : b \hookrightarrow B$ induces

$$\iota_b^! [\overline{M}_{0,m}(\mathcal{X}/B, \beta)]^{\text{vir}} = [\overline{M}_{0,m}(X_b, \beta)]^{\text{vir}}.$$

Here $\iota_b^!$ is the refined Gysin pullback. In particular, intersection numbers of $[\overline{M}_{0,m}(X, \beta)]^{\text{vir}}$ against cohomology classes pulled back from X via the evaluation maps

$$\text{ev}_i : \overline{M}_{0,m}(X, \beta) \rightarrow X, \quad (f, x_1, \dots, x_m) \mapsto f(x_i)$$

are invariant under deformations of (X, β) which keep β of Hodge type.

1.3. Gromov–Witten correspondence. Let X, β be as in Section 1.1. The evaluation maps from the 2-pointed moduli space

$$\begin{array}{ccc} & \overline{M}_{0,2}(X, \beta) & \\ \text{ev}_1 \swarrow & & \searrow \text{ev}_2 \\ X & & X \end{array}$$

induce an action on cohomology:

$$\text{GW}_\beta : H^i(X, \mathbb{Q}) \rightarrow H^i(X, \mathbb{Q}), \quad \gamma \mapsto \text{ev}_{2*}(\text{ev}_1^* \gamma \cap [\overline{M}_{0,2}(X, \beta)]^{\text{vir}}). \quad (8)$$

We call (8) the *Gromov–Witten correspondence*.

We introduce a factorization of (8) as follows. Consider the diagram

$$\begin{array}{ccc} \overline{M}_{0,1}(X, \beta) & \xrightarrow{\text{ev}} & X \\ \downarrow p & & \\ \overline{M}_{0,0}(X, \beta) & & \end{array} \quad (9)$$

⁷We have suppressed an application of Poincaré duality here. Same with the definition of GW_β and Φ_2 in Section 1.3 below.

with p the forgetful map (which is flat). We define morphisms

$$\begin{aligned}\Phi_1 : H^i(X, \mathbb{Q}) &\rightarrow H_{4n-2-i}(\overline{M}_{0,0}(X, \beta), \mathbb{Q}), \quad \gamma \mapsto p_*(\text{ev}^*\gamma \cap [\overline{M}_{0,1}(X, \beta)]^{\text{vir}}), \\ \Phi_2 = \text{ev}_*p^* : H_{4n-2-i}(\overline{M}_{0,0}(X, \beta), \mathbb{Q}) &\rightarrow H^i(X, \mathbb{Q}).\end{aligned}$$

Since β is irreducible, there is a Cartesian diagram of forgetful maps

$$\begin{array}{ccc} & \overline{M}_{0,2}(X, \beta) & \\ & \swarrow \quad \searrow & \\ \overline{M}_{0,1}(X, \beta) & & \overline{M}_{0,1}(X, \beta) \\ & \searrow \quad \swarrow & \\ & \overline{M}_{0,0}(X, \beta) & \end{array}$$

Hence the Gromov–Witten correspondence (8) factors as

$$\text{GW}_\beta = \Phi_2 \circ \Phi_1 : H^i(X, \mathbb{Q}) \rightarrow H^i(X, \mathbb{Q}). \quad (10)$$

1.4. Hodge classes. Now let (X, H) be a very general polarized holomorphic symplectic 4-fold of $K3^{[2]}$ type. It is shown in [24, Section 3] that the Hodge classes in $H^4(X, \mathbb{Q})$ are spanned by H^2 and $c_2(X)$.

A surface $\Sigma \subset X$ is *Lagrangian* if the holomorphic 2-form σ on X restricts to zero on Σ . The class of any Lagrangian surface is a positive multiple of

$$v_X = 5H^2 - \frac{1}{6}(H, H)c_2(X) \in H^4(X, \mathbb{Q}), \quad (11)$$

where $(-, -)$ is the Beauville–Bogomolov form on $H^2(X, \mathbb{Z})$.⁸

Proposition 1.2. *If (X, H) is very general of $K3^{[2]}$ type and $\beta \in H_2(X, \mathbb{Z})$ is the primitive curve class, then for any Hodge class $\alpha \in H^4(X, \mathbb{Q})$, the class*

$$\text{GW}_\beta(\alpha) \in H^4(X, \mathbb{Q})$$

is proportional to v_X .

Proof. We use the factorization (10). For any Hodge class $\alpha \in H^4(X, \mathbb{Q})$, the class

$$\Phi_1(\alpha) \in H_2(\overline{M}_{0,0}(X, \beta), \mathbb{Q})$$

is represented by curves. Hence $\text{GW}_\beta(\alpha)$ can be expressed as a linear combination of classes of the form

$$[\text{ev}(p^{-1}(C))] \in H^4(X, \mathbb{Q})$$

with $C \subset \overline{M}_{0,0}(X, \beta)$ a curve.

Moreover, we have

$$\text{ev}^*\sigma = p^*\sigma'$$

⁸This follows from a direct calculation of the constraint $[\Sigma] \cdot \sigma = 0 \in H^6(X, \mathbb{Q})$. The class v_X was first calculated by Markman.

for some holomorphic 2-form σ' on $\overline{M}_{0,0}(X, \beta)$. Hence any surface of the form $\text{ev}(p^{-1}(C))$ is Lagrangian, and the proposition follows. \square

Proposition 1.2 implies that the class v_X in (11) is an eigenvector of the Gromov–Witten correspondence

$$\text{GW}_\beta : H^4(X, \mathbb{Q}) \rightarrow H^4(X, \mathbb{Q}).$$

An explicit formula for GW_β was calculated in [23] and is recalled in Section 2.5.

2. GROMOV–WITTEN CALCULATIONS

In this Section, we prove Theorem 0.1 using formulas for the 1-pointed Gromov–Witten class in the $K3^{[n]}$ case based on [23]. We also present formulas for the Gromov–Witten correspondence in the $K3^{[2]}$ case, which will be used in Section 3.

2.1. Quasi-Jacobi forms. Jacobi forms are holomorphic functions in variables⁹ $(\tau, z) \in \mathbb{H} \times \mathbb{C}$ with modular properties; see [9] for an introduction. Here we will consider Jacobi forms as formal power series in the variables

$$q = e^{2\pi i\tau}, \quad y = -e^{2\pi iz}$$

expanded in the region $|q| < |y| < 1$.

Recall the Jacobi theta function

$$\Theta(q, y) = (y^{1/2} + y^{-1/2}) \prod_{m \geq 1} \frac{(1 + yq^m)(1 + y^{-1}q^m)}{(1 - q^m)^2}$$

and the Weierstraß elliptic function

$$\wp(q, y) = \frac{1}{12} - \frac{y}{(1+y)^2} + \sum_{m \geq 1} \sum_{d|m} d((-y)^d - 2 + (-y)^{-d})q^m.$$

Define Jacobi forms $\phi_{k,1}$ of *weight* k and *index* 1 by

$$\phi_{-2,1}(q, y) = \Theta(q, y)^2, \quad \phi_{0,1}(q, y) = 12\Theta(q, y)^2\wp(q, y).$$

We also require the weight k and index 0 Eisenstein series

$$E_k(q) = 1 - \frac{2k}{B_k} \sum_{m \geq 1} \sum_{d|m} d^{k-1}q^m, \quad k = 2, 4, 6,$$

where the B_k are the Bernoulli numbers, and the modular discriminant

$$\Delta(q) = \frac{E_4^3 - E_6^2}{1728} = q \prod_{m \geq 1} (1 - q^m)^{24}.$$

⁹Let $\mathbb{H} = \{\tau \in \mathbb{C} : \text{Im}(\tau) > 0\}$ denote the upper half-plane.

We define the ring of quasi-Jacobi forms of even weight as the free polynomial algebra

$$\mathcal{J} = \mathbb{Q}[E_2, E_4, E_6, \phi_{-2,1}, \phi_{0,1}].$$

The weight/index assignments to the generators induce a bigrading

$$\mathcal{J} = \bigoplus_{k \in \mathbb{Z}} \bigoplus_{m \geq 0} \mathcal{J}_{k,m}$$

by weight k and index m .

Lemma 2.1 ([9, Theorem 2.2]). *Let $\phi \in \mathcal{J}_{*,m}$ be a quasi-Jacobi form of index $m \geq 1$. For all $d, r \in \mathbb{Z}$, the coefficient $[\phi]_{q^d y^r}$ only depends on $2d - \frac{r^2}{2m}$ and the set $\{\pm[r]\}$, where $[r] \in \mathbb{Z}/2m\mathbb{Z}$ is the residue of r .*

By Lemma 2.1, we may denote the $q^d y^r$ -coefficient of ϕ by

$$\phi \left[2d - \frac{r^2}{2m}, \pm[r] \right] = [\phi]_{q^d y^r}. \quad (12)$$

If ϕ is of index 0, we set $\phi[2d, 0] = [\phi]_{q^d}$. Lemma 2.1 remains valid if we replace ϕ by $f(q)\phi$ for any Laurent series $f(q)$, and we keep the notation as in (12) for the coefficients.

We will mainly focus on the quasi-Jacobi form

$$\phi = \left(-\wp + \frac{1}{12} E_2 \right) \Theta^2. \quad (13)$$

The following are some positivity results.

Lemma 2.2. *Let ϕ be as in (13). Then $\phi[D] \geq 0$ for all D and*

$$\phi[D] > 0 \iff D = 2n - \frac{r^2}{2} \geq 0 \text{ for some } n, r \in \mathbb{Z}.$$

Proof. By the Jacobi triple product, we have $\Theta = \vartheta_1/\eta^3$ where

$$\vartheta_1(q, y) = \sum_{n \in \mathbb{Z} + \frac{1}{2}} y^n q^{\frac{1}{2}n^2}, \quad \eta(q) = q^{\frac{1}{24}} \prod_{n \geq 1} (1 - q^n).$$

If we write $\Theta = \sum_{n,r} c(n, r) q^n y^r$, we therefore get

$$c(n, r) > 0 \iff \left(r \in \frac{1}{2}\mathbb{Z} \setminus \mathbb{Z} \text{ and } 2n \geq r^2 - \frac{1}{4} \right)$$

and $c(n, r) = 0$ otherwise. By the explicit expressions for the action of differential operators on quasi-Jacobi forms in [23, Appendix B], we have the identity

$$\phi = \Theta^2 D_y^2 \log \Theta = D_y^2(\Theta)\Theta - D_y(\Theta)^2.$$

Hence

$$\begin{aligned} [\phi]_{q^ny^k} &= \sum_{\substack{n=n_1+n_2 \\ k=k_1+k_2}} c(n_1, k_1)c(n_2, k_2)(k_1^2 - k_1k_2) \\ &= \frac{1}{2} \sum_{\substack{n=n_1+n_2 \\ k=k_1+k_2}} c(n_1, k_1)c(n_2, k_2)(k_1 - k_2)^2 \geq 0. \end{aligned} \quad (14)$$

Since ϕ is quasi-Jacobi, the coefficient $[\phi]_{q^ny^k}$ only depends on $4n - k^2$, hence we may assume $k \in \{0, 1\}$. The result now follows from (14) by a direct check. \square

2.2. Beauville–Bogomolov form. Let X be a holomorphic symplectic variety of dimension $2n$. The Beauville–Bogomolov form on $H^2(X, \mathbb{Z})$ induces an embedding

$$H^2(X, \mathbb{Z}) \hookrightarrow H_2(X, \mathbb{Z}), \quad \alpha \mapsto (\alpha, -),$$

which is an isomorphism after tensoring with \mathbb{Q} . Let

$$(-, -) : H_2(X, \mathbb{Z}) \times H_2(X, \mathbb{Z}) \rightarrow \mathbb{Q} \quad (15)$$

denote the unique \mathbb{Q} -valued extension of the Beauville–Bogomolov form.

If X is of $K3^{[n]}$ type with $n \geq 2$, there is an isomorphism of abelian groups

$$r : H_2(X, \mathbb{Z})/H^2(X, \mathbb{Z}) \rightarrow \mathbb{Z}/(2n - 2)\mathbb{Z}$$

such that $r(\alpha) = 1$ for some $\alpha \in H_2(X, \mathbb{Z})$ with $(\alpha, \alpha) = \frac{1}{2-2n}$. The morphism r is unique up to multiplication by ± 1 .

2.3. Curve classes. Consider a pair (X, β) where X is a holomorphic symplectic variety of $K3^{[n]}$ type, and $\beta \in H_2(X, \mathbb{Z})$ is a primitive curve class. The curve class β has the following invariants:

- (i) the Beauville–Bogomolov norm $(\beta, \beta) \in \mathbb{Q}$, and
- (ii) the residue $[\beta] \in H_2(X, \mathbb{Z})/H^2(X, \mathbb{Z})$.

The *residue set* of β is the subset

$$\pm[\beta] = \{\pm r([\beta])\} \subset \mathbb{Z}/(2n - 2)\mathbb{Z}.$$

It is independent of the choice of map r . If $n = 1$, we set $\pm[\beta] = 0$.

Given a (quasi-)Jacobi form ϕ of index $m = n - 1$, we define

$$\phi_\beta = \phi[(\beta, \beta), \pm[\beta]].$$

By Markman [19] (see also [22, Lemma 23]), two pairs (X, β) and (X', β') are deformation equivalent through a family of holomorphic symplectic manifolds which keeps the curve class of Hodge type if and only if the norms and the residue sets of β and β' agree. Hence, by identifying $H^*(X)$ with $H^*(X')$

via parallel transport and by property (c) of the virtual fundamental class, the Gromov–Witten invariants of the pairs (X, β) and (X', β') are equal.¹⁰

2.4. Proof of Theorem 0.1. Recall from (13) the quasi-Jacobi form ϕ .

Theorem 2.3 ([23]). *Let X be a holomorphic symplectic variety of $K3^{[n]}$ type, and let $\beta \in H_2(X, \mathbb{Z})$ be a primitive curve class. Then we have*

$$\mathrm{ev}_*[\overline{M}_{0,1}(X, \beta)]^{\mathrm{vir}} = \left(\frac{\phi^{n-1}}{\Delta} \right)_{\beta} h \in H^2(X, \mathbb{Q})$$

where $h = (\beta, -) \in H^2(X, \mathbb{Q})$ is the dual of β with respect to (15).

For the readers' convenience, we provide a proof of Theorem 2.3 at the end of this section. Theorem 2.3 together with the positivity of the Fourier coefficients of ϕ implies Theorem 0.1.

Proof of Theorem 0.1. By Lemma 2.2 the criterion in Theorem 0.1 holds if and only if

$$\left(\frac{\phi^{n-1}}{\Delta} \right)_{\beta} > 0,$$

hence by Theorem 2.3 if and only if the pushforward $\mathrm{ev}_*[\overline{M}_{0,1}(X, \beta)]^{\mathrm{vir}}$ is nontrivial. Since the pushforward is a class in $H^2(X, \mathbb{Q})$ supported on a uniruled subvariety, the first claim follows. The second claim follows from Proposition 1.1 and property (b) of the virtual fundamental class. \square

In the $K3^{[2]}$ case, we define

$$f = \frac{\phi}{\Delta} = \left(-\wp + \frac{1}{12}E_2 \right) \frac{\Theta^2}{\Delta}.$$

The first few values of f_{β} are listed in the following table.¹¹

(β, β)	$-\frac{5}{2}$	-2	$-\frac{1}{2}$	0	$\frac{3}{2}$	2	$\frac{7}{2}$	4	$\frac{11}{2}$	6
f_{β}	0	1	4	30	120	504	1980	6160	23576	60720

TABLE 1. The first few multiplicities of uniruled divisors for $K3^{[2]}$.

¹⁰The (reduced) virtual fundamental class can also be defined via symplectic geometry and the twistor space of X ; see [6]. Hence, the Gromov–Witten invariants are invariant also under (nonnecessarily algebraic) symplectic deformations of (X, β) which keep β of Hodge type. The invariance under nonalgebraic deformations is not needed for our application to the Fano variety of lines in a cubic 4-fold.

¹¹When $n = 2$, the value $(\beta, \beta) \in \mathbb{Q}$ uniquely determines $\pm[\beta] \subset \mathbb{Z}/2\mathbb{Z}$.

2.5. Gromov–Witten correspondence. In this section, we specialize to the $K3^{[2]}$ case. Recall the Gromov–Witten correspondence GW_β in (8). We also define

$$g = \left(-\frac{12}{5}\wp - E_2 \right) \frac{\Theta^2}{\Delta}.$$

Theorem 2.4 ([23]). *Let X be a holomorphic symplectic 4-fold of $K3^{[2]}$ type, and let $\beta \in H_2(X, \mathbb{Z})$ be a primitive curve class. If $(\beta, \beta) \neq 0$, then GW_β is diagonalizable with eigenvalues*

$$\lambda_0 = 0, \quad \lambda_1 = (\beta, \beta)f_\beta, \quad \lambda_2 = (\beta, \beta)g_\beta,$$

and eigenspaces

$$V_{\lambda_1} = \mathbb{Q}\langle h, h^3, (he_i)_{i=1, \dots, 22} \rangle, \quad V_{\lambda_2} = \mathbb{Q}v.$$

Here $h = (\beta, -) \in H^2(X, \mathbb{Q})$ is the dual of β with respect to (15), $\{e_i\}_{i=1, \dots, 22}$ is a basis of the orthogonal of h in $H^2(X, \mathbb{Q})$, and

$$v = 5h^2 - \frac{1}{6}(\beta, \beta)c_2(X) \in H^4(X, \mathbb{Q}).$$

One can show that the eigenvalues λ_1, λ_2 are integral, and if $(\beta, \beta) > 0$ then $\lambda_2 > \lambda_1 > 0$. The first few eigenvalues are listed in Table 2.

(β, β)	$-\frac{5}{2}$	-2	$-\frac{1}{2}$	0	$\frac{3}{2}$	2	$\frac{7}{2}$	4	$\frac{11}{2}$	6
λ_1	0	-2	-2	0	180	1008	6930	24640	129668	364320
λ_2	3	0	0	0	945	3840	53760	138240	1237005	2661120

TABLE 2. The first eigenvalues of GW_β for $K3^{[2]}$.

2.6. Proof of Theorem 2.3. A very general pair (X, β) has Picard rank 1.¹² Hence there exists $N_\beta \in \mathbb{Q}$ such that

$$\text{ev}_*[\overline{M}_{0,1}(X, \beta)]^{\text{vir}} = N_\beta h \in H^2(X, \mathbb{Q}).$$

By specialization, this also holds for any pair (X, β) as in Theorem 2.3.

We will evaluate N_β on the Hilbert scheme of n points on an elliptic $K3$ surface \mathcal{S} with a section. By Section 2.3, we may assume

$$\beta = \mathbf{B} + (d+1)\mathbf{F} + r\mathbf{A} \in H_2(\text{Hilb}^n(\mathcal{S}), \mathbb{Z}), \quad d \geq -1, \quad r \in \mathbb{Z}, \quad (16)$$

where $\mathbf{B}, \mathbf{F} \in H_2(\mathcal{S}, \mathbb{Z})$ are the classes of the section and fiber of the elliptic fibration, and $\mathbf{A} \in H_2(\text{Hilb}^n(\mathcal{S}), \mathbb{Z})$ is the class of an exceptional curve (for $n \geq 2$). Here we apply the natural identification

$$H_2(\text{Hilb}^n(\mathcal{S}), \mathbb{Z}) \simeq H_2(\mathcal{S}, \mathbb{Z}) \oplus \mathbb{Z}\mathbf{A}.$$

¹²In this statement, we allow X to be a holomorphic symplectic manifold.

Let $F_0 \subset S$ be a nonsingular fiber, and let $x_1, \dots, x_{n-1} \in S \setminus F_0$ be distinct points. Consider the curve

$$C = \{x_1 + \dots + x_{n-1} + x' : x' \in F_0\} \subset \text{Hilb}^n(S).$$

Then $\int_{[C]} h = 1$ and hence by the first equation in [23, Theorem 2], we find

$$N_\beta = \int_{[\overline{M}_{0,1}(X,\beta)]^{\text{vir}}} \text{ev}^*[C] = \left[\frac{\phi^{n-1}}{\Delta} \right]_{q^d y^r} = \left(\frac{\phi^{n-1}}{\Delta} \right)_\beta. \quad \square$$

2.7. Proof of Theorem 2.4. Consider the 2-pointed class

$$Z_\beta = \text{ev}_{12*}[\overline{M}_{0,2}(X, \beta)]^{\text{vir}} \in H^8(X \times X, \mathbb{Q}).$$

By the divisor equation [10] and Theorem 2.3, we have

$$\int_{Z_\beta} \gamma \otimes \delta = \left(\int_\beta \delta \int_\gamma h \right) f_\beta$$

for all $\delta \in H^2(X, \mathbb{Q})$ and $\gamma \in H^6(X, \mathbb{Q})$.¹³ Hence

$$\text{GW}_\beta(\delta) = \left(\int_\beta \delta \right) f_\beta h \in H^2(X, \mathbb{Q}),$$

$$\text{GW}_\beta(\gamma) = \left(\int_\gamma h \right) f_\beta \beta \in H^6(X, \mathbb{Q}).$$

Now consider the (4, 4)-Künneth factor of Z_β ,

$$Z_\beta^{4,4} \in H^4(X) \otimes H^4(X).$$

By monodromy invariance under the group $\text{SO}(H^2(X, \mathbb{C}), h)$, we have

$$\begin{aligned} Z_\beta^{4,4} = & a_\beta h^2 \otimes h^2 + b_\beta (h^2 \otimes c_2(X) + c_2(X) \otimes h^2) + c_\beta c_2(X) \otimes c_2(X) \\ & + d_\beta (h \otimes h)_{c_{BB}} + e_\beta [\Delta_X]^{4,4} \end{aligned}$$

for some $a_\beta, b_\beta, c_\beta, d_\beta, e_\beta \in \mathbb{Q}$; see [13, Section 4]. Here

$$c_{BB} \in \text{Sym}^2(H^2(X, \mathbb{Q})) \subset H^2(X, \mathbb{Q}) \otimes H^2(X, \mathbb{Q})$$

is the inverse of the Beauville–Bogomolov class.

Since $\int_{Z_\beta} \sigma^2 \otimes \bar{\sigma}^2 = 0$, we have $e_\beta = 0$. Also, since the Gromov–Witten correspondence is equivariant with respect to multiplication by σ , we find

$$\text{GW}_\beta(h\sigma) = \text{GW}_\beta(h)\sigma = (\beta, \beta) f_\beta h \sigma.$$

Hence $d_\beta = f_\beta$. Together with Proposition 1.2 and $\int_X v^2 = 48(\beta, \beta)^2 \neq 0$, this implies

$$Z_\beta^{4,4} = \psi_\beta \frac{v \otimes v}{48(\beta, \beta)^2} + f_\beta (h \otimes h) \left(c_{BB} - \frac{h \otimes h}{(\beta, \beta)} \right) \quad (17)$$

for some $\psi_\beta \in \mathbb{Q}$. It remains to determine ψ_β .

¹³We have suppressed an application of Poincaré duality here.

As in the proof of Theorem 2.3, let S be an elliptic $K3$ surface with a section, and let β be as in (16). Consider the fiber class of the Lagrangian fibration $\text{Hilb}^2(S) \rightarrow \mathbb{P}^2$ induced by the elliptic fibration $S \rightarrow \mathbb{P}^1$,

$$\mathbf{L} \in H^4(\text{Hilb}^2(S), \mathbb{Q}).$$

We have

$$\int_{\text{Hilb}^2(S)} h^2 \mathbf{L} = 2, \quad \int_{\text{Hilb}^2(S)} v \mathbf{L} = 10, \quad \int_{\text{Hilb}^2(S) \times \text{Hilb}^2(S)} (h\mathbf{L} \otimes h\mathbf{L}) c_{BB} = 0.$$

Then [23, Theorem 1] and (17) imply the relation

$$\left(\frac{\Theta^2}{\Delta} \right)_\beta = \int_{Z_\beta} \mathbf{L} \otimes \mathbf{L} = \frac{10^2}{48(\beta, \beta)^2} \psi_\beta - \frac{2^2}{(\beta, \beta)} f_\beta.$$

Hence

$$\psi_\beta = \frac{12(\beta, \beta)}{25} \left(4f + \mathcal{H}_1 \left(\frac{\Theta^2}{\Delta} \right) \right)_\beta$$

where

$$\mathcal{H}_m = 2q \frac{d}{dq} - \frac{1}{2m} \left(y \frac{d}{dy} \right)^2, \quad m \geq 1$$

is the *heat operator*. Explicit formulas for the derivatives of Jacobi forms can be found in [23, Appendix B], and this yields $\psi_\beta = (\beta, \beta)g_\beta$ as desired. \square

3. RATIONAL CURVES IN THE FANO VARIETIES OF LINES

We give the proof of Theorem 0.2. From now on, let F be the Fano variety of lines in a very general cubic 4-fold Y , and let $\beta \in H_2(F, \mathbb{Z})$ be the primitive curve class.

3.1. Degeneracy locus. The variety F is naturally embedded in the Grassmannian $\text{Gr}(2, 6)$. Let \mathcal{U} and \mathcal{Q} be the tautological bundles of ranks 2 and 4 with the short exact sequence

$$0 \rightarrow \mathcal{U} \rightarrow \mathbb{C}^6 \otimes \mathcal{O}_{\text{Gr}(2,6)} \rightarrow \mathcal{Q} \rightarrow 0.$$

We use $\mathcal{U}_F, \mathcal{Q}_F$ to denote the restriction of \mathcal{U}, \mathcal{Q} on F . Let $H = c_1(\mathcal{U}_F^*)$ be the hyperplane class on F with respect to the Plücker embedding. By [4], the primitive curve class $\beta \in H_2(F, \mathbb{Z})$ is characterized by $\int_\beta H = 3$.

The indeterminacy locus S of the rational map (2) consists of lines $l \subset Y$ with normal bundle

$$\mathcal{N}_{l/Y} = \mathcal{O}_l(-1) \oplus \mathcal{O}_l(1)^{\oplus 2}.$$

For every line $l \subset Y$ corresponding to $s \in S$, there is a pencil of planes tangent to Y along l . The residual lines of this pencil form the rational curve $\phi(p^{-1}(s)) \subset F$. By [1, Proposition 6], we have

$$\int_{[\phi(p^{-1}(s))]} H = 3.$$

Hence the curve $\phi(p^{-1}(s))$ lies in the primitive curve class β . Moreover, by the calculations in [1, Theorem 8], we find

$$\phi_*[D] = 60H \in H^2(F, \mathbb{Q}). \quad (18)$$

In [1], the surface S is shown to be nonsingular, and is expressed as the (rank ≤ 2) degeneracy locus of the (sheafified) Gauss map

$$g : \text{Sym}^2(\mathcal{U}_F) \rightarrow \mathcal{Q}_F^*$$

associated to the cubic Y . Let $\pi : \mathbb{P}\text{Sym}^2(\mathcal{U}_F) \rightarrow F$ be the \mathbb{P}^2 -bundle and let h be the relative hyperplane class. Then S is isomorphic to the zero locus S' of a section of the rank 4 vector bundle $\pi^*\mathcal{Q}_F^* \otimes \mathcal{O}(h)$ on $\mathbb{P}\text{Sym}^2(\mathcal{U}_F)$. Let $H_{S'}, h_{S'}$ be the restrictions of the divisor classes π^*H, h on S' .

Lemma 3.1. *We have*

$$\int_{S'} H_{S'}^2 = \int_{S'} H_{S'} h_{S'} = \int_{S'} h_{S'}^2 = 315.$$

Proof. Let $c = c_2(\mathcal{U}_F^*) \in H^4(F, \mathbb{Q})$. Since $S' \subset \mathbb{P}\text{Sym}^2(\mathcal{U}_F)$ is the zero locus of a section of the vector bundle $\pi^*\mathcal{Q}_F^* \otimes \mathcal{O}(h)$, a direct calculation yields

$$\begin{aligned} [S'] &= c_4(\mathcal{Q}_F^* \otimes \mathcal{O}(h)) = 5(\pi^*H^2 - \pi^*c)h^2 - \frac{35}{6}\pi^*H^3 \cdot h + \frac{10}{3}\pi^*H^4 \\ &\in H^8(\mathbb{P}\text{Sym}^2(\mathcal{U}_F), \mathbb{Q}). \end{aligned}$$

The lemma follows from the projection formula, the intersection numbers calculated in [1, Lemma 4], and the projective bundle formula associated to $\pi : \mathbb{P}\text{Sym}^2(\mathcal{U}_F) \rightarrow F$,

$$h^3 = 3\pi^*H \cdot h^2 - (2\pi^*H^2 + 4\pi^*c)h + \frac{5}{3}\pi^*H^3 \in H^6(\mathbb{P}\text{Sym}^2(\mathcal{U}_F), \mathbb{Q}). \quad \square$$

From the lemma we can deduce the following.

Lemma 3.2. *We have $H_{S'} = h_{S'} \in H^2(S', \mathbb{Q})$.*

Proof. Let $U \subset \mathbb{P}(H^0(\mathbb{P}^5, \mathcal{O}(3)))$ be the open locus parametrizing nonsingular cubic 4-folds and consider the incidence correspondence

$$I = \{(Y, \ell) : \ell \subset Y \text{ corresponding to } s \in S\} \subset U \times \text{Gr}(2, 6).$$

By [1, Proof of Lemma 1] and general properties of determinantal varieties, the fibers of the projection $I \rightarrow \text{Gr}(2, 6)$ are irreducible. Using the homogeneity of $\text{Gr}(2, 6)$ we find that I is irreducible. In particular, the monodromy of the projection $I \rightarrow U$ acts transitively on the set of connected components of a very general fiber. Since the restriction $\pi|_{S'} : S' \rightarrow S$ is an isomorphism, the same applies to the monodromy of S' .

Let S_i be the connected components of S' and write

$$H_{S'} - h_{S'} = \sum_i a_i$$

with $a_i \in H^2(S_i, \mathbb{Q})$. The classes $H_{S'}$ and $h_{S'}$ are monodromy invariant since they are restricted from $\text{Gr}(2, 6)$ and $\mathbb{P}\text{Sym}^2(\mathcal{U}_F)$. Hence for every monodromy operator $g : H^*(S', \mathbb{Q}) \rightarrow H^*(S', \mathbb{Q})$ that sends the i -th to the j -th component, we find $ga_i = a_j$ and

$$\int_{S'} H_{S'} \cdot a_i = \int_{S'} gH_{S'} \cdot ga_i = \int_{S'} H_{S'} \cdot a_j.$$

In particular, the intersection number $\int_{S'} H_{S'} \cdot a_i$ is independent of i .

By Lemma 3.1 we have $\int_{S'} H_{S'}(H_{S'} - h_{S'}) = 0$, which together with the above implies that $\int_{S'} H_{S'} \cdot a_i = 0$ for all i . Now $\int_{S'} (H_{S'} - h_{S'})^2 = 0$ and the Hodge index theorem force $a_i = 0$. \square

Corollary 3.3. *If H_S is the restriction of H to S , then we have*

$$c_1(S) = -3H_S \in H^2(S, \mathbb{Q}).$$

Proof. The surface S' is the zero locus of a regular section of the vector bundle $\pi^* \mathcal{Q}_F^* \otimes \mathcal{O}(h)$. Hence by the adjunction formula and Lemma 3.2, the first Chern class $c_1(S)$ is proportional to H_S . The coefficient is determined by a calculation of intersection numbers; see [1, Remark in Section 2]. \square

3.2. Divisorial contribution. By Proposition 1.1, the moduli space of stable maps $\overline{M}_{0,1}(F, \beta)$ is pure of dimension 3. Recall the decomposition (1),

$$\overline{M}_{0,1}(F, \beta) = M^0 \cup M^1,$$

such that a general fiber of $\text{ev} : M^i \rightarrow \text{ev}(M^i) \subset F$ is of dimension i . We first analyze the component M^0 .

By construction, the family of maps $p : D \rightarrow S$ in (3) has a factorization

$$\phi : D \rightarrow M^0 \xrightarrow{\text{ev}} F.$$

We have seen in (18) that

$$\phi_*[D] = 60H \in H^2(F, \mathbb{Q}).$$

On the other hand, by Theorem 2.3¹⁴ together with property (b) of the virtual fundamental class, we find

$$\text{ev}_*[M^0] = \text{ev}_*[\overline{M}_{0,1}(F, \beta)] = \text{ev}_*[\overline{M}_{0,1}(F, \beta)]^{\text{vir}} = 60H \in H^2(F, \mathbb{Q}).$$

To conclude $M^0 = D$, it suffices to prove the following proposition.

Proposition 3.4. *For a very general F , each $s \in S$ yields a distinct rational curve $\phi(p^{-1}(s)) \subset F$.*

¹⁴By [4], we have $(\beta, \beta) = \frac{3}{2}$ and $(\beta, -) = \frac{1}{2}H \in H^2(F, \mathbb{Q})$.

Proof. Let $s_1, s_2 \in S$ be two distinct points and suppose

$$\phi(p^{-1}(s_1)) = \phi(p^{-1}(s_2)) \subset F.$$

For $i = 1, 2$, let $l_i \subset Y$ be the line corresponding to s_i , and let $P_i \subset \mathbb{P}^5$ be the 3-dimensional linear subspace spanned by the tangent planes along l_i . Then necessarily $P_1 = P_2$. Otherwise, the intersection $P_1 \cap P_2$ is a plane that contains all lines in Y corresponding to the points on $\phi(p^{-1}(s_i))$. The fact that Y contains a plane violates the very general assumption. We also know $l_1 \cap l_2 = \emptyset$. Otherwise, the plane spanned by l_1 and l_2 is tangent to Y along both l_1 and l_2 , which is impossible.

Consider the Gauss map¹⁵ associated to the cubic Y ,

$$\mathcal{D} : \mathbb{P}^5 \rightarrow \mathbb{P}^{5*}.$$

By definition, the image $\mathcal{D}(l_i) \subset \mathbb{P}^{5*}$ is a line which is dual to $P_i \subset \mathbb{P}^5$. Following the argument of Clemens and Griffiths [7, Section 6], we may assume that l_1, l_2 are given by the equations

$$\begin{aligned} X_2 = X_3 = X_4 = X_5 = 0, \\ X_0 = X_1 = X_4 = X_5 = 0. \end{aligned}$$

Then the condition $P_1 = P_2$ forces $\mathcal{D}(l_1) = \mathcal{D}(l_2)$ to be given by the equations

$$X_0^* = X_1^* = X_2^* = X_3^* = 0.$$

As a result, the cubic polynomial of Y takes the form

$$\begin{aligned} X_4 Q_4^1(X_0, X_1) + X_5 Q_5^1(X_0, X_1) \\ + X_4 Q_4^2(X_2, X_3) + X_5 Q_5^2(X_2, X_3) + R_1 + R_2. \end{aligned} \quad (19)$$

Here the Q_i^j are quadratic polynomials, R_1 consists of terms of degree at least 2 in $\{X_4, X_5\}$, and R_2 consists of terms of degree 1 in each of $\{X_0, X_1\}, \{X_2, X_3\}, \{X_4, X_5\}$. The total number of possibly nonzero coefficients in (19) is

$$4 \cdot 3 + (4 \cdot 3 + 4) + 2 \cdot 2 \cdot 2 = 36.$$

On the other hand, the subgroup of $\mathrm{GL}(\mathbb{C}^6)$ fixing two disjoint lines in \mathbb{P}^5 is of dimension

$$4 + 4 + 3 \cdot 4 = 20,$$

resulting in a locus of dimension $36 - 20 = 16$ in the moduli space of cubic 4-folds. This again contradicts the very general assumption of Y . \square

¹⁵It is called the *dual mapping* in [7].

3.3. Non-contribution. We use the Gromov–Witten correspondence introduced in (8) to eliminate the component M^1 . Recall that by property (b) of the virtual fundamental class, the class $[\overline{M}_{0,2}(F, \beta)]^{\text{vir}}$ in (8) equals the ordinary fundamental class.

We begin by calculating the contribution of $M^0 = D$ to the Gromov–Witten correspondence

$$\text{GW}_\beta : H^4(F, \mathbb{Q}) \rightarrow H^4(F, \mathbb{Q}). \quad (20)$$

Recall the diagram (3) and consider morphisms

$$\begin{aligned} \Phi_1^D &= p_*\phi^* : H^4(F, \mathbb{Q}) \rightarrow H^2(S, \mathbb{Q}), \\ \Phi_2^D &= \phi_*p^* : H^2(S, \mathbb{Q}) \rightarrow H^4(F, \mathbb{Q}). \end{aligned}$$

Comparing with (9) and (10), we see that $\Phi_2^D \circ \Phi_1^D = \phi_*p^*p_*\phi^*$ gives the contribution of D to the Gromov–Witten correspondence (20).

Let $c = c_2(\mathcal{U}_F^*) \in H^4(F, \mathbb{Q})$. Using the short exact sequence

$$0 \rightarrow T_F \rightarrow T_{\text{Gr}(2,6)}|_F \rightarrow \text{Sym}^3(\mathcal{U}_F^*) \rightarrow 0,$$

we find

$$8c = 5H^2 - c_2(F) = v_F \in H^4(F, \mathbb{Q})$$

where v_F is the class defined in (11).¹⁶ There is the following explicit calculation.

Proposition 3.5. *We have*

$$\phi_*p^*p_*\phi^*c = 945c \in H^4(F, \mathbb{Q}).$$

Proof. The argument in Proposition 1.2 shows that c is an eigenvector of $\phi_*p^*p_*\phi^*$. To determine the eigenvalue, it suffices to compute the intersection number

$$\int_F \phi_*p^*p_*\phi^*c \cdot H^2. \quad (21)$$

By the projection formula, we have

$$\begin{aligned} \int_F \phi_*p^*p_*\phi^*c \cdot H^2 &= \int_D p^*p_*\phi^*c \cdot \phi^*H^2 \\ &= \int_S p_*\phi^*c \cdot p_*\phi^*H^2 = \int_F \phi_*p^*p_*\phi^*H^2 \cdot c. \end{aligned}$$

Again by the argument in Proposition 1.2, we know that $\phi_*p^*p_*\phi^*H^2$ is proportional to c . Hence we can deduce the intersection number (21) by calculating instead

$$\int_F \phi_*p^*p_*\phi^*H^2 \cdot H^2 = \int_S (p_*\phi^*H^2)^2.$$

¹⁶The proportionality of c and v_F also follows from the fact that c is represented by a rational (hence Lagrangian) surface.

Let ξ be the relative hyperplane class of the projective bundle

$$p : D = \mathbb{P}(\mathcal{N}_{S/F}) \rightarrow S.$$

By [1, Proposition 6] and the projective bundle formula, we find

$$p_*\phi^*H^2 = p_*(7p^*H_S + 3\xi)^2 = 42H_S - 9c_1(\mathcal{N}_{S/F}) \in H^2(S, \mathbb{Q}),$$

where H_S is the restriction of H to S . Moreover, Corollary 3.3 yields

$$c_1(\mathcal{N}_{S/F}) = -c_1(S) = 3H_S \in H^2(S, \mathbb{Q}).$$

Hence we obtain

$$p_*\phi^*H^2 = 15H_S \in H^2(S, \mathbb{Q}).$$

Applying Lemma 3.1, we find the intersection number

$$\int_F \phi_*p^*p_*\phi^*H^2 \cdot H^2 = \int_S (p_*\phi^*H^2)^2 = 15^2 \cdot 315 = 70875.$$

Finally, by the intersection numbers calculated in [1, Lemma 4], we have

$$\int_F \phi_*p^*p_*\phi^*c \cdot H^2 = \int_F \phi_*p^*p_*\phi^*H^2 \cdot c = 70875 \cdot \frac{27}{45} = 42525$$

and hence

$$\phi_*p^*p_*\phi^*c = \frac{42525}{45}c = 945c \in H^4(F, \mathbb{Q}). \quad \square$$

The eigenvalue in Proposition 3.5 coincides with the one in Theorem 2.4,

$$\text{GW}_\beta(c) = 945c \in H^4(F, \mathbb{Q}).$$

Hence the final step is to show that if the component M^1 is nonempty, then it has to contribute nontrivially to the Gromov–Witten correspondence (20).

If $M' \subset M^1$ is a nonempty irreducible component, consider the restriction of (9)

$$\begin{array}{ccc} M' & \xrightarrow{\text{ev}} & F \\ \downarrow p & & \\ T' & & \end{array}$$

where $T' \subset p(M^1) \subset \overline{M}_{0,0}(F, \beta)$ is the base of M' . We define morphisms

$$\begin{aligned} \Phi_1^{M'} : H^4(F, \mathbb{Q}) &\rightarrow H_2(T', \mathbb{Q}), \quad \gamma \mapsto p_*(\text{ev}^*\gamma \cap [M']), \\ \Phi_2^{M'} = \text{ev}_*p^* : H_2(T', \mathbb{Q}) &\rightarrow H^4(F, \mathbb{Q}). \end{aligned}$$

By definition, the composition $\Phi_2^{M'} \circ \Phi_1^{M'}$ gives the contribution of M' to the Gromov–Witten correspondence (20).

Proposition 3.6. *If $M' \subset M^1$ is a nonempty irreducible component, then we have*

$$\Phi_2^{M'} \circ \Phi_1^{M'}(c) = Nc \in H^4(F, \mathbb{Q})$$

for some $N > 0$.

Proof. Let $Z' = \text{ev}(M')$ with $\iota : Z' \hookrightarrow F$ the embedding. Consider the following diagram

$$\begin{array}{ccccc}
\widetilde{M}' & \xrightarrow{\widetilde{\text{ev}}} & \widetilde{Z}' & & \\
\downarrow \tau & & \downarrow & \searrow \tilde{\iota} & \\
M' & \xrightarrow{\text{ev}} & Z' & \xrightarrow{\iota} & F, \\
\downarrow p & & & & \\
T' & & & &
\end{array}$$

where \widetilde{M}' and \widetilde{Z}' are simultaneous resolutions of M' and Z' .

We calculate $\Phi_1^{M'}(c) \in H_2(T', \mathbb{Q})$. By the projection formula, we have¹⁷

$$\begin{aligned}
\Phi_1^{M'}(c) &= p_*(\text{ev}^* \iota^* c \cap [M']) \\
&= p_* \tau_* \tau^* \text{ev}^* \iota^* c \\
&= p_* \tau_* \widetilde{\text{ev}}^* \iota^* c \in H_2(T', \mathbb{Q}).
\end{aligned}$$

Since Z' is Lagrangian, we find

$$[Z'] = \tilde{\iota}_*[\widetilde{Z}'] = N'c \in H^4(F, \mathbb{Q})$$

for some $N' > 0$. The intersection number $\int_F c^2 = 27$ calculated in [1, Lemma 4] then implies

$$\tilde{\iota}^* c = 27N'[\tilde{x}] \in H^4(\widetilde{Z}', \mathbb{Q})$$

for any point $\tilde{x} \in \widetilde{Z}'$. This yields

$$\Phi_1^{M'}(c) = 27N' p_* \tau_* \widetilde{\text{ev}}^*[\tilde{x}] = 27N'[V_x] \in H_2(T', \mathbb{Q}),$$

where $V_x \subset T'$ parametrizes rational curves through a general point $x \in Z'$. In particular, we see that $\Phi_1^{M'}(c) \in H_2(T', \mathbb{Q})$ is an effective curve class.

As a result, the class

$$\Phi_2^{M'} \circ \Phi_1^{M'}(c) = \text{ev}_* p^* \Phi_1^{M'}(c) \in H^4(F, \mathbb{Q})$$

is an effective sum of classes of Lagrangian surfaces, and hence a positive multiple of c . \square

We conclude $M^1 = \emptyset$, and the proof of Theorem 0.2 is complete.

APPENDIX A. SKETCH OF A CLASSICAL PROOF OF THEOREM 0.2

We sketch a proof of Theorem 0.2 via the classical geometry of cubic hypersurfaces. Let $Y \subset \mathbb{P}^5$ be a very general cubic 4-fold, and let F be the Fano variety of lines in Y .

¹⁷Since \widetilde{M}' is nonsingular, we have suppressed an application of Poincaré duality here.

Consider the correspondence given by the universal family

$$\begin{array}{ccc} \mathcal{L} & \xrightarrow{q_Y} & Y \\ \downarrow q_F & & \\ F & & \end{array}$$

A rational curve $R \subset F$ corresponds to a surface $Z = q_Y(q_F^{-1}(R)) \subset Y$. If R lies in the primitive curve class of F , then we have

$$[Z] = H_Y^2 \in H^4(Y, \mathbb{Z})$$

with H_Y the hyperplane class on Y .

Step 1. Let $j : Y \hookrightarrow \mathbb{P}^5$ be the embedding. Since the surface $j(Z) \subset \mathbb{P}^5$ is of degree 3, we know from [12, Page 173] that $j(Z)$ lies in a hyperplane $\mathbb{P}^4 \subset \mathbb{P}^5$. Hence Z is contained in the hyperplane section

$$Y' = Y \cap \mathbb{P}^4 \subset \mathbb{P}^4.$$

Step 2. By [12, Page 525, Proposition], the surface $Z \subset Y'$ belongs to one of the following classes:

- (i) a cubic rational normal scroll;
- (ii) a cone over a twisted cubic curve;
- (iii) a cubic surface given by a hyperplane section of $Y' \subset \mathbb{P}^4$.

Since (i) and (ii) cannot hold for a very general¹⁸ cubic 4-fold, we find that Z is a cubic surface of the form

$$Z = Y \cap \mathbb{P}^3.$$

Step 3. The singularities of cubic surfaces were classified long ago; see [8, Chapter 9] and [17, Section 2]. Since Z is integral, it satisfies one of the following conditions:

- (i) Z has rational double point singularities;
- (ii) Z has a simple elliptic singularity;
- (iii) Z is integral but not normal.

By definition, the surface Z is swept out by a 1-dimensional family of lines parameterized by a rational curve. Hence we may narrow down to case (iii).

¹⁸Case (i) corresponds to the divisor \mathcal{C}_{12} in the moduli space of cubic 4-folds; see [14]. Case (ii) is a degeneration of (i), and can be argued by a similar dimension count.

Step 4. By further classification results (see [17, Section 2.3]), the surface Z is projectively equivalent to one of the four surfaces with explicit equations:

$$\begin{aligned} X_0^2 X_1 + X_2^2 X_3 &= 0, \\ X_0 X_1 X_2 + X_0^2 X_3 + X_1^3 &= 0, \\ X_1^3 + X_2^3 + X_1 X_2 X_3 &= 0, \\ X_1^3 + X_2^2 X_3 &= 0. \end{aligned}$$

In each of the four cases, the singular locus of Z is a line $l \subset Z$, and the 1-dimensional family of lines covering Z is given by the residual lines of the planes containing l . Hence we conclude that all rational curves in the primitive curve class of F are given by the uniruled divisor (3). The uniqueness part of Theorem 0.2 follows from Proposition 3.4.

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UNIVERSITÄT BONN, MATHEMATISCHES INSTITUT
E-mail address: `georgo@math.uni-bonn.de`

MASSACHUSETTS INSTITUTE OF TECHNOLOGY, DEPARTMENT OF MATHEMATICS
E-mail address: `jlshen@mit.edu`

PEKING UNIVERSITY, BEIJING INTERNATIONAL CENTER FOR MATHEMATICAL RESEARCH
E-mail address: `qizheng@math.pku.edu.cn`