Kaneko-Zagier equations for Jacobi forms and curve counting on CHL manifolds

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Based on the joint works:
arXiv:1811.06102 (with Jim Bryan),
arXiv:2007.03489 (with Jan-Willem van Ittersum and Aaron Pixton),
arXiv:1411.1514 (with Rahul Pandharipande)

Short Overview

Abstract

I will discuss counting curves on $K3 \times \mathbb{P}^1$ with relative conditions at $K3 \times 0$ and $K3 \times \infty$. Main features:

- Structure constants are quasi-Jacobi forms
- g-twisted traces $\rightsquigarrow$ CHL geometry
- Relations to Conway Moonshine VOAs (??)

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Let $q = e^{2\pi i \tau}$ where $\tau \in \mathbb{H}$. Given $k > 2$ consider the Serre derivative

$$\vartheta_k = D_\tau - \frac{k}{12} E_2(\tau)$$

where $D_\tau = q\frac{d}{dq}$ and $E_2 = 1 - 24\sum_{n\geq 1} \sigma(n) q^n$.

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Lemma

$\vartheta_k : \mathrm{Mod}_k \to \mathrm{Mod}_{k+2}$

Proof.

The transformation law $E_2(-1/\tau) = \tau^2 E_2(\tau) - 6i\tau/\pi$ cancels the quasi-modularity arising by differentiating by τ. □

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Definition (Kaneko-Zagier equation)

$$\vartheta_{k+2}\vartheta_k f_k(\tau) = \frac{k(k+2)}{144} E_4(\tau) f_k(\tau)$$

Theorem (Kaneko-Zagier)

For every $k \not\equiv 2 \,(mod\, 3)$, the solution $f_k = 1 + \ldots$ to the KZ equation is a modular form of weight k.

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Proof.

Let $\wp(x,\tau)$ be the Weierstraß function and define the formal power series

$$f(x) = \left(\frac{\wp'(x)}{-2} \right)^{-1/3}.$$

Let $\Phi(y)$ be formal inverse of f in x, i.e. $\Phi(f(x)) = x$. Then the coefficients of the expansion

$$\Phi(y) = \sum_{k \geq 1} \frac{f_{k-1}}{k} y^k$$

are modular forms and the desired solutions (invert the diff eqn). □

Remark. KZ equation is essentially unique second-order ODE with modular solution. Relations to characters of simple modules for rational vertex operator algebras (Kaneko,Nagatomo, Sakai, 2013)

Jacobi KZ equation (with van Ittersum, Aaron Pixton)

Let $z \in \mathbb{C}$ elliptic variable, $p = e^{2\pi i z}$. Define the theta function

$$\vartheta_1(z,\tau) = \sum_{\nu \in \mathbb{Z}+\frac{1}{2}} (-1)^{\lfloor \nu \rfloor} p^{\nu} q^{\nu^2/2}$$

and the renormalization

$$\Theta(z,\tau) = \frac{\vartheta_1(z,\tau)}{\eta^3(\tau)}.$$

Let

$$F(z) := \frac{D_\tau^2 \Theta(z)}{\Theta(z)} = -\sum_{n \geq 1} \sum_{d|n} (n/d)^3 (p^{d/2} - p^{-d/2})^2 q^n,$$

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Definition (vIOP, Kaneko-Zagier equation for Jacobi forms)

For any $m \in \mathbb{Z}$ we consider:

$$\begin{cases} D_\tau^2 \varphi_m = m^2 F \varphi_m \\ \varphi_m = (p^{m/2} - p^{-m/2}) + O(q) \end{cases}$$

The solutions

With $A = D_z\Theta/\Theta$ one has

$$
\begin{aligned}
\varphi_1 &= \Theta \\
\varphi_2 &= 2\Theta^2 A \\
\varphi_3 &= \frac{9}{2}\Theta^3 A^2 - \frac{3}{2}\Theta^3 \wp \\
\varphi_4 &= \frac{32}{3}\Theta^4 A^3 - 8\Theta^4 A\wp - \frac{2}{3}\Theta^4 D_z(\wp) \\
\varphi_5 &= \frac{625}{24}\Theta^5 A^4 - \frac{125}{4}\Theta^5 A^2 \wp + \frac{15}{8}\Theta^5 \wp^2 - \frac{25}{6}\Theta^5 A D_z(\wp) + \frac{5}{2}\Theta^5 G_4 \\
&\ldots
\end{aligned}
$$

General solution

Define the series

$$f(x) = \frac{\Theta(x)}{\Theta(x+z)} = \frac{x}{\Theta(z)} + \dots$$

Let $\Phi(y)$ be formal inverse of f in x, i.e. $\Phi(f(x)) = x$. Consider the expansion

$$\Phi(y) = \sum_{m=1}^{\infty} \frac{\varphi_m}{m} y^m.$$

Theorem (vIOP)

The functions φ_m defined above are solutions of the Jacobi KZ equation.

Corollary

Every φ_m is a quasi-Jacobi form of index $|m|/2$ and weight -1.

Second differential equation

Define a second set of functions $\varphi_{m,n}(z,\tau)$ for $m, n \in \mathbb{Z}$ as follows:

$$\begin{cases} D_\tau \varphi_{m,n} = mn\varphi_m \varphi_n F + (D_\tau \varphi_m)(D_\tau \varphi_n) \\ \varphi_{m,n} = O(q) \end{cases}$$

Theorem (vIOP)

For all $m, n \in \mathbb{Z}$ the difference

$$\varphi_{m,n} - |n|\delta_{m+n,0}$$

is a quasi-Jacobi form of weight 0 and index $\frac{1}{2}(|m| + |n|)$.

Proof.

For $m \neq -n$ this is easy:

$$\varphi_{m,n} = \frac{m}{m+n}\varphi_m D_\tau(\varphi_n) + \frac{n}{m+n} D_\tau(\varphi_m)\varphi_n$$

For $m = -n$ this becomes extremely subtle. □

Back to geometry: Overview

Let $g : S \to S$ be a symplectic automorphism of order N of a K3 surface. The associated CHL Calabi-Yau threefold is:

$$X_g = (S \times E)/\langle g \times \tau_N \rangle.$$

String theory of X_g studied by David, Jatkar, Sen, .. many others. Our strategy to evaluate its Gromov-Witten theory mathematically:

- Degenerate one period of the torus E:

$$X_g \rightsquigarrow (S \times \mathbb{P}^1)/\sim, \quad (s,0) \sim (gs, \infty)$$

- Express GW theory of $K3 \times \mathbb{P}^1$ as operator on Fock space
- g-twisted traces give topological string partition function

Fock Space

Consider the (negative) K3 lattice

$$\Lambda = U^{\oplus 4} \oplus E_8^{\oplus 2}$$

and the associated Fock space

$$\mathcal{F}_{K3} = \bigotimes_{m \geq 1} \mathrm{Sym}^{\bullet}(\Lambda_{\mathbb{Q}} q^i).$$

It is acted on by the Heisenberg operators $\alpha_i(\gamma)$ for $i \in \mathbb{Z}$ and $\gamma \in \Lambda$ subject to the commutation rule:

$$[\alpha_m(\gamma), \alpha_n(\gamma')] = m\delta_{m+n,0}(\gamma, \gamma')\mathrm{id}_{\mathcal{F}}.$$

The Fock space $\mathcal{F}_{K3}$ is freely generated from the action of the creation operators $\alpha_i(\gamma)$, $i < 0$ on the vaccuum vector $|\varnothing >$.

Operators (based on 1406.1139)

For every element $\beta \in \Lambda$ define the 'vertex operator'

$$\omega_\beta : \mathcal{F}_{K3} \to \mathcal{F}_{K3} \otimes \frac{\mathbb{Q}[p^{\pm 1/2}]}{(p^{1/2} - p^{-1/2})^2}$$

as the $q^{-\frac{1}{2}(\beta,\beta)}$ coefficient of

$$: \frac{1}{\Theta^2 \eta^{24}} \exp\left(\sum_{m \neq 0} \alpha_m(\beta) \frac{\varphi_m}{m} \right) \exp\left(\frac{1}{2} \sum_{m,n \in \mathbb{Z} \setminus \{0\}} \alpha_m \alpha_n(\Delta) \frac{\varphi_{m,n}}{mn} \right) :$$

Here

- $\Delta \in \Lambda \otimes \Lambda$ is the class of the diagonal,
- $: - :$ stands is the normal ordered product,
- $\varphi_m, \varphi_{m,n}$ are the functions defined previously.
- dependence on p, q is surpressed

Example

- If $(\beta, \beta) > 2$, then $\omega_\beta = 0$.
- If $(\beta, \beta) = 2$

$$\omega_\beta = : \left[\frac{1}{(s - s^{-1})^2} \exp\left(\sum_{m \neq 0} \frac{(s^m - s^{-m})}{m} \alpha_m(\beta) \right) \right] :$$

  where $p = s^2$. This can be expressed in terms of action of $\hat{gl}(2)$ (Maulik-Oblomkov).

- If $(\beta, \beta) = 0$,

$$\omega_\beta = : \exp\left(\sum_{m \neq 0} \frac{s^m - s^{-m}}{m} \alpha_m(\beta) \right) \cdot \Big[(2s^{-2} + 20 + 2s^2)/(s - s^{-1})^2 - \sum_{m \neq 0} m(s^m - s^{-m}) \alpha_m(\beta) - \frac{1}{2} \sum_{\ell, m \neq 0} (s^\ell - s^{-\ell})(s^m - s^{-m}) \alpha_\ell \alpha_m(\Delta) \Big] :$$

Conjecture

Assume β primitive. For any $\mu, \nu \in \mathcal{F}_n$ we have

$$\mathrm{Coeff}_{p^k}\left(\mu \,\middle|\, \omega_\beta \nu\right)_{\mathcal{F}_n} = (-1)^{k-1} \mathsf{PT}^{K3\times\mathbb{P}^1}_{k+n,(\beta,n)}(\mu,\nu)$$

under suitable identification of boundary conditions: $H^*(\mathrm{Hilb}_n K3) \cong \mathcal{F}_n$.

Conjecturally this yields full solution to Pandharipande-Thomas theory of $K3 \times \mathbb{P}^1$ relative to the divisors $K3 \times 0$ and $K3 \times \infty$.

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Upshot: The PT theory of all CHL models becomes computable:

$$\begin{aligned} \mathsf{PT}^{X_g}_{k,(\beta,n)} &= \frac{1}{N} \sum_{\substack{\tilde{\beta}\in H_2(S,\mathbb{Z}) \\ P(\tilde{\beta})=\beta}} \mathsf{PT}^{K3\times\mathbb{P}^1}_{k+n,(\tilde{\beta},n)}(\Gamma_g) \\ &= \frac{1}{N} \sum_{P(\tilde{\beta})=\beta} (-1)^{k+1} \mathrm{Coeff}_{p^k}\left((\Gamma_g)_1 \,\middle|\, \omega_{\tilde{\beta}} (\Gamma_g)_2\right)_{\mathcal{F}_n} \\ &= (-1)^{k+1} \frac{1}{N} \mathrm{Coeff}_{p^k} \sum_{P(\tilde{\beta})=\beta} \mathrm{Tr}_{\mathcal{F}_n}(g\omega_{\tilde{\beta}}). \end{aligned}$$

Back to CHL models

Given $g : S \to S$ symplectic automorphism, let:

- $\Lambda = H^*(S, \mathbb{Z})$ as before.
- $L = H^2(S, \mathbb{Z})^g$ invariant lattice
- $P = \frac{1}{N} \sum_{i=0}^{N-1} g^i$

$$H_2(X_g, \mathbb{Z})/\text{torsion} \cong \text{Im}(P : L \to L \otimes \mathbb{Q}) \oplus \mathbb{Z}$$
$$\cong (L^g)^\vee \oplus \mathbb{Z}$$

Lemma

Assume $\gamma \in (L^g)^\vee$ primitive. Then $\mathrm{PT}^{X_g}_{k,(\gamma,n)}$ only depends on (X_g, γ) via

$$s = \gamma^2, \qquad \alpha = [\gamma] \in (L^g)^\vee / L^g,$$

and $g : L \to L$ up to conjugation. We write $\mathrm{PT}_{k,s,n,\alpha}$ instead.

CHL models II

Definition

Given $\alpha \in (L^g)^\vee / L^g$ define

$$Z_\alpha = \sum_{k,s,n} \mathsf{PT}_{k,s,n,\alpha} (-1)^{k+1} p^k t^{n-1} q^{s/2}.$$

Conjecture

Every Z_α is a Siegel modular form of genus 2 (for a congruence subgroup).

Computation scheme:

$$Z_\alpha = \frac{1}{N} \sum_s \sum_{P(\tilde{\beta})=\beta_s} \mathrm{Tr}_{\mathcal{F}}(g \omega_{\tilde{\beta}} t^{\mathbb{N}-1}).$$

where $\mathbb{N}$ is the energy operator: $\mathbb{N}|_{\mathcal{F}_n} = n \cdot \mathrm{id}_{\mathcal{F}_n}$, and $\beta_s \in (L^g)^\vee$ is a primitive class of square $\beta_s^2 = s$ and $[\beta_s] = \alpha$.

Example: $N = 1$

Simplest case: $g = \mathrm{id}$, then $X_g = K3 \times E$. It is well-known:

$$Z^{K3\times E} = \frac{1}{\chi_{10}\begin{pmatrix} \tau & z \\ z & \tilde{\tau} \end{pmatrix}}$$

where $p = e^{2\pi i z}$, $q = e^{2\pi i \tau}$ and $t = e^{2\pi i \tilde{\tau}}$.
Hence on expects:

$$\mathrm{Tr}_{\mathcal{F}}\, t^{\mathbb{N}-1} \omega_\beta = \mathrm{Coeff}_{q^{\beta^2/2}} \left[\frac{1}{\chi_{10}(\Omega)} \right]$$

$\rightarrow$ Strong evidence available (exact expressions later)

Example: $N = 2$

In order 2 the involution g interchanging the E_8 factors.

$$L^g = E_8(-2) \oplus U^3$$

$$D(L^g) = (L^g)^\vee / L^g = D(E_8(2)) = \mathbb{Z}_2^8$$

The Weyl group acts on the discriminant group with three orbits $0, \alpha_1, \alpha_2$. If $[\gamma] \neq 0$, then its orbit is determined by γ^2. Hence we obtain two cases:

- If $[\gamma] = 0$ (untwisted case) $Z_{\text{untwisted}} = Z_0^{N=2}$
- If $[\gamma] \neq 0$ (twisted case) $Z_{\text{twisted}} = Z_{\alpha_1}^{N=2} + Z_{\alpha_2}^{N=2}$

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$$Z_{\text{twisted}} = \frac{1}{2^4} \frac{1}{\Phi_{6,3}(\Omega)}$$

(Ref: Jatkar and A. Sen, Dyon spectrum in CHL models;
David, D. P. Jatkar and A. Sen Product Representation of Dyon Partition Function in CHL Models)

Example: $N = 2$ continued

By joint work with Bryan one has:

$$Z_{\text{untwisted}} = -\frac{1}{2}\left(\frac{1}{\Phi_{6,0}} + \frac{1}{2^4\,\Phi_{6,3}} + \frac{1}{2^4\,\Phi_{6,4}}\right)$$

(the simplified formula here is from [Fischbach, Klemm, Nega - Lost Chapters in CHL Black Holes ..])

$$\frac{1}{\Phi_{6,0}} = \frac{\theta_{0000}^2\theta_{0001}^2\theta_{0010}^2\theta_{0011}^2}{\chi_{10}} \qquad (1)$$

$$\frac{1}{\Phi_{6,3}} = \frac{\theta_{0000}^2\theta_{0010}^2\theta_{0100}^2\theta_{0110}^2}{\chi_{10}} \qquad (2)$$

$$\frac{1}{\Phi_{6,4}} = -\frac{\theta_{0001}^2\theta_{0011}^2\theta_{0100}^2\theta_{0110}^2}{\chi_{10}}\,. \qquad (3)$$

The computation scheme

$$Z_\alpha^N = \frac{1}{N}\sum_s \sum_{P(\tilde\beta)=\beta_s} \mathrm{Tr}_{\mathcal{F}}(g\omega_\beta t^{\mathbb{N}-1}).$$

yields conjectural identities relating Siegel modular forms with the functions φ_{mn}. Already $g = \mathrm{id}$ is very interesting, it gives the following: Define functions $L(z,\tau,\tilde\tau)$ and $M(z,\tau,\tilde\tau)$ by

$$L = \sum_{r\geq 1}(-1)^{r-1}\sum_{d_1,\ldots,d_r\in\mathbb{Z}\setminus 0}\frac{1}{|d_1|\cdots|d_r|}\frac{\tilde q^{|d_1|}}{1-\tilde q^{|d_1|}}\cdots\frac{\tilde q^{|d_r|}}{1-\tilde q^{|d_r|}}\varphi_{d_1}\varphi_{-d_1,d_2}\cdots\varphi_{-d_{r-1},d_r}\varphi_{-d_r}$$

$$M = \sum_{r\geq 1}\frac{(-1)^{r-1}}{r}\sum_{d_1,\ldots,d_r\in\mathbb{Z}\setminus 0}\frac{1}{|d_1|\cdots|d_r|}\frac{\tilde q^{|d_1|}}{1-\tilde q^{|d_1|}}\cdots\frac{\tilde q^{|d_r|}}{1-\tilde q^{|d_r|}}\varphi_{d_1,-d_2}\cdots\varphi_{d_r,-d_1}$$

Conjecture

$$\sum_{k\geq 0}\frac{1}{k!}D_\tau^k\left(\frac{L^k}{\exp(M)^{12}\Theta(z,\tau)^2\Delta(\tau)\Delta(\tilde\tau)}\right) = \frac{1}{\chi_{10}\begin{pmatrix}\tau & z\\ z & \tilde\tau\end{pmatrix}}$$

Main open questions:

- What is the algebra structure of the operators ω_β ?
- Are there relations to the Conway Moonshine VOA?
- Mathematical proofs
- Non-commutative CHL models

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Thank you! (the end)

(Kummer K3 surface, source:mo-labs.com)