Multiple cover formulas for K3 geometries, wall-crossing, and Quot schemes

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Abstract

Let S be a K3 surface. We study the reduced Donaldson-Thomas theory of the cap $(S \times \mathbb{P}^1)/S_\infty$ by a second cosection argument. We obtain four main results:
(i) A multiple cover formula for the rank 1 Donaldson-Thomas theory of $S \times E$, leading to a complete solution of this theory. (ii) Evaluation of the wall-crossing term in Nesterov's quasimap wall-crossing between the punctual Hilbert schemes and Donaldson-Thomas theory of $S \times \text{Curve}$. (iii) A multiple cover formula for the genus 0 Gromov-Witten theory of punctual Hilbert schemes. (iv) Explicit evaluations of virtual Euler numbers of Quot schemes of stable sheaves on K3 surfaces.

1 Introduction

1.1 Overview

Let S be a K3 surface. In this paper we consider three different types of counting theories:

- Gromov-Witten theory of moduli spaces of stable sheaves on S,
- Donaldson-Thomas theory of $S \times E$, where E is an elliptic curve,
- Virtual Euler characteristics of Quot schemes of stable sheaves.

As usual for K3 geometries, in all three theories the standard virtual fundamental classes of the moduli spaces vanish. Instead the counting theories are defined by a reduced virtual class. This deviation from the standard theory leads to surprising additional structure of the invariants. Two of them are taken up in this work. First, one expects a multiple cover formula that expresses counts in imprimitive (curve) classes in terms of those for primitive classes [13, 16, 41, 33, 27]. In this paper we prove such multiple cover formulas for the genus 0 Gromov-Witten theory of the punctual Hilbert schemes and the rank 1 Donaldson-Thomas theory of $S \times E$. In particular, together with the results of [34] the latter determines all rank 1 reduced Donaldson-Thomas invariants of the (non-strict) Calabi-Yau threefold $S \times E$. The second structural result concerns wall-crossing formulas, where due to ϵ-calculus [35, 32], one expects the vanishing of almost all wall-crossing contributions. We will consider the case of Nesterov's quasimap wall-crossing between moduli spaces of sheaves on K3 surfaces, in particular the punctual Hilbert schemes, and Donaldson-Thomas theory. Nesterov shows that these theories are related by a single wall-crossing term. We prove that this single term is precisely the virtual Euler number of the Quot scheme. We then use this connection to determine these virtual Euler numbers explicitly. The outcome is an intimate connection between all three counting theories above.

1.2 Three theories

1.2.1 Punctual Hilbert schemes

For the first geometry, consider the Hilbert scheme $S^{[n]}$ of n points on the K3 surface S. There exists a canonical isomorphism

$$H_2(S^{[n]}, \mathbb{Z}) \cong H_2(S, \mathbb{Z}) \oplus \mathbb{Z}A$$

where A is the extremal curve of the Hilbert-Chow morphism $S^{[n]} \to \mathrm{Sym}^n(S)$, [26, Sec.1.2].

Let $\beta \in H_2(S,\mathbb{Z})$ be an effective curve class, and let $\overline{M}_E(S^{[n]}, \beta + mA)$ be the moduli space of (unmarked) degree $\beta + mA$ stable maps from nodal degenerations of the elliptic curve E to $S^{[n]}$, see [33, 24, 39] for details. The moduli space is of reduced[1] virtual dimension 0. We define the Gromov-Witten count of elliptic curves in $S^{[n]}$ of class $\beta + mA$ with fixed j-invariant by

$$\mathsf{GW}^{S^{[n]}}_{E,\beta,m} = \int_{[\overline{M}_E(S^{[n]},\beta+mA)]^{\mathsf{vir}}} 1. \tag{1}$$

The invariant $\mathsf{GW}^{S^{[n]}}_{E,\beta,m}$ is related to the actual enumerative count of elliptic curves with fixed j-invariant by a (conjectural) genus 0 correction term, see [24].

1.2.2 $S \times E$

In this second geometry we consider the Hilbert scheme of curves in $S \times E$:

$$\mathrm{Hilb}_m(S \times E, (\beta, n)) = \{Z \subset S \times E \,|\, \mathsf{ch}_3(\mathcal{O}_Z) = m, [Z] = (\beta, n)\},$$

where we use the identification given by the Künneth decomposition

$$H_2(S \times E, \mathbb{Z}) \cong H_2(S,\mathbb{Z}) \oplus \mathbb{Z}E.$$

The elliptic curve (viewed as a group) acts on the Hilbert scheme by translation. The stack quotient $\mathrm{Hilb}_m(S \times E, (\beta, n))/E$ is of reduced virtual dimension 0 (in fact, an étale cover carries a symmetric perfect obstruction theory [28]), so we can define:

$$\mathsf{DT}^{S\times E}_{m,(\beta,n)} = \int_{[\mathrm{Hilb}_m(S\times E,(\beta,n))/E]^{\mathrm{vir}}} 1.$$

The number $\mathsf{DT}_{n,\beta,m} \in \mathbb{Q}$ is the Donaldson-Thomas count of curves in $S \times E$ in class $(\beta, n[\mathbb{P}^1])$ up to translation.

1.2.3 Quot schemes

Out third geometries are the Quot schemes. Let $F \in \mathrm{Coh}(S)$ be a coherent sheaf of positive rank which is Gieseker stable with respect to some polarization. We consider the Quot scheme

$$\mathrm{Quot}(F, u) = \{F \twoheadrightarrow Q \,|\, v(Q) = u\}$$

where $v(Q) := \mathsf{ch}(Q)\sqrt{\mathrm{td}_S}$ is the Mukai vector. The moduli space has a reduced perfect obstruction theory with virtual tangent bundle $T^{\mathsf{vir}} = R\,\mathrm{Hom}_S(\mathcal{K}, \mathcal{Q}) + \mathcal{O}$, see Section 3.2. The virtual Euler characteristic of the moduli space is defined after Fantechi and Göttsche [6] to be:

$$e^{\mathsf{vir}}(\mathrm{Quot}(F,u)) = \int_{[\mathrm{Quot}(F,u)]^{\mathrm{vir}}} c_{\mathsf{vd}}(T^{\mathsf{vir}})$$

where $\mathsf{vd} = \mathsf{rk}(T^{\mathsf{vir}})$ is the virtual dimension. If F is the structure sheaf these Euler characteristics were studied by Oprea and Pandharipande in [36] and are related to the Kawai-Yoshioka formula [14]. If $F = I_\eta$ is the ideal sheaf of a length n subscheme $\eta \in S^{[n]}$, and $u = (0, \beta, m)$ we write

$$\mathsf{Q}_{n,(\beta,m)} := e^{\mathsf{vir}}\left(\mathrm{Quot}(I_\eta, (0,\beta,m))\right).$$

[1] We will denote the reduced virtual fundamental classes in this paper by $[-]^{\mathsf{vir}}$. The (almost always vanishing) ordinary virtual class associated to the standard perfect obstruction theory will be denoted by $[-]^{\mathsf{std}}$.

1.2.4 Wall-crossing

Denis Nesterov in [22, 23] uses quasimaps to prove that

$$\mathsf{DT}^{S\times E}_{m,(\beta,n)} = \mathsf{GW}^{S^{[n]}}_{E,\beta,m} + (\text{Wall-crossing correction}),$$

where the wall-crossing term is the contribution from the extremal component in the relative Donaldson-Thomas theory of $S \times \mathbb{P}^1/S_\infty$. Our first main result is to make this wall-crossing term more explicit and relate it to the Quot scheme.

Theorem 1.1.

$$\mathsf{DT}^{S\times E}_{m,(\beta,n)} = \mathsf{GW}^{S^{[n]}}_{E,\beta,m} - \chi(S^{[n]}) \sum_{r|(\beta,m)} \frac{1}{r}(-1)^m \mathsf{Q}_{n,\frac{1}{r}(\beta,m)}$$

Our second result is a complete evaluation of the wall-crossing term:

Theorem 1.2. *The invariant $\mathsf{Q}_{n,(\beta,m)}$ only depends on the square $\beta\cdot\beta = 2h-2$. Moreover, if we write $\mathsf{Q}_{n,h,m} := \mathsf{Q}_{n,(\beta,m)}$ for this value, we have*

$$\sum_{h\geq 0}\sum_{m\in\mathbb{Z}} \mathsf{Q}_{n,h,m} p^m q^{h-1} = \frac{\mathbf{G}(p,q)^n}{\Theta(p,q)^2\Delta(q)}$$

where we let

$$\Theta(p,q) = (p^{1/2} - p^{-1/2}) \prod_{m\geq 1} \frac{(1-pq^m)(1-p^{-1}q^m)}{(1-q^m)^2}$$

$$\Delta(\tau) = q\prod_{n\geq 1}(1-q^n)^{24}$$

and $\mathbf{G}(p,q) = -\Theta(p,q)^2 \left(p\frac{d}{dp}\right)^2 \log(\Theta(p,q))$.

Remark 1.1. The case $n = 0$ has been obtained previously in [36, Thm 21].

Remark 1.2. Let $\beta_h \in \mathrm{Pic}(S)$ be a primitive effective class of square $2h-2$. Consider the generating series

$$\begin{aligned} \mathsf{DT}_n(p,q) &= \sum_{h\geq 0}\sum_{m\in\mathbb{Z}} \mathsf{DT}^{S\times E}_{m,(\beta_h,n} q^{h-1}(-p)^m \\ \mathsf{H}_n(p,q) &= \sum_{h\geq 0}\sum_{m\in\mathbb{Z}} \mathsf{GW}^{S^{[n]}}_{E,\beta_h+mA} q^{h-1}(-p)^m \end{aligned} \tag{2}$$

The following evaluation was proven in [34]:

$$\sum_{n\geq 0} \mathsf{DT}_n(p,q)\tilde{q}^{n-1} = -\frac{1}{\chi_{10}(p,q,\tilde{q})} \tag{3}$$

where χ_{10} is the Igusa cusp form (as in [33, Eqn. (12)]). We obtain the complete evaluation

$$\sum_{n\geq 0} \mathsf{H}_n(p,q)\tilde{q}^{h-1} = -\frac{1}{\chi_{10}(p,q,\tilde{q})} + \frac{1}{\Theta^2\Delta}\frac{1}{\tilde{q}}\prod_{n\geq 1}\frac{1}{(1-(\tilde{q}\mathbf{G})^n)^{24}}$$

which was conjectured in [33, Conj.A].

1.3 Multiple cover formula: $S \times E$

The main tool we will employ in this paper is a second-cosection argument for the Donaldson-Thomas theory of the cap $(S \times \mathbb{P}^1)/S_\infty$. It will also imply the following multiple cover formula, conjectured on the Gromov-Witten side in [33, Conj. B].

Define the (Fourier) coefficients of the Igusa cusp form by

$$c(h,n,m) := \left[\frac{1}{\chi_{10}(p,q,\tilde{q})} \right]_{q^h \tilde{q}^n p^m}.$$

By the evaluation (3) it is given up to a (confusing) index shift by the Donaldson-Thomas theory of $S \times E$ for a primitive class β_h of square $2h-2$,

$$\mathsf{DT}^{S\times E}_{m,(\beta_h,n)} = (-1)^{m+1} c(h-1,n-1,m)$$

Theorem 1.3. *For any effective class $\beta \in \mathrm{Pic}(S)$,*

$$\mathsf{DT}^{S\times E}_{m,(\beta,n)} = (-1)^{m+1} \sum_{r|(m,\beta)} \frac{1}{r} c\left(\frac{1}{2}(\beta/r)^2,\, n-1,\, m/r \right)$$

1.4 Multiple cover formula: Hilb

We consider Gromov-Witten invariants of the Hilbert scheme $S^{[n]}$ more generally. For classes $\gamma_1, \ldots, \gamma_N \in H^*(S^{[n]})$ and a tautological class $\alpha \in H^*(\overline{M}_{g,N})$ they are defined by

$$\big\langle \alpha; \gamma_1, \ldots \gamma_N \big\rangle^{S^{[n]}}_{g,\beta+mA} = \int_{[\overline{M}_{g,n}(S^{[n]},\beta+mA)]} \pi^*(\alpha) \prod_i \mathrm{ev}_i^*(\gamma_i).$$

where the integral is over the reduced virtual class and π is the forgetful morphism to the moduli space $\overline{M}_{g,N}$ of stable curves. A general multiple cover formula for these counts was conjectured in [27, Conj.B]. We state an equivalent special case of the conjecture here: (The special case is equivalent to the general case by the deformation theory of hyperkähler varieties, see [27, Lemma.3].) For every divisor $r|\beta$ let S_r be a K3 surface and let

$$\varphi_r : H^2(S,\mathbb{R}) \to H^2(S_r,\mathbb{R})$$

be a *real* isometry such that $\varphi_r(\beta/r) \in H_2(S_r,\mathbb{Z})$ is a primitive effective curve class. We extend φ_r to the full cohomology lattice by $\varphi_r(\mathsf{p}) = \mathsf{p}$ and $\varphi_r(1) = 1$, where $\mathsf{p} \in H^4(S,\mathbb{Z})$ is the class of a point. By acting factorwise in the Nakajima operators, the isometry φ_r then naturally induces an isomorphism on the cohomology of the Hilbert schemes (see (28) below for details):

$$\varphi_r : H^*(S^{[n]}) \to H^*(S_r^{[n]}).$$

Conjecture 1.3. *We have*

$$\big\langle \alpha; \gamma_1, \ldots \gamma_N \big\rangle^{S^{[n]}}_{g,\beta+mA} = \sum_{r|(\beta,m)} r^{3g-3+N-\deg(\alpha)} (-1)^{m+\frac{m}{r}} \big\langle \alpha; \varphi_r(\gamma_1), \ldots \varphi_r(\gamma_N) \big\rangle^{S^{[n]}}_{g,\varphi_r(\frac{\beta}{r})+\frac{m}{r}A}.$$

Our main result here is the following:

Theorem 1.4. *Conjecture 1.3 holds for $g = 0$ and $N \leq 3$.*

In particular, the theorem expresses the structure constants of the reduced quantum cohomology of $S^{[n]}$ for arbitrary degree in terms of those for primitive degree. Moreover, the counts $\mathsf{GW}^E_{E,\beta,m}$ in (1) can be computed in terms of genus 0 invariants (by degenerating E), and so Theorem 1.4 also implies a multiple cover formula for these types of invariants. Since the Gromov-Witten theory of $S^{[n]}$ vanishes for $g > 1$ if $n \geq 3$ (and for $g > 2$ if $n = 2$) by dimension reasons, this proves a large chunk of the general conjecture.

Theorem 1.4 also gives a new proof of the classical Yau-Zaslow formula governing genus 0 counts on K3 surfaces. The previous proofs given in [16] and [41] both relied on the Gromov-Witten/Noether-Lefschetz correspondence while ours does not.

1.5 Higher rank

Nesterov's wall-crossing also applies to moduli spaces of higher rank sheaves. Consider the lattice $\Lambda = H^*(S,\mathbb{Z})$ endowed with the Mukai pairing

$$(x \cdot y) := -\int_S x^\vee y,$$

where, if we decompose an element $x \in \Lambda$ according to degree as (ρ, D, n), we have written $x^\vee = (\rho, -D, n)$. Let $M(v)$ be a moduli space of Gieseker stable sheaves F on S of positive rank and Mukai vector $v(F) = \mathsf{ch}(F)\sqrt{\mathrm{td}_S} = v$. We assume that stability and semi-stability agrees for sheaves in class v, so that $M(v)$ is proper. Assume also that there exists an algebraic class $y \in K_{\mathrm{alg}}(S)$ such that $v \cdot v(y) = 1$, which implies that $M(v)$ is fine.[2] We refer to [11, Sec.6] for the construction and the properties of $M(v)$. By work of Mukai there exists a canonical isomorphism (see Section 2):

$$\theta : (v^\perp)^\vee \xrightarrow{\cong} H_2(M(v),\mathbb{Z}).$$

For $w' \in H_2(M(v),\mathbb{Z})$ we define parallel to before:

$$\mathsf{GW}^{M(v)}_{E,w'} = \int_{[\overline{M}_E(M(v),w')]^{\mathsf{vir}}} 1$$

Following [22, Section 3], let also $M_{v,w}(S\times E)$ be the moduli space parametrizing torsion free sheaves G of fixed determinant on $S \times E$ whose restriction to the generic fiber over the elliptic curve is Gieseker-stable, and which have Mukai vector

$$\mathsf{ch}(G)\sqrt{\mathrm{td}_S} = v + w \cdot \omega.$$

Here $\omega \in H^2(E,\mathbb{Z})$ is the point class and we have suppressed pullbacks by the projections to S and E. We assume that $w \neq 0$ and define the counts:

$$\mathsf{DT}^{S\times E}_{(v,w)} = \int_{[M_{(v,w)}(S\times E)/E]^{\mathrm{vir}}} 1.$$

Theorem 1.5. *Assume that $w \cdot v(y) = 0$. For any fixed $F \in M(v)$ we have that*

$$\mathsf{DT}^{S\times E}_{(v,w)} = \mathsf{GW}^{M(v)}_{E,w'} - \chi(S^{[n]}) \sum_{r|w} \frac{1}{r}(-1)^{w\cdot v} e^{\mathrm{vir}}(\mathrm{Quot}(F,u_r))$$

where

- *$w' = -\langle w, -\rangle : v^\perp \to \mathbb{Z}$ is the homology class induced by w,*
- *$u_r = -w/r - s_r v$ for the unique integer $s_r \in \mathbb{Z}$ such that $0 \leq \mathsf{rk}(u_r) \leq \mathsf{rk}(v) - 1$.*

Because $v \cdot v(y) = 1$, the condition $w \cdot v(y) = 0$ can always be achieved by replacing w by $w + \ell v$ for some $\ell \in \mathbb{Z}$. Since the stability condition on $S \times E$ is invariant under tensoring by line bundles pulled back from E, $\mathsf{DT}^{S\times E}_{(v,w)}$ is invariant under this replacement.

1.6 Open questions

Let S be a smooth projective surface with $H^0(S, \mathcal{O}(-K_S)) \neq 0$. Let $F \in \mathrm{Coh}(S)$ be a Gieseker stable sheaf (with respect to some polarization). Then the Quot schemes $\mathrm{Quot}(F,u)$ carry a perfect obstruction theory, see Section 3.1.

Problem 1.4. *Compute the virtual Euler number $e^{\mathsf{vir}}(\mathrm{Quot}(F,u))$.*

[2] We expect that this condition can be removed eventually.

Even for F the ideal sheaf of a length n subscheme this question needs (to the best of the authors knowledge) further investigation. In case $n = 0$, that is quotients of the structure sheaf, we refer to [36] for some results. More generally, we can ask for the computation of the wall-crossing corrections in Nesterov's wall-crossing formula [22].

For K3 surfaces in upcoming work [25] the higher rank Donaldson-Thomas theory of $S \times E$ is related to the rank 1 theory by derived auto-equivalences and wall-crossing. It will show that $\mathsf{DT}^{S\times E}_{(v,w)}$ as we defined it above only depends on the pairings $v \cdot v$, $v \cdot w$ $w \cdot w$ and the divisibility $\mathrm{div}(v \wedge w)$. Moreover, by deformation theory of hyperkähler varieties (the global Torelli theorem) the counts $\mathsf{GW}^{M(v)}_{E,w'}$ also only depends on the same data. Hence by Theorem 1.5 one finds the following:

Theorem 1.6. *(Dependent on [25]) Let F be a Gieseker stable sheaf of positive rank and primitive Mukai vector $v = v(F)$ on a K3 surface S. Assume there exists a class $y \in K_{\mathrm{alg}}(S)$ such that $v \cdot y = 1$, and that $\mathrm{Quot}(F, u)$ is non-empty. Then the virtual Euler characteristic $e^{vir}(\mathrm{Quot}(F, u))$ only depends on the following pairings in the Mukai lattice:*

$$v \cdot v, \quad u \cdot v, \quad u \cdot u.$$

Together with Theorem 1.2 this determines the Euler numbers $e^{\mathrm{vir}}(\mathrm{Quot}(F, u))$ (since any such pair (v, w) is isometric to a pair $((1, 0, 1 - n),(0, \beta, m))$. However, it would be useful to have a more direct way to prove Theorem 1.6, since this would give another way to relate higher rank DT theory of $S \times E$ to rank 1. The natural pathway to proving the theorem is to apply an auto-equivalence which identifies $\mathrm{Quot}(F, u)$ with a Quot scheme of a rank 1 object in the derived category, where the quotients are taken in the heart of some Bridgeland stability conditions. The theorem would then boil down to showing that the virtual Euler number of the Quot scheme stays invariant under change of hearts (the invariance under changing F is provided already by deformation equivalence).

The paper [33] proposed 8 different conjectures related to counting in K3 geometries. They were labeled

A, B, C1, C2, D, E, F, G.

This paper here in combination with [22, 23, 34] tackles Conjectures A and B of [33]. (Strictly speaking, we obtain Conjecture B only for DT invariants.) The same results also immediately imply Conjecture G. Conjecture C1 was proven by T. Buelles [4]. The most difficult of the conjectures appear to be Conjectures C2 (multiple cover formula for Gromov-Witten theory of K3 surfaces, divisibility 2 solved in [1]), and Conjecture D (GW/DT correspondence for imprimitive classes). The remaining conjectures concern the matrix of quantum multiplication with a divisor on $S^{[n]}$. Conjecture E here was partially resolved in the work [12], which provided an explicit candidate for the matrix. This makes Conjecture F now the most accessible candidate on the list.

1.7 Plan of the paper

In Section 2 we discuss our conventions for degree and prove a few basic lemmas about it. In Section 3 we explain a basic universality result for descendent integrals over nested Hilbert schemes (based on work of Gholampour and Thomas). We then express the virtual numbers of the Quot schemes $\mathrm{Quot}(F, u)$ in rank 1 as tautological integrals over the Hilbert scheme and use a degeneration argument to show that their generating series is of a certain form. Section 4 concern the Donaldson-Thomas theory of the cap $S \times \mathbb{P}^1/S_\infty$ and is the heart of the paper. We analyze the obstruction theory on the extremal component of the fixed locus by proving both vanishing of the contribution of most components and relate the remaining terms to the Quot integrals. The multiple cover formulas are taken up in Section 5. In Section 6 we put everything together and prove the theorems announced above.

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2 Definitions of degree

2.1 Mukai vector on K3 surfaces

Let S be a K3 surface and consider the lattice $\Lambda = H^*(S, \mathbb{Z})$ endowed with the Mukai pairing

$$(x \cdot y) := -\int_S x^\vee y,$$

where, if we decompose an element $x \in \Lambda$ according to degree as (ρ, D, n), we have written $x^\vee = (\rho, -D, n)$. We will also write

$$\mathsf{rk}(x) = \rho, \quad c_1(x) = D, \quad v_2(x) = n.$$

Given a sheaf or complex E on S the Mukai vector of E is defined by

$$v(E) = \sqrt{\mathrm{td}_S} \cdot \mathsf{ch}(E) \in \Lambda.$$

The relationship to the Euler characteristic is $\chi(E, F) = -v(E) \cdot v(F)$.

2.2 Mukai vector on $S \times C$

Let C be a smooth curve. We naturally decompose the even cohomology

$$H^{2*}(S \times C, \mathbb{Z}) = H^*(S, \mathbb{Z})1_C \oplus H^*(S, \mathbb{Z})\omega$$

where $1_C, \omega \in H^*(C)$ is the unit and the class of a point respectively. We denote the Mukai vector of a sheaf F on $S \times C$ by

$$\mathsf{ch}(F)\sqrt{\mathrm{td}_S} = v(F) + w(F)\omega = (v(F), w(F)).$$

2.3 Quasimap degree

Let $M(v)$ be a proper moduli space of Gieseker-stable sheaves in Mukai vector v and let $M(v) \subset \mathrm{Coh}_{\mathfrak{r}}(v)$ be the rigidified stack of coherent sheaves of Mukai vector v in which it $M(v)$ is embedded as an open substack. We assume that we are given an algebraic class $y \in K_{\mathrm{alg}}(S)$ with $v \cdot v(y) = 1$. The class y defines canonically a universal family $\mathbb{G}$ over

$\mathrm{Coh}_{\mathfrak{r}}(v)$, which has the following property[3]: If we define the morphism

$$\lambda : K_{\mathrm{alg}}(S) \to \mathrm{Pic}(\mathrm{Coh}_{\mathfrak{r}}(v)), \quad \lambda(u) = \det \pi_*(\mathbb{G} \otimes \pi_S^*(u))$$

where π, π_S are the projections of $\mathrm{Coh}_{\mathfrak{r}}(v) \times S$ onto the factors, then

$$\lambda(-y^\vee) = \mathcal{O}. \tag{4}$$

From a hyperkähler point of view the most natural way to construct cohomology classes on the stack $\mathrm{Coh}_{\mathfrak{r}}(v)$ is given by the Mukai morphism

$$\theta : v^\perp \to H^2(\mathrm{Coh}_{\mathfrak{r}}(v)), \quad x \mapsto \left[\pi_*\left(\mathsf{ch}(\mathbb{G})\sqrt{\mathrm{td}_S} \cdot x^\vee\right)\right]_1,$$

where $[-]_k$ stands for taking the complex degree k component of a cohomology class, i.e. the component in H^{2k}. By restricting the image to $M(v)$ we obtain an isomorphism of lattices:

$$\theta : v^\perp \xrightarrow{\cong} H^2(M(v), \mathbb{Z}),$$

where the right hand side carries the Beauville-Bogomolov-Fujiki form, see [11, Sec.6.2].

We define the degree of a map $f : C \to \mathrm{Coh}_{\mathfrak{r}}(v)$ to be the morphism

$$\deg(f) : v^\perp \to \mathbb{Z}$$

given by $\deg(f)(y) = \int_C f^*(\theta(y))$.

Lemma 2.1. *Let $f : C \to \mathrm{Coh}_{\mathfrak{r}}(v)$ be a quasimap (see [22, Sec.3.1]) and let F be the associated sheaf on $S \times C$ determined by the pullback of the universal family $\mathbb{G}$.*

Then $v(F) = v$ and the class $w(F)$ is determined by

$$\deg(f) = -\langle w(F), -\rangle \ \in \mathrm{Hom}(v^\perp, \mathbb{Z}), \tag{5}$$

where $\langle x, -\rangle$ is the operator of pairing with x in the Mukai lattice, and

$$w(F) \cdot v(y) = 0.$$

Proof. By restricting F to a fiber over C, we find $v(F) = v$. Let $x \in v^\perp$. Then

$$\begin{aligned}\deg(f)(x) &= \int_{S\times C} \mathsf{ch}(F)\pi_S^*(x^\vee)\sqrt{\mathrm{td}_S} \\ &= \int_{S\times C} (v(F) + w(F)\omega)\pi_S^*(x^\vee) \\ &= \int_S w(F)x^\vee \\ &= -(w(F)\cdot x).\end{aligned}$$

This shows (5). Similarly, by (4) we have

$$0 = \int_C c_1(f^*\lambda(-y^\vee)) = \int_{S\times C} \mathsf{ch}(F \otimes \pi_S^*(-y^\vee))\mathrm{td}_S = \int_S w(F)\cdot v(-y^\vee) = w(F)\cdot v(y).$$

Since $v(y) \cdot v = 1$, we see that $w(F)$ is determined by $\deg(f)$ and $w(F) \cdot v(y) = 0$. ☐

Remark 2.2. The divisibility $\mathrm{div}(\alpha)$ of a vector $\alpha \in (v^\perp)^\vee$ is the largest positive integer k such that $\alpha/k \in (v^\perp)^\vee$. Since $H^*(S, \mathbb{Z}) \cong v^\perp \oplus \mathbb{Z}v(y)$ we have $\mathrm{div}(\deg(f)) = \mathrm{div}(w(F))$.

[3] Let $\mathrm{Coh}(v)$ be the stack of coherent sheaves on S and consider the $\mathbb{G}_m$-gerbe $\mathrm{Coh}(v) \to \mathrm{Coh}_{\mathfrak{r}}(v)$. Let $\mathbb{F}$ be the universal sheaf on $\mathrm{Coh}(v) \times S$ (which always exists and is canonical) and for $u \in K_{\mathrm{alg}}(S)$ let $\lambda_{\mathbb{F}}(u) = \det \pi_*(\mathbb{F} \otimes \pi_S^*(u)) \in \mathrm{Pic}(\mathrm{Coh}(S))$. Then $\mathbb{F} \otimes \lambda_{\mathbb{F}}(-y^\vee)^{-1}$ has $\mathbb{G}_m$-weight zero and hence descends as the universal sheaf $\mathbb{G}$ to $\mathrm{Coh}_{\mathfrak{r}}(v) \times S$.

There is a subtle point: The construction of the line bundles $\lambda_{\mathbb{F}}(u)$ and the universal sheaves $\mathbb{G}$ require a resolution of $\pi_*(\mathbb{F} \otimes \pi_S^*(u))$ which exists a priori only over finite type subschemes. Hence $\mathbb{G}$ can be defined only over finite type open substacks of $\mathrm{Coh}_{\mathfrak{r}}(v)$, therefore globally only in an inductive way. Since the quasimaps we consider have fixed degree, they can be shown to map to a sufficiently large finite type substack of $\mathrm{Coh}_{\mathfrak{r}}(v)$ [22]. Hence for our purposes we may assume that $\mathbb{G}$ and $\lambda(u)$ are globally defined. We refer to [22] for a discussion on this point and the exact conventions that we follow.

3 Quot scheme integrals

3.1 Perfect obstruction theory

Let S be a smooth projective surface and let $F \in \mathrm{Coh}(S)$ be a coherent sheaf which is of positive rank and Gieseker stable with respect to some ample class. Consider the Quot scheme $\mathrm{Quot}(F)$ and let π, π_S be the projections of $\mathrm{Quot}(F) \times S$ to the factors. We denote the universal quotient sequence on $\mathrm{Quot}(F) \times S$ by

$$0 \to \mathcal{K} \to \pi_S^*(F) \to \mathcal{Q} \to 0.$$

For sheaves (or complexes, or K-theory classes) $\mathcal{F}_1, \mathcal{F}_2$ on $\mathrm{Quot}(F) \times S$ we write

$$R\,\mathrm{Hom}_S(\mathcal{F}_1, \mathcal{F}_2) = R\pi_* R\mathscr{H}om(\mathcal{F}_1, \mathcal{F}_2).$$

Lemma 3.1. *Assume that $H^0(\mathcal{O}(-K_S)) \neq 0$. Then the Quot scheme $\mathrm{Quot}(F)$ admits a (canonical) perfect obstruction theory with virtual tangent bundle $T^{\mathsf{std}} = R\,\mathrm{Hom}_S(\mathcal{K}, \mathcal{Q})$.*

Proof. (Sketch) Consider a short exact sequence $0 \to K \to F \to Q \to 0$ defining a point in $\mathrm{Quot}(F)$, where $Q \neq F$, and apply $\mathrm{Hom}(-, Q)$. This gives

$$\mathrm{Ext}^1(K, Q) \to \mathrm{Ext}^2(Q, Q) \to \mathrm{Ext}^2(F, Q) \to \mathrm{Ext}^2(K, Q) \to 0. \tag{6}$$

Given a morphism $s : Q \to F$ we have a surjection $F \twoheadrightarrow Q \twoheadrightarrow \mathrm{Im}(s) \subset F$, which by stability of F shows that $\mathrm{Im}(s) = 0$, so $s = 0$. Hence $\mathrm{Hom}(Q, F) = 0$. By our assumption on S there exists an effective divisor $D \in |-K_S|$. Applying $\mathrm{Hom}(-, F)$ to $0 \to Q \to Q(D) \to Q|_D \to 0$ implies then that $\mathrm{Hom}(Q(D), F) = 0$. We conclude that $\mathrm{Ext}^2(F, Q) = \mathrm{Hom}(Q, F \otimes K_S)^\vee = \mathrm{Hom}(Q(D), F)^\vee$ vanishes. The existence then follows by standard methods, e.g. [19, 36]. □

Remark 3.2. The same argument works for any surface S if F is the ideal sheaf of a zero-dimensional subscheme.

3.2 K3 surfaces

From now on let S be a K3 surface. Let $\mathrm{Quot}(F, u)$ be the Quot scheme parametrizing quotients $F \to Q$ with Mukai vector $v(Q) = u$. We always assume that u is chosen such that the Quot scheme is non-empty and that $u \notin \{v(F), 0\}$.

Lemma 3.3. *The canonical perfect obstruction theory on $\mathrm{Quot}(F, u)$ admits a surjective cosection $h^1(T^{\mathsf{std}}) \to \mathcal{O}$.*

Proof. We compose the first map in (6) with the trace map

$$H^1(R\,\mathrm{Hom}(K, Q)) = \mathrm{Ext}^1(K, Q) \to \mathrm{Ext}^2(Q, Q) \xrightarrow{tr} H^2(S, \mathcal{O}_S) = \mathcal{O}.$$

Since the trace map is Serre dual to the inclusion $H^0(S, \mathcal{O}_S) \hookrightarrow \mathrm{Hom}(Q, Q)$, it is surjective. □

Hence the standard virtual class $[\mathrm{Quot}(F, u)]^{\mathsf{std}}$ which is of dimension $\chi(K, Q) = u \cdot (u - v)$ vanishes. Using co-section localization by Kiem-Li [15] we obtain a reduced virtual cycle:

$$[\mathrm{Quot}(F, u)]^{\mathsf{vir}} \in A_{\mathsf{vd}}(\mathrm{Quot}(F, u)).$$

It is associated to the (reduced) virtual tangent bundle $T^{\mathsf{vir}}_{\mathrm{Quot}(F,u)} = R\,\mathrm{Hom}_S(\mathcal{K}, \mathcal{Q}) + \mathcal{O}$, and hence of dimension $\mathsf{vd} = \mathsf{rk}(T^{\mathsf{vir}}_{\mathrm{Quot}(F,u)}) = u \cdot (u - v) + 1$. We write

$$e^{\mathsf{vir}}(\mathrm{Quot}(F, u)) = \int_{[\mathrm{Quot}(F,u)]^{\mathsf{vir}}} c_{\mathsf{vd}}(T^{\mathsf{vir}}_{\mathrm{Quot}(F,u)})$$

for the virtual Euler characteristic of the moduli space in the sense of Fantechi–Göttsche [6].

3.3 Nested Hilbert schemes

Let $\beta \in \mathrm{NS}(S)$ be an effective curve class and let $n_1, n_2 \geq 0$ be integers. Consider the nested Hilbert scheme, where we follow the notation of [9],

$$S_\beta^{[n_1,n_2]} = \{I_1(-D) \subseteq I_2 \subset \mathcal{O}_S : [D] = \beta, \mathrm{length}(\mathcal{O}_S/I_i) = n_i.\}.$$

There exists a natural embedding

$$j : S_\beta^{[n_1,n_2]} \hookrightarrow S^{[n_1]} \times S^{[n_2]} \times \mathbb{P} \tag{7}$$

where the linear system in class β is denoted by

$$\mathbb{P} = \mathbb{P}(H^0(\mathcal{O}(\beta)))$$

Let $\mathcal{I}_i \subset \mathcal{O}$ be the ideal sheaf of the universal subscheme $\mathcal{Z}_i \subset S^{[n_i]} \times S$.

Theorem 3.1 ([9]). *There exists a (reduced) perfect obstruction theory on $S_\beta^{[n_1,n_2]}$ with virtual tangent bundle*

$$T^{\mathsf{vir}}_{S_\beta^{[n_1,n_2]}} = -R\,\mathrm{Hom}_S(\mathcal{I}_1, \mathcal{I}_1)_0 - R\,\mathrm{Hom}_S(\mathcal{I}_2, \mathcal{I}_2)_0 + R\,\mathrm{Hom}_S(\mathcal{I}_1, \mathcal{I}_2 \otimes \mathcal{O}(\beta)) \otimes \mathcal{O}_{\mathbb{P}}(1) - \mathcal{O}$$

where $R\,\mathrm{Hom}_S(\mathcal{I}_i, \mathcal{I}_i)_0 = \mathrm{Cone}(R\Gamma(S, \mathcal{O}_S) \to R\,\mathrm{Hom}_S(\mathcal{I}_i, \mathcal{I}_i))$.

The associated virtual cycle $[S_\beta^{[n_1,n_2]}]^{vir}$ is of dimension $n_1 + n_2 + \beta^2/2 + 1$ and satisfies

$$\begin{aligned} &j_*[S_\beta^{[n_1,n_2]}]^{vir} \\ &= c_{n_1+n_2}\Big(R\Gamma(\mathcal{O}(\beta)) \otimes \mathcal{O}_{\mathbb{P}}(1) - R\,\mathrm{Hom}_S(\mathcal{I}_1, \mathcal{I}_2 \otimes \mathcal{O}(\beta)) \otimes \mathcal{O}_{\mathbb{P}}(1)\Big) \cdot c_1(\mathcal{O}_{\mathbb{P}^1}(1))^{h^1(\mathcal{O}(\beta))} \end{aligned}$$

Proof. The first claim follows directly from Theorem 4.16 in [9], which also shows the second claim whenever $H^1(\mathcal{O}(\beta)) = 0$ (we can take $A = 0$). In the general case, where $H^1(\mathcal{O}(\beta))$ may be non-zero[4], we apply Corollary 4.22 of [9] which gives:

$$j_*[S_\beta^{[n_1,n_2]}]^{\mathrm{vir}} = c_{n_1+n_2}(R\Gamma(\mathcal{O}(\beta)) \otimes \mathcal{O}_{\mathbb{P}}(1) - R\,\mathrm{Hom}_S(\mathcal{I}_1, \mathcal{I}_2 \otimes \mathcal{O}(\beta)) \otimes \mathcal{O}_{\mathbb{P}}(1)) \cdot [S_\beta]^{\mathsf{vir}} \tag{8}$$

where for a class $\gamma \in \mathrm{NS}(S)$ we write $S_\gamma = \mathbb{P}(H^0(\mathcal{O}(\gamma)))$ for the Hilbert scheme of curves in class γ (which of course for a K3 surface is just the linear system). The virtual class $[S_\beta]^{\mathsf{vir}}$ in (8) is the natural one appearing in Seiberg-Witten theory and can be identified with the first degeneracy locus of the complex $R\Gamma(\mathcal{O}(\beta))$. Concretely, it is described as follows (see also [17, Sec.2]). Choose a fixed ample divisor $A \subset S$ such that $H^{\geq 1}(\mathcal{O}(\beta + A)) = 0$. Let $\gamma = \beta + [A] \in \mathrm{NS}(S)$. There exists an embedding

$$S_\beta \to S_\gamma, \quad C \mapsto C + A.$$

Its image consists of those divisors $D = \{s = 0\}$ which contain A, or equivalently, for which the composition $\mathcal{O}_S \xrightarrow{s} \mathcal{O}_S(\gamma) \to \mathcal{O}_S(\gamma)|_A$ vanishes. Globally, let $\mathcal{D} \subset S_\gamma \times S$ be the universal divisor and let $\pi : S_\gamma \times S \to S_\gamma$ be the projection. There exists a universal section $s : \mathcal{O} \to \mathcal{O}(\mathcal{D})$ which yields the sequence $\mathcal{O} \to \mathcal{O}(\mathcal{D}) \to \mathcal{O}(\mathcal{D})|_{S_\gamma \times A}$. Pushing forward, and using that $\mathcal{O}(\mathcal{D}) = \pi_S^*(\mathcal{O}(\gamma)) \otimes \mathcal{O}_{S_\gamma}(1)$ we find that S_β is naturally cut out by a section of

$$\pi_*(\mathcal{O}(\mathcal{D})|_{S_\gamma \times A}) = H^0(\mathcal{O}(\beta + A)|A) \otimes \mathcal{O}(1).$$

The associated virtual class $[S_\beta]^{\mathsf{vir}}$ is the localized Euler class. Using the sequence

$$0 \to H^0(\mathcal{O}(\beta)) \xrightarrow{f} H^0(\mathcal{O}(\beta + A)) \to H^0(\mathcal{O}(\beta + A)|_A) \to H^1(\mathcal{O}(\beta)) \to 0$$

and that S_β is cut out by Cokernel(f) $\otimes\, \mathcal{O}(1)$ one obtains that

$$S_\beta^{\mathsf{vir}} = e(H^1(\mathcal{O}(\beta)) \otimes \mathcal{O}(1)) \cap [S_\beta].$$

This shows the claim. ∎

[4] A basic example is an elliptic K3 surface $S \to \mathbb{P}^1$ and $\beta = mf$ where f is the fiber class. The linear system $|\mathcal{O}(\beta)|$ is of dimension m, while $\chi(\mathcal{O}(\beta)) = 1$.

Let $\mathcal{Z} \subset S^{[n]} \times S$ be the universal subscheme of the Hilbert scheme of points $S^{[n]}$. Given $\alpha \in H^*(S)$ we define the descendants

$$\tau_k^{S^{[n]}}(\alpha) := \pi_*(\mathsf{ch}_{2+k}(\mathcal{O}_{\mathcal{Z}})\pi_S^*(\alpha))$$

where π, π_S are the projections of $S^{[n]} \times S$ to the factors.

We consider integrals over $S_\beta^{[n_1,n_2]}$ of polynomials in the pullback of the classes

$$\tau_{k_1}^{S^{[n_1]}}(\alpha_1), \quad \tau_{k_2}^{S^{[n_2]}}(\alpha_2), \quad z = c_1(\mathcal{O}_{\mathbb{P}}(1)), \quad k_i \geq 0, \alpha_i \in H^*(S),$$

where the pullback is by the composition of the inclusion (7) with the projection to the factors. By Theorem 3.1 we can reduce any such integral to an integral of such type of classes on $S^{[n_1]} \times S^{[n_2]} \times \mathbb{P}$. Integrating out $\mathbb{P}$ and using [5] one obtains:

Theorem 3.2. *(Universality) Let P be a polynomial. Let $\alpha_{i,1}, \alpha_{j_2} \in H^*(S)$ be homogeneous classes and let $k_{i,1}, k_{j,2} \geq 0$ be integers. Then the integral*

$$\int_{\left[S_\beta^{[n_1,n_2]}\right]^{vir}} P(\tau_{k_{i,1}}^{S^{[n_1]}}(\alpha_{i,1}),\ \tau_{k_{j,2}}^{S^{[n_2]}}(\alpha_{j,2}),\ z)$$

depends upon $(S, \beta, \alpha_{i,1}, \alpha_{j,2})$ only through the intersection pairings of the classes $\beta, \alpha_{i_1}, \alpha_{j_2}, 1, \mathsf{p}$.

3.4 Universality

Let $\eta \in S^{[n]}$ be a fixed length n subscheme and let $I_\eta \subset \mathcal{O}_S$ be its ideal sheaf. We consider the Quot scheme $\mathrm{Quot}(I_\eta, u)$ for Mukai vector $u = (0, \beta, m)$ with $\beta \in \mathrm{NS}(S)$ effective.

The Quot scheme $\mathrm{Quot}(I_\eta, u)$ parametrizes sequences of the form

$$0 \to I_z(-\beta) \to I_\eta \to Q \to 0$$

for some $z \in S^{[n_1]}$, where one computes that

$$n_1 = m + n + \beta^2/2.$$

Hence the Quot scheme is naturally a subscheme of the nested Hilbert scheme:

$$\mathrm{Quot}(I_\eta, (0, \beta, m)) = \pi_2^{-1}(\eta) \subset S_\beta^{[n_1,n_2]} \tag{9}$$

where $\pi_2 : S_\beta^{[n_1,n_2]} \to S^{[n_2]}$ is the projection and $n_2 = n$.

Lemma 3.4. *We have the following comparison of virtual cycles:*

$$[\mathrm{Quot}(I_\eta, u)]^{\mathsf{vir}} = \iota_\eta^! [S_\beta^{[n_1,n_2]}]^{vir}.$$

where $\iota_\eta : \{\eta\} \to S^{[n_2]}$ is the inclusion.

Proof. Let $\mathcal{I}$ denote the universal ideal sheaf on $S^{[n]} \times S$ and let $\pi : S^{[n]} \times S \to S^{[n]}$ be the projection. We can identify the π-relative Quot scheme $\mathrm{Quot}(\mathcal{I}/S^{[n]}, u)$ (whose fiber over a point $\eta \in S^{[n]}$ is $\mathrm{Quot}(I_\eta, u)$) with the nested Hilbert scheme:

$$\mathrm{Quot}(\mathcal{I}/S^{[n]}, u) = S_\beta^{[n_1,n_2]}. \tag{10}$$

The left hand side carries the natural perfect obstruction theory of Lemma 3.1 taken relative to the base $S^{[n]}$. Its virtual tangent bundle is

$$T^{\mathsf{vir}}_{\mathrm{Quot}(\mathcal{I}/S^{[n]}, u)} = R\,\mathrm{Hom}_S(\mathcal{K}, \mathcal{Q}) + \mathcal{O} + \mathrm{Ext}^1_S(\mathcal{I}, \mathcal{I})$$

and its virtual class satisfies

$$\iota_\eta^![\mathrm{Quot}(\mathcal{I}/S^{[n]},u)]^{\mathsf{vir}} = [\mathrm{Quot}(\mathcal{I}_\eta,u)]^{\mathsf{vir}}.$$

Under the identification (10) we have

$$\mathcal{K} = \mathcal{I}_1(-\beta)\otimes\mathcal{O}_{\mathbb{P}}(-1), \quad \mathcal{Q} = \mathcal{I}_2 - \mathcal{K}. \tag{11}$$

By the first claim in Theorem 3.1 a small calculation shows that

$$T^{\mathsf{vir}}_{\mathrm{Quot}(\mathcal{I}/S^{[n]},u)} = T^{\mathsf{vir}}_{S_\beta^{[n_1,n_2]}}.$$

Since the virtual class depends on the perfect obstruction theory only through the K-theory class of the virtual tangent bundle we get

$$[\mathrm{Quot}(\mathcal{I}/S^{[n]},u)]^{\mathsf{vir}} = [S_\beta^{[n_1,n_2]}]^{\mathsf{vir}}.$$

□

By Theorem 3.1, the above lemma and (11) we obtain:

$$\begin{aligned} e^{\mathsf{vir}}(\mathrm{Quot}(I_\eta,u)) &= \int_{[\mathrm{Quot}(F,u)]^{\mathsf{vir}}} c_{\beta^2+m+1}\left(R\,\mathrm{Hom}_S(\mathcal{K},\mathcal{Q})+\mathcal{O}\right) \\ &= \int_{[S_\beta^{[n_1,n_2]}]^{\mathsf{vir}}} c_{\beta^2+m+1}\left(R\,\mathrm{Hom}_S(\mathcal{K},\mathcal{Q})+\mathcal{O}\right)\pi_2^*[F]. \end{aligned} \tag{12}$$

Applying Grothendieck-Riemann-Roch to rewrite the term $e\left(R\,\mathrm{Hom}_S(\mathcal{K},\mathcal{Q})+\mathcal{O}\right)$ in descendants, and applying Theorem 3.2 we conclude the following:

Proposition 3.5. *The integrals $\mathsf{Q}_{n,(\beta,m)} = e^{\mathsf{vir}}(\mathrm{Quot}(I_\eta,u))$ only depends on β via the square $\beta\cdot\beta = 2h-2$. We write $\mathsf{Q}_{n,h,m} := \mathsf{Q}_{n,\beta,m}$ from now on.*

Remark 3.6. One can be more explicit in what kind of tautological integral over $S^{[n_1]}$ one obtains when computing $\mathsf{Q}_{n,(\beta,m)}$. By similar arguments as in [36] one proves:

$$e^{\mathrm{red}}(\mathrm{Quot}(I_\eta,u)) = \mathrm{Coeff}_{t^{n_1+n_2}z^{-1}} \int_{S^{[n_1]}} c(T_{S^{[n_1]}})\frac{c_t(x^{[n_1]}\otimes e^z)}{c(x^{[n_1]}\otimes e^z)}\left(\frac{1+z}{z}\right)^{h+1-n_2}(z^{-1}+t)^{n_2}$$

where z,t are formal variables, $x = \mathcal{O}_S(-\beta) - \mathcal{O}_\eta$, and $x^{[n]} = \pi_*(\pi_S^*(x)\otimes\mathcal{O}_{\mathcal{Z}}) \in K_{\mathrm{alg}}(S^{[n]})$ is the tautological class associated to x. If $n_2 = 0$, the above discussion specializes to Section 5.2 in [36]. We do not need this expression later on.

3.5 Multiplicativity

Our goal here is to prove the following structural statement for $\mathsf{Q}_{n,h,m}$. Define the series

$$\mathsf{Q}_n(p,q) = \sum_{h\geq 0}\sum_{m\in\mathbb{Z}} \mathsf{Q}_{n,h,m}q^{h-1}p^m$$

Proposition 3.7. *There exists power series $F_1 \in \mathbb{Q}((p,p^{-1}))[[q]]$ and $F_2 \in q^{-1}\mathbb{Q}((p,p^{-1}))[[q]]$ such that for any $n \geq 0$ we have*

$$\mathsf{Q}_n(p,q) = F_1^n F_2.$$

Proof. Let $S \to \mathbb{P}^1$ be an elliptic K3 surface with section $B \subset S$ and take $\beta_h = B + hF$. Let $E \subset S$ be a fixed smooth fiber over the point $x \in \mathbb{P}^1$. We apply the Li-Wu degeneration formula [18] to the degeneration

$$S \rightsquigarrow S \cup_E (\mathbb{P}^1\times E)\cup_E \ldots \cup_E (\mathbb{P}^1\times E) \tag{13}$$

where there are $n+1$ copies of $\mathbb{P}^1 \times E$ and the i-th copy is glued along the divisor[5] E_∞ to the divisor E_0 in the $(i+1)$-th copy (for $i = 1, \ldots, n$). Moreover, S is glued along E to E_0 in the first copy. This type of degeneration plays a crucial role in [20].

Let $p \in \mathbb{P}^1 \times E$ be a point disjoint from the relative divisor $E_{0,\infty} = \{0, \infty\} \times E$, and let $v_{d,m} = (0, [\mathbb{P}^1] + d[E], m)$. Let $\mathrm{Quot}_{\mathbb{P}^1 \times E/E_{0,\infty}}(I_p, v_{d,m})$ be the Quot scheme of the relative pair $(\mathbb{P}^1 \times E, E_{0,\infty})$. It parametrizes quotients $r^* I_p \twoheadrightarrow Q$ where $r_* \mathsf{ch}(Q) = v_{d,m}$ and r is the canonical projection from an expanded degeneration to $\mathbb{P}^1 \times E$ along $E_{0,\infty}$, see [18]. Since the support of Q meets the relative divisor E_0 with multiplicity 1 there exists evaluation morphisms $\mathrm{ev}_0 : \mathrm{Quot}_{\mathbb{P}^1 \times E/E_{0,\infty}}(I_p, v_{d,m}) \to E_0$. Using the obstruction theory of Lemma 3.1, the fiber

$$\mathrm{Quot}_{\mathbb{P}^1 \times E/E_{0,\infty}}(I_p, v_{d,m})_0 = \mathrm{ev}_0^{-1}(0_E)$$

is seen to have virtual tangent bundle $R\,\mathrm{Hom}_{\mathbb{P}^1 \times E}(\mathcal{K}, \mathcal{Q}) - \mathcal{O}$. Define the generating series:

$$\mathsf{Q}_{\mathsf{p}}(\mathbb{P}^1 \times E/E_{0,\infty}) = \sum_{d \geq 0} \sum_{m \in \mathbb{Z}} q^d p^m e^{\mathsf{vir}} \left(\mathrm{Quot}_{\mathbb{P}^1 \times E/E_{0,\infty}}(I_p, v_{d,m})_0 \right)$$

Similarly, let $\mathrm{Quot}_{\mathbb{P}^1 \times E/E_0}(\mathcal{O}_{\mathbb{P}^1 \times E}, v_{d,m})_0$ be the relative Quot scheme on $\mathbb{P}^1 \times E/E_0$ which parametrizes quotients $\mathcal{O} \to Q$ with $r_* \mathsf{ch}(Q) = v_{d,m}$ such that the restriction to the relative fiber $(\mathcal{O}_{\mathbb{P}^1 \times E} \to Q)|_{E_0}$ is isomorphic to $\mathcal{O}_{E_0} \to \mathcal{O}_0$. Define

$$\mathsf{Q}(\mathbb{P}^1 \times E/E_0) = \sum_{d \geq 0} \sum_{m \in \mathbb{Z}} q^d p^m e^{\mathsf{vir}} \left(\mathrm{Quot}_{\mathbb{P}^1 \times E/E_0}(\mathcal{O}_{\mathbb{P}^1 \times E}, v_{d,m})_0 \right).$$

Finally, let $\mathrm{Quot}_{S/E}(\mathcal{O}_S, (0, \beta_h, m)$ parametrize quotients $\mathcal{O} \twoheadrightarrow Q$ on the pair (S, E) with $\mathsf{ch}(Q) = (0, \beta_h, m)$. Define

$$\mathsf{Q}(S/E) = \sum_{h \geq 0} \sum_{m \in \mathbb{Z}} q^{h-1} p^m e^{\mathsf{vir}} \left(\mathrm{Quot}_{S/E}(\mathcal{O}_S, (0, \beta_h, m)) \right).$$

Let I_η be the ideal sheaf of a length n subscheme $\eta = x_1 + \ldots + x_n$ for distinct points $x_i \in S$. We can choose a simple degeneration $\pi : \mathcal{S} \to C$ over a smooth curve C together with disjoint sections $p_i : C \to \mathcal{S}$ such that

(i) Over the point $c_0 \in C$, $(\pi^{-1}(c_0), p_1(c_0), \ldots, p_n(c_0)) = (S, x_1, \ldots, x_n)$

(ii) Over the point $c_1 \in C$, the fiber $\pi^{-1}(c_1)$ is the surface on the right of (13), and $p_i(c_1)$ is a point on the i-th copy of $\mathbb{P}^1 \times E$ away from the relative divisors.

We then apply the Li-Wu degeneration formula to $\mathrm{Quot}_{\mathcal{S} \to C}(\mathbb{I}, (0, \beta_h, m))$, the Quot scheme relative to the base C, where $\mathbb{I}$ is the ideal sheaf of the union $\cup_i p_i(C)$. One finds that:

$$\mathsf{Q}_n(p, q) = \mathsf{Q}(S/E) \cdot \left(\mathsf{Q}_{\mathsf{p}}(\mathbb{P}^1 \times E/E_{0,\infty}) \right)^n \cdot \mathsf{Q}(\mathbb{P}^1 \times E/E_0),$$

which implies the claim. (The main point is that the integrand splits nicely: If $0 \to K \to I_{p_1, \ldots, p_n} \to Q \to 0$ is a quotient sequence on the right side of (13), and Q_i, K_i are the restriction to the i-th component (with S the 0-th component), then by applying $\mathrm{Hom}(K, -)$ to the sequence $0 \to Q \to \sum_{i=0}^{n+1} Q_i \to \sum_{j=0}^{n} Q|_{x_j} \to 0$ and using adjunction one finds that

$$R\,\mathrm{Hom}(K, Q) + \mathcal{O} = (R\,\mathrm{Hom}(K_0, Q_0) + \mathcal{O}) + \sum_{i=1}^{n+1} (R\,\mathrm{Hom}(K_i, Q_i) - \mathcal{O}).\ )$$

□

[5] We write E_z to denote the fiber over $z \in \mathbb{P}^1$ of the projection $E \times \mathbb{P}^1 \to \mathbb{P}^1$.

4 Analysis of the cap geometry

4.1 Overview

Let S be a K3 surface and let $\Lambda = H^*(S, \mathbb{Z})$ be the Mukai lattice, see Section 2. Let $v \in \Lambda$ be a primitive vector of positive rank satisfying $v \cdot v = 2n - 2$. Let H be a polarization on S and assume that the moduli space of H-Gieseker stable sheaves $M(v)$ is proper. We also fix a class $y \in K_{\mathrm{alg}}(S)$ such that $y \cdot v = 1$, which implies that $M(v)$ is fine and admits a canonical universal family (see Section 2.3).

Consider the moduli space

$$M_{v,w}(S \times \mathbb{P}^1 / S_\infty)$$

parametrizing torsion-free, generically H-stable sheaves E of fixed determinant[6] on the relative geometry $(S \times \mathbb{P}^1, S_\infty)$ with Mukai vector $v(E) = (v, w)$. By [22] the moduli space is proper, and because of the existence of the class y, it is fine and admits a canonical universal family. By Lemma 2.1 the class w satisfies

$$w \cdot v(y) = 0. \tag{14}$$

Let $\mathcal{E}$ be the universal sheaf over $M_{v,w}(S \times \mathbb{P}^1/S_\infty)$. The standard (non-reduced) perfect obstruction theory of $M_{v,w}(S \times \mathbb{P}^1/S_\infty)$ has virtual tangent bundle

$$T^{\mathrm{vir}} = R\,\mathrm{Hom}_{S \times \mathbb{P}^1}(\mathcal{E}, \mathcal{E})_0[1]$$

and is of virtual dimension $v^2 + \chi(S, \mathcal{O}_S) = 2n$. The associated standard virtual class vanishes because of the existence of a cosection [23]. By work of Kiem-Li [15] there exists a reduced virtual class. The reduced virtual dimension is $2n + 1$.

The group $\mathbb{C}^*$ acts on the base $\mathbb{P}^1$ with tangent weight $-t$ at the point $0 \in \mathbb{P}^1$, where we let $\mathfrak{t}$ be the trivial line bundle with $\mathbb{C}^*$-action of weight 1 and set $t = c_1(\mathfrak{t})$. We obtain an induced action on the moduli space $M_{v,w}(S \times \mathbb{P}^1/S_\infty)$. Let

$$M_{\mathrm{ext}} \subset M_{v,w}(S \times \mathbb{P}^1/S_\infty)^{\mathbb{C}^*} \tag{15}$$

be the component of the fixed locus which parametrizes sheaves on $S \times \mathbb{P}^1$ (i.e. sheaves on an expanded degeneration are excluded). We call M_{ext} the *extremal* component.

The goal of this section is to analyze the contribution of this extremal component to the Donaldson-Thomas invariant. We first prove that only 'single-jump' loci contribute to the invariants, and then relate these contributions to Quot scheme integrals. This determines the wall-crossing term in the quasimap wall-crossing.

4.2 Virtual class

Let $E \in M_{\mathrm{ext}}$ and write

$$\mathbb{P}^1 = \mathbb{A}^1_\infty \cup \mathbb{A}^1$$

where $\mathbb{A}^1_\infty, \mathbb{A}^1$ is the standard affine chart around ∞ and 0 respectively. Since E is generically stable and its restriction to S_∞ is stable, we have

$$E|_{S \times \mathbb{A}^1_\infty} = \pi_S^*(F)$$

for some stable sheaf $F \in M(v)$. Over 0, we identify the equivariant sheaf $E|_{S \times \mathbb{A}^1}$ with a graded $\mathcal{O}_S[x]$-module on S,

$$E|_{S \times \mathbb{A}^1} = \bigoplus_{i \geq i_0} E_i \mathfrak{t}^i \tag{16}$$

[6] We refer to [22, Section 3] for the precise definition of generically H-stable and how the determinant is fixed. Essentially, generically H-stable is the condition that the restriction of the sheaf to the generic fiber over $\mathbb{P}^1$ (or in case that the sheaf is defined over a degeneration of $\mathbb{P}^1$, the restrictions to all generic fibers of this degeneration) is H-stable. The sheaf E has fixed determinant if $\det p_*(E \cdot \pi_S^*(y)) \cong \mathcal{O}$, where p is the projection to the curve.

for some $E_i \in \mathrm{Coh}(S)$ and $i_0 \in \mathbb{Z}$. Since $E|_{S\times\mathbb{A}^1}$ is finitely generated, there exists r such that $E_{i_0+r} = E_{i_0+r+1} = \ldots =: F$. Hence our sheaf takes the form

$$E|_{S\times\mathbb{A}^1} = E_{i_0}\mathfrak{t}^{i_0} \oplus E_{i_0+1}\mathfrak{t}^{i_0+1} \oplus \cdots \oplus E_{i_0+r-1}\mathfrak{t}^{i_0+r-1} \oplus F\mathfrak{t}^{i_0+r} \oplus F\mathfrak{t}^{i_0+r+1} \oplus F\mathfrak{t}^{i_0+r+2} \oplus \ldots .$$

By the assumption that E is torsion-free, multiplication by x yields the injective morphisms

$$f_i : E_i \hookrightarrow E_{i+1}.$$

Thus associated to E we have the flag of subsheaves

$$E_\bullet = (E_{i_0} \subseteq E_{i_0+1} \subseteq E_{i_0+2} \subseteq \ldots \subseteq E_{i_0+r} = F).$$

The stabilization parameter r can be chosen uniformly on each connected component of M_{ext} (because it only depends on the Chern classes $\mathsf{ch}(E_i)$). Since there are only finitely many connected components, we hence may choose an r that is a stabilization parameter for all sheaves $E \in M_{\mathrm{ext}}$.

Proposition 4.1. *(a) The fixed part of the restriction of T^{vir} to M_{ext} is given by*

$$\left(T^{vir}|_{M_{\mathrm{ext}}}\right)^{\mathrm{fixed}} \cong \mathrm{Cone}\left(\bigoplus_{i=i_0}^{i_0+r} R\,\mathrm{Hom}_S(E_i,E_i)_0 \xrightarrow{\delta} \bigoplus_{i=i_0}^{i_0+r-1} R\,\mathrm{Hom}_S(E_i,E_{i+1})_0\right) \tag{17}$$

where

$$R\,\mathrm{Hom}_S(E_i,E_i)_0 = \mathrm{Cone}(R\Gamma(S,\mathcal{O}_S) \xrightarrow{\mathrm{id}} R\,\mathrm{Hom}_S(E_i,E_i)) \tag{18}$$

$$R\,\mathrm{Hom}_S(E_i,E_{i+1})_0 = \mathrm{Cone}(R\Gamma(S,\mathcal{O}_S) \xrightarrow{\mathrm{id}} R\,\mathrm{Hom}_S(E_i,E_i) \xrightarrow{f_i\circ(-)} R\,\mathrm{Hom}_S(E_i,E_{i+1})) \tag{19}$$

and δ is induced by the map that sends a tuple $(\alpha_i)_i \in \oplus_i R\,\mathrm{Hom}(E_i,E_i)$ to $(\alpha_{i+1}\circ f_i - f_i\circ\alpha_i)_i$ where $f_i : E_i \to E_{i+1}$ is the inclusion map.

(b) The K-theory class of the moving part of the restriction of T^{vir} to M_{ext} is

$$\left(T^{vir}|_{M_{\mathrm{ext}}}\right)^{\mathrm{mov}} = \sum_{i\geq i_0}\sum_{k\geq 1}\begin{pmatrix} -\mathfrak{t}^{-k}\otimes R\,\mathrm{Hom}_S(E_{i+k}-E_{i+k-1},E_i) \\ +\mathfrak{t}^{k}\otimes R\,\mathrm{Hom}_S(E_{i+k+1}-E_{i+k},E_i)^\vee \end{pmatrix}.$$

Remark 4.2. Here and in what follows in this section, we will denote the sheaves on the moduli space by its fibers over closed points. So $R\,\mathrm{Hom}_S(E_i,E_i)$ stands for $R\,\mathrm{Hom}_S(\mathcal{E}_i,\mathcal{E}_i)$ where $\mathcal{E}_i$ is the i-th summand in the decomposition of the universal sheaf $\mathcal{E}|_{M_{\mathrm{ext}}\times S\times\mathbb{A}^1}$ under the decomposition (16). As before we write $R\,\mathrm{Hom}_S(-,-) = \pi_*\mathscr{H}om(-,-)$ where π is the projection away from S.

Remark 4.3. The right hand side of (17) is the natural perfect obstruction theory appearing in the deformation theory of flags of sheaves $E_\bullet$. Indeed, assuming $i_0 = 0$ for simplicity and taking the long exact sequence in cohomology yields:

$$\bigoplus_{i=0}^{r-1}\mathrm{Hom}(E_i,E_{i+1})/\mathbb{C}f_i \to T^{\mathsf{vir}}_{M_{\mathrm{ext}},[E]} \xrightarrow{\gamma} \bigoplus_{i=0}^{r}\mathrm{Ext}^1(E_i,E_i)$$
$$\xrightarrow{\delta} \bigoplus_{i=0}^{r-1}\mathrm{Ext}^1(E_i,E_{i+1})_0 \to \mathrm{Obs}^{\mathsf{vir}}_{M_{\mathrm{ext}},[E]} \to \ldots .$$

The first term parametrize deformations of the maps $f_i : E_i \to E_{i+1}$. The map γ sends a deformation of the flag $E_\bullet$ to the deformation of the individual terms E_i in the flag. Given a deformation of the individual terms $(\alpha_i) \in \oplus_i \operatorname{Ext}^1(E_i, E_i)$, its image under δ vanishes if and only if the diagram

$$\begin{array}{ccc} E_i & \xrightarrow{f_i} & E_{i+1} \\ \downarrow{\scriptstyle \alpha_i} & & \downarrow{\scriptstyle \alpha_{i+1}} \\ E_i[1] & \xrightarrow{f_i} & E_{i+1}[1] \end{array}$$

commutes, hence if and only if the deformations are compatible with f_i. We refer to [8] for a discussion.

Proof. We linearize the line bundle $\mathcal{O}_{\mathbb{P}^1}(-1)$ such that it has weight $\mathfrak{t}$ over $0 \in \mathbb{P}^1$ (and hence weight 0 over $\infty \in \mathbb{P}^1$). By replacing E by $E(k)$ for appropriate k (and raising r if needed) we can assume that

$$E|_{S\times\mathbb{A}^1} = E_0 \oplus E_1\mathfrak{t}^1 \oplus \cdots \oplus E_{r-1}\mathfrak{t}^{r-1} \otimes F\mathfrak{t}^r \oplus F\mathfrak{t}^{r+1} \oplus F\mathfrak{t}^{r+2} \oplus \ldots .$$

for the restriction of the universal family to M_{ext}.

Let $\iota : S \to X := S \times \mathbb{P}^1$ be the inclusion of the fiber over 0. We argue now similarly to [7, Prop.3.12]. The idea is to peel off one factor of E_i at a time. Concretely, define a sequence of sheaves $E^{(j)}$ inductively by $E^{(0)} := E$ and by the short exact sequence

$$0 \to E^{(j+1)}(-1) \to E^{(j)} \to \iota_* E_j \to 0. \tag{20}$$

Since the flag $E_\bullet$ stabilizes at the r-th step, we have

$$E^{(r)} = \pi_S^*(F).$$

Lemma 4.4. *For any $A \in D^b(S)$ we have*

$$R\operatorname{Hom}_X(E^{(j)}, \iota_* A) = R\operatorname{Hom}_S(E_j, A) + \sum_{k\geq 1} \mathfrak{t}^{-k} \otimes R\operatorname{Hom}_S(E_{j+k} - E_{j+k-1}, A).$$

Proof. Since $E_\bullet$ stabilizes the sum on the right hand side has only finitely many non-zero terms, so the claim is well-defined. We argue by induction. First apply $R\operatorname{Hom}(-, \iota_* A)$ to the sequence (20), then we use adjunction with respect to ι and the well-known fact (e.g. [10, Cor.11.4]) that for any $B \in D^b(S)$ we have the distinguished triangle

$$B(-S_0)[1] \to L\iota^*\iota_* B \to B \to B(-S_0)[2].$$

This yields:

$$R\operatorname{Hom}_X(E^{(j)}, \iota_* A) = R\operatorname{Hom}_S(E_j, A) - R\operatorname{Hom}_S(E_j, A) \otimes \mathfrak{t}^{-1} + \mathfrak{t}^{-1} \otimes R\operatorname{Hom}_X(E^{(j+1)}, \iota_* A)$$

from which the claim follows by induction. ☐

Lemma 4.5. *For any $A \in D^b(S)$ and j we have*

$$R\operatorname{Hom}_X(\iota_* A, E^{(j)}) = (R\operatorname{Hom}_X(E^{(j)}, \iota_* A[3]) \otimes \mathfrak{t})^\vee.$$

Proof. By Serre duality we have

$$R\operatorname{Hom}_X(\iota_* A, E^{(j)}) = R\operatorname{Hom}_X(E^{(j)}, \iota_* A \otimes \omega_X[3])^\vee = (R\operatorname{Hom}_X(E^{(j)}, \iota_* A[3]) \otimes \mathfrak{t})^\vee,$$

where we used that $\omega_X|_{S_0} = \Omega_{\mathbb{P}^1,0} \otimes \mathcal{O}_S = \mathfrak{t} \otimes \mathcal{O}_S$. ☐

Lemma 4.6.

$$R\operatorname{Hom}_X(E^{(j)}, E^{(j)})^{\text{fixed}}[1] \cong \operatorname{Cone}\left(\bigoplus_{i=j}^{r} R\operatorname{Hom}_S(E_i, E_i) \xrightarrow{\delta} \bigoplus_{i=j}^{r-1} R\operatorname{Hom}_S(E_i, E_{i+1}) \right).$$

Proof. By applying $R\operatorname{Hom}(E^{(j)},-)$ to (20) we obtain the distinguished triangle

$$R\operatorname{Hom}_X(E^{(j)},\iota_*E_j)\to R\operatorname{Hom}_X(E^{(j)},E^{(j+1)}(-1))[1]\to R\operatorname{Hom}_X(E^{(j)},E^{(j)})[1].$$

By Lemma 4.4 we have

$$R\operatorname{Hom}_X(E^{(j)},\iota_*E_j)^{\mathrm{fixed}}=R\operatorname{Hom}_S(E_j,E_j).$$

For the second term we apply $R\operatorname{Hom}(-,E^{(j+1)}(-1))$ to (20) and obtain the distinguished triangle

$$R\operatorname{Hom}_X(E^{(j+1)},E^{(j+1)})\to R\operatorname{Hom}_X(\iota_*E_j,E^{(j+1)}(-1))[1]\to R\operatorname{Hom}_X(E^{(j)},E^{(j+1)}(-1))[1].$$

By Lemma 4.5 we have

$$\begin{aligned}R\operatorname{Hom}_X(\iota_*E_j,E^{(j+1)}(-1))&=R\operatorname{Hom}_X(\iota_*E_j(1),E^{(j+1)})\\&=R\operatorname{Hom}_X(\iota_*E_j,E^{(j+1)})\otimes\mathfrak{t}\\&=R\operatorname{Hom}_X(E^{(j+1)},\iota_*E_j[3])^\vee.\end{aligned}$$

Taking the fixed part, using Lemma 4.4 and Serre duality on S we get

$$R\operatorname{Hom}_X(\iota_*E_j,E^{(j+1)}(-1))^{\mathrm{fixed}}[1]=R\operatorname{Hom}(E_j,E_{j+1}).$$

Taking both statements together we end up with the distinguished triangle:

$$\begin{aligned}&R\operatorname{Hom}_S(E_j,E_j)\oplus R\operatorname{Hom}_X(E^{(j+1)},E^{(j+1)})^{\mathrm{fixed}}\\&\qquad\to R\operatorname{Hom}(E_j,E_{j+1})\to R\operatorname{Hom}_X(E^{(j)},E^{(j)})^{\mathrm{fixed}}[1].\end{aligned}$$

The last piece of information we need is that

$$R\operatorname{Hom}_X(E^{(r)},E^{(r)})=R\operatorname{Hom}_S(F,\pi_*\pi^*(F))=\operatorname{Hom}_S(F,F).$$

Hence by iterating the above argument, the claim now follows by induction, see also [7, 8, 9] for a discussion of the maps. □

Lemma 4.7. *In K-theory we have:*

$$R\operatorname{Hom}_X(E^{(j)},E^{(j)})^{\mathrm{mov}}[1]=\sum_{i\geq j}\sum_{k\geq 1}\begin{pmatrix}-\mathfrak{t}^{-k}\otimes R\operatorname{Hom}_S(E_{i+k}-E_{i+k-1},E_i)\\+\mathfrak{t}^{k}\otimes R\operatorname{Hom}_S(E_{i+k+1}-E_{i+k},E_i)^\vee\end{pmatrix}.$$

Proof. By the same argument as in Lemma 4.6 but now taking the moving part. □

We complete the proof of Proposition 4.1. The first part follows from Lemma 4.6, taking the tracefree part and arguing as in [8, Proof Thm.7.1]. The second part follows directly from Lemma 4.7 by taking $j=0$ (since the trance part $R\Gamma(X,\mathcal{O}_X)=R\Gamma(S,\mathcal{O}_S)$ is $\mathbb{C}^*$-fixed). □

4.3 Second cosection

Proposition 4.8. *Let $N\subset M_{\mathrm{ext}}$ be a connected component and let $s=|\{i:E_i\neq E_{i+1}\}|$ be the number of non-trivial steps in the flag $E_\bullet$. If $s\geq 2$, then the virtual class of the fixed perfect-obstruction theory on N vanishes:*

$$[N]^{\mathsf{vir}}=0.$$

Proof. The strategy is to construct a second, linearly independent cosection of the fixed obstruction theory of the component N. Since there is then a second trivial piece in the obstruction theory, the virtual class vanishes: $[N]^{\mathrm{vir}} = 0$. This idea was pioneered in the proof of the Katz-Klemm-Vafa conjecture by Pandharipande and Thomas in [41]. It can be viewed as the source of all multiple cover behaviour on the sheaf side.

In rank 1 the existence of a cosection can be seen quite easily. Indeed, if $\mathsf{rk}(v) = 1$, we can assume that v is the Mukai vector of the ideal sheaf of some length zero subscheme, so that $E_r \subset \mathcal{O}_S$ is an ideal sheaf, and hence all E_i are ideal sheaves of some curves. By definition (see (19)) we have the exact sequence

$$\begin{aligned} 0 \to \mathrm{Ext}^1(E_i, E_{i+1}) \to \mathrm{Ext}^1(E_i, E_{i+1})_0 \\ \xrightarrow{h_i} H^2(S, \mathcal{O}_S) \to \mathrm{Ext}^2(E_i, E_{i+1}) \to \mathrm{Ext}^2(E_i, E_{i+1})_0 \to 0. \end{aligned} \tag{21}$$

Since the E_i are stable one has that

$$\mathrm{Ext}^2(E_i, E_{i+1}) = \mathrm{Hom}(E_{i+1}, E_i)^\vee = \begin{cases} 0 & \text{if } E_i \neq E_{i+1} \\ \mathbb{C} & \text{if } E_i = E_{i+1}. \end{cases}.$$

If $E_i \neq E_{i+1}$ we hence find that h_i is surjective. Consider the diagram:

$$\begin{array}{ccccc} \bigoplus_{i=0}^{r} R\,\mathrm{Hom}_S(E_i, E_i)_0 & \xrightarrow{\delta} & \bigoplus_{i=0}^{r-1} R\,\mathrm{Hom}_S(E_i, E_{i+1})_0 & \longrightarrow & h^1(T_N^{\mathrm{vir}}) \longrightarrow 0 \\ & & \downarrow h & \swarrow \exists & \\ & & \mathcal{O}^{\oplus s} & & \end{array}.$$

where $h = \oplus_{i:E_i \neq E_{i+1}}(h_i \circ \mathrm{pr}_i)$ and pr_i is the projection to the i-th summand. By the previous argument h is surjective. By the claim below, the map above factors through a surjection $h^1(T_N^{\mathrm{vir}}) \to \mathcal{O}^s$. We see that the obstruction sheaf has s trivial summands. The reduced obstruction theory removes one of these summands. Hence if $s \geq 2$, there is a positive number of trivial summands and the virtual class vanishes.

Claim. The composition $h \circ \delta$ vanishes.

Proof of Claim. Given $(\alpha_i) \in \oplus_i \mathrm{Ext}^1(E_i, E_i)$ we need to show that $h(\alpha_{i+1} \circ f_i - f_i \circ \alpha_i)$ vanishes. By (19), $f_i \circ \alpha$ lies in the image of the map $\mathrm{Ext}^1(E_i, E_{i+1}) \to \mathrm{Ext}^1(E_i, E_{i+1})_0$, which using (21) shows that $h_i(f_i \circ \alpha) = 0$. Similarly, given $\alpha_{i+1} \in \mathrm{Ext}^1(E_{i+1}, E_{i+1})$ the composition $\alpha_{i+1} \circ f_i$ lies in $\mathrm{Ext}^1(E_i, E_{i+1})$. □

In higher rank the above naive approach does not work since the individual sheaves E_i can behave quite badly: they do not have to be semi-stable, and there can be maps $E_{i+1} \to E_i$. Instead, we will imitate the arguments of [41]. If $E \in M_{\mathrm{ext}}$, there exists a canonical injection $E \hookrightarrow \pi_S^*(F)$. Let Q be the cokernel. We can hence view E as the 'stable pair':

$$E = [\pi_S^*(F) \xrightarrow{\varphi} Q].$$

The cokernel Q is supported over $S \times \mathrm{Spec}(\mathbb{C}[x]/(x^r))$ and is identified there with

$$Q = F/E_0 \oplus F/E_1 \mathfrak{t}^1 \oplus \cdots \oplus F/E_{r-1} \mathfrak{t}^{r-1}\,.$$

Then applying the construction of [41, Sec.5.4] to Q (and deforming the section φ as in Sec.5.5 of *loc.cit.*) yields an explicit first-order deformation of Q of weight[7] -1. Arguing as in [41, Prop.12] shows that this vector field is linearly independent from the translational shift (that gives the reduced obstruction theory), if and only if Q is uniformly r-times thickened, hence if and only if $E_0 = \ldots = E_{r-1}$, hence only if there is at most 1 step. By Serre duality we hence obtain the second independent cosection whenever $s \geq 2$. □

[7] In [41] the vector field has weight 1. We have the sign difference to our case because we let out torus act with tangent weight -1 at $0 \in \mathbb{A}^1$ whereas in [41] it is with weight 1.

4.4 The contributions from the single-step locus

We consider the components $M_r^{\text{1-step}} \subset M_{\text{ext}}$ with a single step and a uniform thickened sheaf of length r, i.e. the component parametrizing sheaves E of the form

$$E|_{S\times\mathbb{A}^1} = K\mathsf{t}^{i_0} \oplus K\mathsf{t}^{i_0+1} \oplus \cdots \oplus K\mathsf{t}^{i_0+r-1} \oplus F\mathsf{t}^{i_0+r} \oplus F\mathsf{t}^{i_0+r+1} \oplus \ldots . \tag{22}$$

We have the (non-equivariant) exact sequence

$$0 \to \pi_S^*(F) \otimes \mathcal{O}_{\mathbb{P}^1}(-r-i_0) \to E|_{S\times\mathbb{A}^1} \to \iota_{r*}\pi_S^*(K) \to 0,$$

where $\iota_r : S \times \operatorname{Spec}(k[x]/x^r) \to S \times \mathbb{P}^1$ is the inclusion. We obtain that

$$w = rv(K) - (r+i_0)v.$$

By taking the pairing with $v(y)$ and using $w \cdot v(y) = 0$ (see (14)) we find that $0 = rv(K) \cdot v(y) - (r+i_0)$, and hence that $i_0 = s_r \cdot r$ for some $s_r \in \mathbb{Z}$. This shows that w is divisibile by r as well. We conclude that:

$$v(F) = v, \quad v(F/K) = -\frac{w}{r} - s_r v =: u$$

Moreover, since F/K is a quotient of the torsion-free F it has to be of rank in the interval $[0, \mathsf{rk}(v)-1]$. We see that s_r is the unique integer such that

$$-\frac{\mathsf{rk}(w)}{r} - s_r\,\mathsf{rk}(v) \in [0, \mathsf{rk}(v)-1]. \tag{23}$$

The inclusion $K \subset F$ defines an element in the Quot scheme of F with quotients of Mukai vector u. Let $\mathcal{F} \in \operatorname{Coh}(M(v) \times S)$ be the universal sheaf over the moduli space. We conclude that the component is the relative Quot scheme:

$$M_r^{\text{1-step}} = \operatorname{Quot}(\mathcal{F}/M(v), u).$$

By Proposition 4.1 we have the virtual normal bundle

$$\begin{aligned} N^{\mathrm{vir}} &= -R\operatorname{Hom}_S(K, F-K)^\vee \otimes (\mathsf{t}^{-1} + \ldots + \mathsf{t}^{-r}) \\ &\quad + R\operatorname{Hom}_S(K, F-K) \otimes (\mathsf{t} + \ldots + \mathsf{t}^{r-1}) \end{aligned}$$

Moreover, we have

$$\begin{aligned} \mathsf{rk}\operatorname{Hom}_S(K, F-K) &= -v(K)\cdot v(F/K) \\ &= v(F/K)^2 - v(F)\cdot v(F/K) \\ &= u^2 - u\cdot v \\ &\equiv u\cdot v \quad \text{(modulo 2)} \\ &\equiv \frac{w\cdot v}{r} \quad \text{(modulo 2)}. \end{aligned}$$

We find that the contribution of $M_r^{\text{1-step}}$ to the virtual class of $M_{v,w}(S\times\mathbb{P}^1/S_\infty)$ in the localization formula is given by:

$$\begin{aligned} &\frac{1}{e_{\mathbb{C}^*}(N^{\mathsf{vir}})} \cdot [M_r^{\text{1-step}}]^{\mathrm{vir}} \\ &= \frac{e_{\mathbb{C}^*}\left(R\operatorname{Hom}(K,F-K)^\vee \otimes (\mathsf{t}^{-1}+\ldots+\mathsf{t}^{-(r-1)})\right)}{e_{\mathbb{C}^*}\left((R\operatorname{Hom}(K,F-K)\otimes(\mathsf{t}+\ldots+\mathsf{t}^{r-1}))\right)} \cdot \\ &\qquad \cdot e_{\mathbb{C}^*}\left(R\operatorname{Hom}_S(K,F-K)^\vee \otimes \mathsf{t}^{-r}\right) \cdot [\operatorname{Quot}(\mathcal{F}/M(v),u)]^{\mathrm{vir}} \\ &= (-1)^{(r-1)\cdot\frac{w\cdot v}{r}} e_{\mathbb{C}^*}\left(R\operatorname{Hom}_S(K,F-K)^\vee \otimes \mathsf{t}^{-r}\right) \cdot [\operatorname{Quot}(\mathcal{F}/M(v),u)]^{\mathrm{vir}} \\ &= (-1)^{(r-1)\cdot\frac{w\cdot v}{r}} \sum_{k\in\mathbb{Z}} (-rt)^{-k} c_{k+u\cdot(u-v)}(R\operatorname{Hom}_S(K,F-K)^\vee) \cdot [\operatorname{Quot}(\mathcal{F}/M(v),u)]^{\mathrm{vir}} \end{aligned} \tag{24}$$

where we used that $e_{\mathbb{C}^*}(V) = (-1)^{\mathsf{rk}(V)} e_{\mathbb{C}^*}(V^\vee)$.

4.5 Contributions from mixed and pure-rubber components

Aside from the extremal locus M_{ext}, there are two other types of components in the fixed locus $M_{v,w}(S\times\mathbb{P}^1/S_\infty)^{\mathbb{C}^*}$:

(i) Components W of the fixed locus which parametrize sheaves supported on an expanded degeneration of $S\times\mathbb{P}^1$, but whose restriction to $S\times\mathbb{P}^1|_{\mathbb{A}^1}$ is *not* the pullback of a sheaf from S. We call these components W of *mixed type*. These components are given as a fiber product over $M(v)$ of an extremal component M'_{ext} (for different Chern characters) and a rubber component $M_{v',w'}(S\times\mathbb{P}^1/S_{0,\infty})^{\sim}$. The perfect obstruction theory decomposes according to this decomposition and one observes that each summand admits a surjective cosection. Since the reduced class is formed by removing one trivial $\mathcal{O}$-summand, one trivial summand remains and the fixed virtual class vanishes. Hence the contribution from this component vanishes.

(ii) The second type of component parametrizes sheaves on an expanded degeneration but whose restriction to $S\times\mathbb{P}^1$ is pulled back from S. We call it of *pure rubber type*. It is given by the rubber space

$$W=M_{v,w}(S\times\mathbb{P}^1/S_{0,\infty})^{\sim}.$$

Because of the $\mathbb{C}^*$-scaling action, the *reduced* virtual class of the fixed obstruction theory is of dimension $v^2+2=2n$. The virtual normal bundle is $N^{\mathrm{vir}}=T_{\mathbb{P}^1,\infty}\otimes\mathcal{L}_0^\vee$ where $\mathcal{L}_0\to W$ is the relative cotangent line bundle over the rubber space (of the marking glued to $\mathbb{P}^1$). Hence one finds the contribution

$$\frac{1}{e_{\mathbb{C}^*}(N^{\mathrm{vir}})}[W]^{\mathrm{vir}}=\frac{1}{t-c_1(\mathcal{L}_0)}[M_{v,w}(S\times\mathbb{P}^1/S_{0,\infty})^{\sim}]^{\mathrm{vir}}. \tag{25}$$

Since the reduced virtual dimension of $[W]^{\mathrm{vir}}$ (which is $2n$) is strictly smaller than that of $M_{v,w}(S\times\mathbb{P}^1/S_\infty)^{\mathbb{C}^*}$ (which is $2n+1$), we will see below that the contribution from $[W]^{\mathrm{vir}}$ to suitable integrals over the full moduli space will vanish.

4.6 Quasimap wall-crossing

We turn to the proof of the quasimap wall-crossing (Theorem 1.5):

Consider the pure rubber component $M_{v,w}(S\times\mathbb{P}^1/S_{0,\infty})^{\sim}$ and the evaluation map

$$\mathrm{ev}_\infty: M_{v,w}(S\times\mathbb{P}^1/S_{0,\infty})^{\sim}\to M(v)$$

given by intersection with the fiber over $\infty\in\mathbb{P}^1$. Let $\mathrm{pt}\in H^{4n}(M(v))$ be the class of a point and consider the integral:

$$\langle \mathrm{pt},1\rangle_{v,w}^{S\times\mathbb{P}^1/S_{0,\infty}}:=\int_{[M_{v,w}(S\times\mathbb{P}^1/S_{0,\infty})^{\sim}]^{\mathrm{vir}}}\mathrm{ev}_\infty^*(\mathrm{pt}).$$

We have the following wall-crossing formula proven by Nesterov:

Theorem 4.1 ([23, Theorem 3.5])**.**

$$\mathsf{DT}^{S\times E}_{(v,w)}=\mathsf{GW}^{M(v)}_{E,w'}+\chi(S^{[n]})\langle \mathrm{pt},1\rangle_{v,w}^{S\times\mathbb{P}^1/S_{0,\infty}}$$

where $w'=-\langle w,-\rangle: v^\perp\to\mathbb{Z}$ is the homology class induced by w,

We prove here the following evaluation of the wall-crossing term:

Theorem 4.2. *For any $F\in M(v)$ we have*

$$\langle \mathrm{pt},1\rangle_{v,w}^{S\times\mathbb{P}^1/S_{0,\infty}}=-\sum_{r|w}\frac{(-1)^{w\cdot v}}{r}e^{\mathrm{vir}}(\mathrm{Quot}(F,u_r))$$

where $u_r=-w/r-s_rv$ for the unique $s_r\in\mathbb{Z}$ such that $0\leq\mathsf{rk}(u_r)\leq\mathsf{rk}(v)-1$

Proof. The moduli space $M_{v,w}(S\times\mathbb{P}^1/S_\infty)$ is proper and has a (reduced) virtual fundamental class of dimension $2n+1$. Hence for dimension reasons we have

$$\int_{[M_{v,w}(S\times\mathbb{P}^1/S_\infty)]^{\mathsf{vir}}} \mathrm{ev}_\infty^*(\mathrm{pt}) = 0.$$

We lift the virtual class to an equivariant class and apply the virtual localization formula. By the discussion in Section 4.5, only the extremal component and the component of pure rubber type contributes in the localization formula, which gives:

$$\int_{[M_{\mathrm{ext}}]^{\mathrm{red}}} \frac{\mathrm{ev}_\infty^*(\mathrm{pt})}{e_{\mathbb{C}^*}(N^{\mathsf{vir}})} + \int_{[M_{v,w}(S\times\mathbb{P}^1/S_{0,\infty})^\sim]^{\mathsf{vir}}} \frac{\mathrm{ev}_\infty^*(\mathrm{pt})}{t - c_1(\mathcal{L}_0)} = 0$$

Taking the t^{-1}-coefficient we obtain

$$\langle \mathrm{pt}, 1\rangle_{v,w}^{S\times\mathbb{P}^1/S_{0,\infty}} = (-1)\cdot \mathrm{Coefficient}_{t^{-1}}\left[\int_{[M_{\mathrm{ext}}]^{\mathrm{red}}} \frac{\mathrm{ev}_\infty^*(\mathrm{pt})}{e_{\mathbb{C}^*}(N^{\mathsf{vir}})}\right].$$

For any $F \in M(v)$ we have

$$\begin{aligned}
&\int_{[M_{\mathrm{ext}}]^{\mathrm{vir}}} \frac{1}{e_{\mathbb{C}^*}(N^{\mathsf{vir}})} \mathrm{ev}_\infty^*(\mathrm{pt}) \\
&\overset{(*)}{=} \sum_{r|w} \int_{[M_r^{\text{1-step}}]^{\mathrm{vir}}} \frac{1}{e_{\mathbb{C}^*}(N^{\mathrm{vir}})} \mathrm{ev}_\infty^*(\mathrm{pt}) \\
&\overset{(**)}{=} \sum_{r|w} (-1)^{(r-1)\cdot\frac{w\cdot v}{r}} \sum_{k\in\mathbb{Z}} (-rt)^{-k} \int_{[\mathrm{Quot}(\mathcal{F}/M(v),u_r)]^{\mathrm{vir}}} c_{k+u_r\cdot(u_r-v)}(R\,\mathrm{Hom}_S(K, F-K)^\vee)\rho^*(\mathrm{pt}) \\
&\overset{(***)}{=} \sum_{r|w} \frac{1}{-rt}(-1)^{(r-1)\frac{w\cdot v}{r}}(-1)^{1+u_r\cdot(u_r-v)} \int_{[\mathrm{Quot}(F,u_r)]^{\mathrm{vir}}} c_{1+u_r\cdot(u_r-v)}\left(R\,\mathrm{Hom}(K,F-K)+\mathcal{O}\right) \\
&= \sum_{r|w} \frac{1}{rt}(-1)^{w\cdot v} e^{\mathrm{vir}}(\mathrm{Quot}(F,u_r)).
\end{aligned}$$

where $(*)$ follows since only the 1-step locus contributes by Proposition 4.8, $(**)$ follows from (24) and by checking that the K-theory class of the fixed obstruction theory given in Proposition 4.1 equals the class of the reduced perfect obstruction theory of the relative Quot scheme $\mathrm{Quot}(\mathcal{F}/M(v))$ and we let $\rho : \mathrm{Quot}(\mathcal{F}/M(v)) \to M(v)$ denote the morphism to the base, and $(***)$ follows since $\mathrm{Quot}(F,u_r)$ has reduced virtual dimension $u_r\cdot(u_r-v)+1$, see Section 3.2. This yields the claim. $\square$

Proof of Theorem 1.5. This follows by combining Theorem 4.1 and Theorem 4.2. $\square$

4.7 Rank 1

Let $n \geq 0$ and assume that

$$v = (1, 0, 1-n).$$

We fix the class $y = -[\mathcal{O}_s]$ for a point $s \in S$ which satisfies $v\cdot v(y) = 1$. The condition $w\cdot v(y) = 0$ then says that w is of rank zero, so it is of the form

$$w = (0, -\beta, -m)$$

for some $\beta \in \mathrm{Pic}(S)$ and $m \in \mathbb{Z}$.

We prove the quasimap wall-crossing in rank 1:

Proof of Theorem 1.1. This follows by specializing Theorem 1.5 to $v = (1, 0, 1 - n)$ and $w = (0, -\beta, -m)$. Indeed, in this case the equality of moduli spaces

$$M_{(v,w)}(S \times E) = \mathrm{Hilb}_m(S \times E, (\beta, n))$$

shows

$$\mathsf{DT}^{S\times E}_{(v,w)} = \mathsf{DT}^{S\times E}_{m,(\beta,n)}.$$

Moreover, the homology class induced by w,

$$w' = -\langle w, - \rangle \in (v^{\perp})^* \cong H_2(S^{[n]}, \mathbb{Z})$$

is precisely $\beta + mA$, see for example [31, Example 2.3].

Finally, for divisors $r|(\beta, m)$ the condition (23) shows that $s_r = 0$, so $i_0 = 0$, and hence $u_r = \frac{1}{r}(0, \beta, m)$ in Theorem 1.5. Inserting everything we obtain

$$\mathsf{DT}^{S\times E}_{m,(\beta,n)} = \mathsf{GW}^{S^{[n]}}_{E,\beta,m} - \chi(S^{[n]}) \sum_{r|(\beta,m)} \frac{1}{r}(-1)^m e^{\mathrm{vir}}(\mathrm{Quot}(I_z, \frac{(0,\beta,m)}{r}))$$

where I_z is the ideal sheaf of a subscheme $z \in S^{[n]}$, which is precisely what was claimed. □

Remark 4.9. For later use we also remark that in rank 1 the 1-step component can be written as the relative Hilbert scheme

$$M_r^{\text{1-step}} = \mathrm{Quot}(\mathcal{I}_{\mathcal{Z}}/S^{[n]}, (0, \beta/r, m/r)) = S^{[n_1,n]}_{\beta/r}$$

where $n_1 = n + \frac{m}{r} + \frac{1}{2}\beta^2/r^2$.

5 Multiple cover conjectures

5.1 Definitions

Let C be a smooth curve with distinct points $z_1, \ldots, z_k \in C$ and consider the Hilbert scheme

$$\mathrm{Hilb} := \mathrm{Hilb}_{n,\beta,m}(S \times C/S_{z_1,\ldots,z_k})$$

parametrizing 1-dimensional subschemes Z of the relative geometry

$$(S \times C, S_{z_1,\ldots,z_k}), \quad S_{z_1,\ldots z_k} = \bigsqcup S \times \{z_i\}.$$

The numerical invariants are determined by the Mukai vector:

$$\mathsf{ch}(I_Z)\sqrt{\mathrm{td}}_S = (v(I_Z), w(I_Z)) = ((1, 0, 1-n), (0, -\beta, -m)).$$

In other words,

$$[Z] = \iota_*\beta + n[C], \quad \mathsf{ch}_3(\mathcal{O}_Z) = m,$$

where $\iota : S = S \times \{z\} \to S \times C$ is the natural inclusion for some $z \in C$.

There exists evaluation maps at the markings

$$\mathrm{ev}_{z_i} : \mathrm{Hilb}_{n,\beta,m}(S \times C/S_{z_1,\ldots,z_k}) \to S^{[n]}.$$

Let $\mathcal{Z} \subset \mathrm{Hilb} \times S \times C$ denote the universal subscheme, and consider the diagram

$$\begin{array}{ccc} \mathrm{Hilb} \times S \times C & \xrightarrow{\pi_X} & S \times C \\ \downarrow{\scriptstyle \pi} & & \\ \mathrm{Hilb}\,. & & \end{array}$$

For any $\gamma \in H^*(S \times C)$ we define the descendants

$$\tau_i(\gamma) = \pi_*(\mathsf{ch}_{2+i}(\mathcal{O}_{\mathcal{Z}})\pi_X^*(\gamma)) \in H^*(\mathrm{Hilb}).$$

If $(\beta, m) \neq 0$, then the moduli space carries a reduced virtual class $[\mathrm{Hilb}]^{\mathsf{vir}}$ of dimension $(2-2g(C))n+1$. Given $\lambda_1, \ldots, \lambda_k \in H^*(S^{[n]})$ and $\beta \neq 0$ we define

$$Z_{(\beta,n)}^{S\times C/S_{z_1,\ldots,z_k}}\left(\prod_i \tau_{k_i}(\gamma_i)\middle|\lambda_1,\ldots,\lambda_k\right) = \sum_{m\in\mathbb{Z}}(-p)^m \int_{[\mathrm{Hilb}]^{\mathrm{vir}}} \prod_i \tau_{k_i}(\gamma_i) \prod_i \mathrm{ev}_{z_i}^*(\lambda_i).$$

For any (β, m) the moduli space Hilb carries also a standard (non-reduced) virtual class $[\mathrm{Hilb}]^{\mathsf{std}}$. By the existence of the non-trivial cosection it vanishes if $(\beta, m) \neq (0,0)$. If we integrate over the standard virtual class on the right hand side above, we decorate Z with a superscript std on the left. In this case, $Z^{\mathsf{std}}(\ldots) \in \mathbb{Q}$.

We state the degeneration formula [18] in the setting of reduced invariants [20]. Related discussions appear in [33, 29] and [26, App.A]. Let $C \rightsquigarrow C_1 \cup_x C_2$ be a degeneration of C. Let

$$\{1,\ldots,k\} = A_1 \sqcup A_2$$

be a partition of the index set of points. We write $z(A_i) = \{z_j | j \in A_i\}$. We choose that the points in A_i specialize to the curve C_i. Fix also a partition of the index set of interior markings

$$\{1,\ldots,\ell\} = B_1 \sqcup B_2.$$

Then for any $\alpha_i \in H^*(S)$ we have

$$Z_{(\beta,n)}^{S\times C/S_{z_1,\ldots,z_k}}\left(\prod_{i=1}^{\ell} \tau_{k_i}(\omega\alpha_i)\middle|\lambda_1,\ldots,\lambda_k\right) =$$

$$Z_{(\beta,n)}^{S\times C_1/S_{z(A_1),x}}\left(\prod_{i\in B_1}\tau_{k_i}(\omega\alpha_i)\middle|\prod_{i\in A_1}\lambda_i,\Delta_1\right) Z_{(0,n)}^{S\times C_2/S_{z(A_2),x},\mathsf{std}}\left(\prod_{i\in B_2}\tau_{k_i}(\omega\alpha_i)\middle|\prod_{i\in A_2}\lambda_i,\Delta_2\right)$$
$$+Z_{(0,n)}^{S\times C_1/S_{z(A_1),x},\mathsf{std}}\left(\prod_{i\in B_1}\tau_{k_i}(\omega\alpha_i)\middle|\prod_{i\in A_1}\lambda_i,\Delta_1\right) Z_{(\beta,n)}^{S\times C_2/S_{z(A_2),x}}\left(\prod_{i\in B_2}\tau_{k_i}(\omega\alpha_i)\middle|\prod_{i\in A_2}\lambda_i,\Delta_2\right)$$

where Δ_1, Δ_2 stands for summing over the Künneth decomposition of the diagonal class

$$[\Delta] \in H^*(S^{[n]} \times S^{[n]}).$$

Alternatively, we can also degenerate C to an irreducible nodal curve and resolve it by a curve C'. Let $x_1, x_2 \in C'$ be the preimage of the node. Write $z = (z_1, \ldots, z_k)$. Then

$$Z_{(\beta,n)}^{S\times C/S_z}\left(\prod_{i=1}^{\ell}\tau_{k_i}(\omega\alpha_i)\middle|\lambda_1,\ldots,\lambda_k\right) = Z_{(\beta,n)}^{S\times C'/S_{z,x_1,x_2}}\left(\prod_{i=1}^{\ell}\tau_{k_i}(\omega\alpha_i)\middle|\lambda_1,\ldots,\lambda_k,\Delta_1,\Delta_2\right).$$

Remark 5.1. We index the generating series Z by $\mathsf{ch}_3(\mathcal{O}_Z)$ in order to avoid extra factors of p in the degeneration formula above.

5.2 Induction scheme

Our goal here is to express invariants of the form

$$Z_{(\beta,n)}^{S\times C/S_{z_1,\ldots,z_k}}(I|\lambda_1,\ldots,\lambda_k), \quad I = \prod_{i=1}^{\ell}\tau_{k_i}(\omega\alpha_i) \tag{26}$$

in terms of invariants of the cap geometry of the form

$$Z^{S\times\mathbb{P}^1/S_\infty}_{(\beta,n)}(I'|\lambda'), \quad I' = \prod_i \tau_{k_i'}(\omega\alpha_i'). \tag{27}$$

The general strategy to do so is well-known and goes back at least to work of Okounkov and Pandharipande in [37] or even [38]. The process is in general highly non-linear. However for reduced theories as considered here the dependence becomes $\mathbb{Q}$-linear. The linearity will allow us to prove the compatibility of the multiple cover formula with this process.

This reduction is based on the following simple but useful lemma which relates descendants and the Nakajima basis of $S^{[n]}$. Given a cohomology weighted partition

$$\mu = (\mu_i, \alpha_i)_{i=1}^{\ell}, \quad \mu_i \geq 0, \alpha_i \in H^*(S)$$

define the special monomial of descendants

$$\tau[\mu] := \prod_{i=1}^{\ell} \tau_{\mu_i - 1}(\alpha_i \cdot \omega).$$

and the Nakajima cycle

$$\mu := \prod_i \mathfrak{q}_{\mu_i}(\gamma_i) 1_{S^{[0]}} \in H^*(S^{[\sum_i \mu_i]})$$

where $\mathfrak{q}_i(\alpha) : H^*(S^{[a]}) \to H^*(S^{[a+i]})$ are the Nakajima creation operators in the convention of [21] (actually the precise convention is not important). Let $\mathcal{B}$ be a basis of $H^*(S)$. We say μ is $\mathcal{B}$-weighted if $\alpha_i \in \mathcal{B}$ for all i.

Lemma 5.2 ([40, Proposition 6]). *The matrix indexed by $\mathcal{B}$-weighted partitions of n with coefficients*

$$Z^{S\times\mathbb{P}^1/S_\infty,\mathsf{std}}_{(0,n)}(\tau[\mu], \nu) \in \mathbb{Q}$$

is invertible.

Proof. We give a sketch of the proof, see [40] for full details. For $(\beta, m) = (0, 0)$, we have that

$$\mathrm{Hilb}_{n,(0,0)}(S \times \mathbb{P}^1/S_{z_1,\ldots,z_k}) = S^{[n]}.$$

The virtual dimension matches the actual dimension, so the virtual class is just the fundamental class. Hence in this case one finds:

$$Z^{S\times\mathbb{P}^1/S_\infty,z_1,\ldots,z_k,\mathsf{std}}_{(0,n)}(\tau[\mu]|\nu_1,\ldots,\nu_k) = \int_{S^{[n]}} \prod_{j=1}^{\ell(\mu)} \tau^{S^{n]}}_{\mu_j-1}(\alpha_j) \cdot \nu_1 \cdots \nu_k$$

where $\tau^{S^{n]}}_k(\alpha) = \pi_*(\mathsf{ch}_{2+k}(\mathcal{O}_{\mathcal{Z}})\pi_S^*(\alpha))$ are the descendants on the Hilbert scheme $S^{[n]}$ (where $\mathcal{Z} \subset S^{[n]} \times S$ is the universal family). The claim hence follows from checking that certain intersection numbers on the Hilbert scheme (determined by a suitable partial ordering) between descendants and Nakajima cycles do not vanish. $\square$

The induction scheme: We reduce the general invariants (26) to invariants (27) by induction on the genus g and the number of relative markings k. If $g(C) > 0$ we degenerate C to a curve with a single node, and apply the degeneration formula in this case. If $g(C) = 0$ and $k \geq 2$, we consider the invariant

$$Z^{S\times\mathbb{P}^1/S_{z_1,\ldots,z_{k-1}}}_{(\beta,n)}(I \cdot \tau[\lambda_k]|\lambda_1,\ldots,\lambda_{k-1}).$$

which is known by the induction hypothesis. The degeneration formula yields:

$$\begin{aligned} &Z^{S\times\mathbb{P}^1/S_{z_1,\dots,z_{k-1}}}_{(\beta,n)}(I\tau[\lambda_k]|\lambda_1,\dots,\lambda_{k-1}) = \\ &Z^{S\times\mathbb{P}^1/S_{z_1,\dots,z_{k-1},x}}_{(\beta,n)}(I|\lambda_1,\dots,\lambda_{k-1},\Delta_1)Z^{S\times\mathbb{P}^1/S_x,\mathsf{std}}_{(0,n)}(\tau[\lambda_k]|\Delta_2) \\ &+Z^{S\times\mathbb{P}^1/S_{z_1,\dots,z_{k-1},x},\mathsf{std}}_{(0,n)}(I|\lambda_1,\dots,\lambda_{k-1},\Delta_1)Z^{S\times\mathbb{P}^1/S_x}_{(\beta,n)}(\tau[\lambda_k]|\Delta_2). \end{aligned}$$

By subtracting the second term on the right of the equality, and using Lemma 5.2 to invert this relation, we see that (26) is a ($\mathbb{Q}$-linear!) combination of terms which are lower in the ordering. Since the base case is $(g(C),k)=(0,1)$, this concludes the scheme.

5.3 Statement and proof of multiple cover formula

Let $\beta\in\mathrm{Pic}(S)$ be an effective class and for every $r|\beta$ let S_r be a K3 surface and

$$\varphi_r : H^2(S,\mathbb{R})\to H^2(S_r,\mathbb{R})$$

be a real isometry such that $\varphi_r(\beta/r)\in H_2(S',\mathbb{Z})$ is a primitive effective curve class. We extend φ_r to the full cohomology lattice by $\varphi_r(\mathsf{p})=\mathsf{p}$ and $\varphi_r(1)=1$. We can further extend φ_r to an action on the cohomology of the Hilbert scheme

$$\varphi_r : H^*(S^{[n]})\to H^*(S_r^{[n]})$$

by letting it act on Nakajima cycles by:

$$\varphi_r(\prod_i \mathfrak{q}_{\mu_i}(\alpha_i)1)=\prod_i \mathfrak{q}_{\mu_i}(\varphi_r(\alpha_i))1. \tag{28}$$

Then φ_r is an isometric ring isomorphism and satisfies[8]

$$\varphi_r(\tau_i(\alpha))=\tau_i(\varphi_r(\alpha)).$$

Theorem 5.1. *We have*

$$\begin{aligned} &Z^{S\times C/S_{z_1,\dots,z_k}}_{(\beta,n)}\left(\prod_{i=1}^{\ell}\tau_{k_i}(\omega\alpha_i)\middle|\lambda_1,\dots,\lambda_k\right) \\ &=\sum_{r|\beta} Z^{S_r\times C/S_{z_1,\dots,z_k}}_{(\varphi_r(\beta/r),n)}\left(\prod_{i=1}^{\ell}\tau_{k_i}(\omega\varphi_r(\alpha_i))\middle|\varphi_r(\lambda_1),\dots,\varphi_r(\lambda_k)\right)(p^k) \end{aligned} \tag{29}$$

Proof. Since φ_r is an isometry the morphism

$$\varphi_r\otimes\varphi_r : H^*(S^{[n]})\otimes H^*(S^{[n]})\to H^*(S_r^{[n]})\otimes H^*(S_r^{[n]})$$

sends $[\Delta_{S^{[n]}}]$ to $[\Delta_{S_r^{[n]}}]$. From this one shows in straightforward manner that (29) is compatible with the degeneration formula. By the induction scheme of the previous section we are hence reduced to proving the statement for the cap $S\times\mathbb{P}^1/S_\infty$.

Consider a class $\lambda\in H^*(S^{[n]})$ and descendants $\tau_{k_i}(\omega\alpha_i)$, all homogeneous, such that

$$\deg(\lambda)+\sum_i \deg\tau_{k_i}(\omega\alpha_i)=2n+1 \tag{30}$$

[8]This is clear if $\varphi_r : H^*(S)\to H^*(S_r)$ is the parallel transport operator of a deformation from S to S_r, and it follows in general by observing that the parallel transport operators are Zariski dense in the space of isometries from $H^2(S,\mathbb{R})\to H^2(S_r,\mathbb{R})$.

where $\deg(\gamma)$ denotes the complex cohomology degree of a class γ, that is $\gamma \in H^{2\deg(\gamma)}$. (Otherwise the invariants below will vanish, so there is nothing to prove.) We apply the virtual localization formula to the series

$$Z^{S\times\mathbb{P}^1/S_\infty}_{(\beta,n)}\left(\prod_i \tau_{k_i}(\omega\alpha_i)\middle|\,\lambda\right)$$

with respect to the scaling action on the $\mathbb{P}^1$, where we lift all the point classes $\omega \in H^2(\mathbb{P}^1)$ to the equivariant class $[\mathbf{0}] \in H^2_{\mathbb{C}^*}(\mathbb{P}^1)$. The formula has contributions from the extremal, the mixed and the pure-rubber components.

By the discussion in Section 4.5 the contribution from the mixed components vanishes. The contribution from the pure-rubber components is

$$Z^{S\times\mathbb{P}^1/S_{0,\infty},\sim}_{(\beta,n)}\left(\frac{1}{t-c_1(\mathcal{L}_0)}\middle|\,\lambda,\gamma\right)$$

where $\sim$ stands for rubber, and the second relative insertion is

$$\gamma = \prod_{i=1}^{\ell} \tau^{S^{[n]}}_{k_i}(\alpha_i) \in H^*(S^{[n]}).$$

Since we have the (non-equivariant) degree $\deg(\gamma)+\deg(\lambda) = 2n+1$ by (30) and the rubber space is of (non-equivariant) virtual dimension $2n$, the above integral vanishes. Only the extremal component contributes in the virtual localization.

Moreover, Proposition 4.8 shows that from the extremal component only the 1-step component can contribute.

We analyze now the contribution from the 1-step component. By equation (24) in Section 4.4 and Remark 4.9 one finds that:

$$Z^{S\times\mathbb{P}^1/S_\infty}_{(\beta,n)}\left(\prod_i \tau_{k_i}(\omega\alpha_i)\middle|\,\lambda\right) = \sum_m (-p)^m \int_{[\mathrm{Hilb}_{n,\beta,m}(S\times\mathbb{P}^1/S_\infty)]^{\mathrm{vir}}} \prod_i \tau_{k_i}(\omega\alpha_i)\,\mathrm{ev}_\infty^*(\lambda)$$

$$= \sum_m (-p)^m \sum_{r|m} (-1)^{(r-1)\frac{m}{r}} \int_{[S^{[n_1,n]}_{\beta/r}]^{\mathrm{vir}}} e_{\mathbb{C}^*}\left(R\operatorname{Hom}_S(\mathcal{K},\mathcal{F}-\mathcal{K})^\vee \otimes \mathfrak{t}^{-r}\right) \prod_i \tau_{k_i}([\mathbf{0}]\alpha_i)\cdot \pi_2^*(\lambda)$$

where $n_1 = n + \frac{m}{r} + \frac{1}{2}\beta^2/r^2$. We analyze the descendent insertion in the next lemma.

Lemma 5.3. *Under the identification $M_r^{\text{1-step}} \cong S^{[n_1,n]}_{\beta/r}$ we have that*

$$\tau_k([\mathbf{0}]\alpha)|_{M_r^{\text{1-step}}} = \sum_{d\geq 0} \sigma_d(\alpha,\beta/r)(rt)^d$$

where $\sigma_d(\alpha,\beta')$ is a universal (i.e. independent of α,β') polynomial of complex cohomological degree $\deg\sigma_d(\alpha,\beta') = \deg\tau_k(\omega\alpha) - d$ in the following variables:

$$\tau^{S^{[n_1]}}_{j_1}(\alpha\beta'^s),\quad \tau^{S^{[n]}}_{j_2}(\beta'),\quad z = c_1(\mathcal{O}_{\mathbb{P}}(1)),\quad \int_S \alpha\beta'^s,\quad s\in\{0,1,2\},\quad j_1,j_2\geq 0.$$

Proof of Lemma 5.3. Let $\mathcal{J}$ denote the universal ideal sheaf over $M_r^{\text{1-step}}$. It sits in an exact sequence

$$0 \to \pi_S^*(\mathcal{I}_2)\otimes\mathcal{O}(-r) \to \mathcal{J} \to \iota_{r*}\pi_S^*(\mathcal{I}_1(-\beta/r)\otimes\mathcal{O}_{\mathbb{P}}(-1)) \to 0,$$

where $\iota_r : S\times \operatorname{Spec}(k[x]/x^r) \to S\times\mathbb{P}^1$ is the inclusion. Since we have the exact sequence

$$0 \to \mathcal{O}_{\mathbb{P}^1}(-r) \to \mathcal{O}_{S\times\mathbb{P}^1} \to \iota_{r*}\mathcal{O} \to 0$$

we obtain $\mathsf{ch}(\iota_{r*}(1))|_{S_0} = 1 - e^{-rt}$ and therefore

$$\mathsf{ch}(\mathcal{J})\cup[\mathbf{0}] = \left(\mathsf{ch}(\mathcal{I}_2)\otimes e^{-rt} + (1-e^{-rt})\mathsf{ch}(\mathcal{I}_1)e^{-\beta/r}e^{-z}\right)\cup[\mathbf{0}].$$

Cupping with $\pi_S^*(\alpha)$ and pushing forward to $M_r^{\text{1-step}}$ we find

$$\begin{aligned}\tau_k([\mathbf{0}]\alpha)|_{M_r^{\text{1-step}}} &= -\pi_*(\mathsf{ch}_{2+k}(\mathcal{J}) \cup [\mathbf{0}]\pi_S^*(\alpha)) \\ &= -\left[\pi_*^{(2)}(\mathsf{ch}(\mathcal{I}_2)\pi_S^*(\alpha))e^{-rt} + (1-e^{-rt})\pi_*^{(1)}\left(\mathsf{ch}(\mathcal{I}_1)\pi_S^*(e^{-\beta/r}\alpha)\right)e^{-z}\right]_{\deg(\tau_k(\alpha\omega))} \\ &= \Bigg[\left(-(\textstyle\int_S\alpha)1 + \displaystyle\sum_{j_2\geq 0}\tau_{j_2}^{S^{[n]}}(\alpha)\right)e^{-rt} \\ &\quad + (1-e^{-rt})\left(-(\textstyle\int_S e^{-\beta/r}\alpha)1 + \displaystyle\sum_{j_1\geq 0}\tau_{j_1}^{S^{[n_1]}}(\alpha e^{-\beta/r})\right)e^{-z}\Bigg]_{\deg(\tau_k(\alpha\omega))}\end{aligned}$$

where $\pi^{(i)} : S^{[n_i]} \times S \to S^{[n_i]}$ are the projections. This concludes the claim. □

We continue the proof of Theorem 5.1. We have $\mathsf{rk}\, R\operatorname{Hom}_S(\mathcal{K}, \mathcal{F}-\mathcal{K})^\vee = (\beta/r)^2 + m/r$, and hence can write

$$e_{\mathbb{C}^*}(R\operatorname{Hom}_S(\mathcal{K}, \mathcal{F}-\mathcal{K})^\vee \otimes \mathfrak{t}^{-r}) = \sum_j c_{(\beta/r)^2+m/r+j}(R\operatorname{Hom}_S(\mathcal{K}, \mathcal{F}-\mathcal{K}))(-rt)^{-j}. \quad (31)$$

Moreover, $[S^{[n_1,n]}_{\beta/r}]^{\mathsf{vir}}$ is of dimension $2n+1+m/r+\beta^2/r^2$. By Lemma 5.3 and since by the degree condition (30) the term $(rt)^{-j}$ coming from the expansion (31) cancels with the term $(rt)^{d_1+\ldots+d_r}$ coming from the expansion of the $\tau_{k_i}([\mathbf{0}]\alpha_i)$, we conclude the following equality of non-equivariant integrals:

$$Z^{S\times\mathbb{P}^1/S_\infty}_{(\beta,n)}\left(\prod_i \tau_{k_i}(\omega\alpha_i)\,\middle|\,\lambda\right) = \sum_m (-p)^m \sum_{r|m}(-1)^{(r-1)\frac{m}{r}}$$

$$\sum_j \sum_{d_1+\ldots+d_\ell=j}(-1)^j\int_{[S^{[n_1,n]}_{\beta/r}]^{\mathsf{vir}}} c_{(\beta/r)^2+m/r+j}(R\operatorname{Hom}_S(\mathcal{K},\mathcal{F}-\mathcal{K})^\vee)\prod_i \sigma_{d_i}(\alpha_i,\beta/r)\cdot\pi_2^*(\lambda)$$

By Theorem 3.2 this integral depends upon S, β/r, α_i and $\lambda = \prod_i \mathfrak{q}_{\lambda_i}(\delta_i)$ only through the intersection pairings of the classes $\beta/r, \alpha_i, \delta_i, 1, \mathsf{p}$. Hence we may replace them by the isometric data given by

$$\varphi_r(\beta/r), \varphi_r(\alpha_i), \varphi_r(\delta_i), 1, \mathsf{p}.$$

Inserting, and applying the above arguments backwards the above becomes

$$= \sum_m (-p)^m \sum_{r|m}(-1)^{(r-1)\frac{m}{r}}\sum_j\sum_{d_1+\ldots+d_\ell=j}$$

$$\int_{[S^{[n_1,n]}_{\varphi_r(\beta/r)}]^{\mathsf{vir}}} c_{\varphi_r(\beta/r)^2+m/r+j}(R\operatorname{Hom}_{S_r}(\mathcal{K},\mathcal{F}-\mathcal{K})^\vee)\prod_i\sigma_{d_i}(\varphi_r(\alpha_i),\varphi_r(\beta/r))\cdot\pi_2^*(\varphi_r(\lambda))$$

$$= \sum_m(-p)^m\sum_{r|m}(-1)^{(r-1)\frac{m}{r}}\int_{[\operatorname{Hilb}_{n,\varphi_r(\beta/r),m/r}(S\times\mathbb{P}^1/S_\infty)]^{\mathrm{vir}}}\prod_i\tau_{k_i}(\omega\varphi_r(\alpha_i))\operatorname{ev}_\infty^*(\varphi_r(\lambda))$$

$$= \sum_{r|m} Z^{S\times\mathbb{P}^1/S_\infty}_{(\varphi_r(\beta/r),n)}\left(\prod_i\tau_{k_i}(\omega\varphi_r(\alpha_i))\,\middle|\,\varphi_r(\lambda)\right)(p^r)$$

which was what we wanted to prove. □

6 Proofs of the remaining main results

6.1 Proof of Theorem 1.2

The first part (the independence from the divisibility) follows by Proposition 3.5. We hence need to evaluate $Q_{n,h,m}$. Recall the generating series $\mathsf{DT}_n(p,q)$ and $\mathsf{H}_n(p,q)$ from (2), and the series

$$\mathsf{Q}_n(p,q) = \sum_{h\geq 0}\sum_{m\in\mathbb{Z}} \mathsf{Q}_{n,h,m} q^{h-1} p^m.$$

By Theorem 1.2 we have for all n the equality

$$\mathsf{DT}_n(p,q) = \mathsf{H}_n(p,q) - \chi(S^{[n]})\mathsf{Q}_n(p,q). \tag{32}$$

Since $S^{[0]} = pt$ and $S^{[1]} = S$, by the Yau-Zaslow formula we have

$$\mathsf{H}_0 = 0, \quad \mathsf{H}_1 = -2\frac{E_2(q)}{\Delta(q)}.$$

On the other hand, by the Katz-Klemm-Vafa formula [20] for $n = 0$, and by [2] and [29] (see also [34]) for $n = 1$ we have that

$$\mathsf{DT}_0 = -\frac{1}{\Theta^2\Delta}, \quad \mathsf{DT}_1 = -24\frac{\wp(p,q)}{\Delta(q)}.$$

where $\wp(p,q)$ is the Weierstraß elliptic function (see [33, Sec.2]).

By Proposition 3.7 we have $\mathsf{Q}_n(p,q) = F_1^n F_2$ for some F_1, F_2. So from case $n = 0$ we conclude that:

$$F_2 = \frac{1}{\Theta^2\Delta}.$$

For the $n = 1$ term we conclude

$$F_1 = \frac{1}{24F_2}(\mathsf{H}_1 - \mathsf{DT}_1) = \Theta^2 \cdot (-\frac{1}{12}E_2 + \wp) = \mathbf{G}(p,q)$$

where we used that $\left(\frac{p}{dp}\right)^2 \log(\Theta(p,q)) = -\wp(p,q) + \frac{1}{12}E_2(q)$, see [33, Equation (11)].[9] $\square$

Remark 6.1. If one had the GW/DT correspondence for the cap geometry $(S \times \mathbb{P}^1)/S_\infty$ the above computation of F_1 would also follow from the results of [29].[10]

6.2 Proof of Theorem 1.3

Define the series

$$Z^{S\times E/E}_{(\beta,n)} = \sum_{m\in\mathbb{Z}} \mathsf{DT}^{S\times E}_{m,(\beta,n)}(-p)^m.$$

Choose a class $D \in H^2(S,\mathbb{Z})$ such that $D\cdot\beta \neq 0$. By [28] we have that

$$Z^{S\times E/E}_{(\beta,n)} = \frac{1}{\beta\cdot D} Z^{S\times E}_{(\beta,n)}(\tau_0(\omega D)).$$

By Theorem 5.1 we hence have that:

$$\begin{aligned} Z^{S\times E/E}_{(\beta,n)} &= \frac{1}{\beta\cdot D} Z^{S\times E}_{(\beta,n)}(\tau_0(\omega D)) \\ &= \frac{1}{D\cdot\beta}\sum_{r|\beta} Z^{S\times E}_{(\varphi_r(\beta/r),n)}(\tau_0(\omega\varphi_r(D)))(p^r) \\ &= \sum_{r|\beta} \frac{\varphi_r(D)\cdot\varphi_r(\beta/r)}{D\cdot\beta} Z^{S\times E/E}_{(\varphi_r(\beta/r),n)}(p^r) \\ &= \sum_{r|\beta}\frac{1}{r} Z^{S\times E/E}_{(\varphi_r(\beta/r),n)}(p^r). \end{aligned}$$

This implies the claim by taking coefficients. $\square$

[9] Note that $F(z,\tau)$ in [33] corresponds to $-i\Theta(p,q)$. The variable convention is the same.
[10] The GW/DT correspondence for $(S\times\mathbb{P}^1)/S_\infty$ was recently proven in [30].

6.3 Proof of Theorem 1.4

Since the multiple cover formula is compatible with the divisor equation and restriction of Gromov-Witten classes to boundary components, it is enough to consider the case $N = 3$ and $\alpha = 1$. By [23, Cor. 4.2] these primary invariants are identical to the DT invariants of the relative geometry $S \times \mathbb{P}^1 / S_{0,1,\infty}$. A small calculation shows that the multiple cover formula given in Theorem 5.1 implies the form of the multiple cover formula given in Theorem 1.4. □

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