

Seminar on Quantum Groups & Quantum Cohomology of Symplectic Varieties

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Univ. Bonn, Winter 19/20, Tuesday 2-4, Room 0.011

Let X be a complex variety with an action by a torus $T = (\mathbb{C}^*)^n$. Since the topological Euler characteristic of \mathbb{C}^* vanishes, the Euler characteristic of X is the Euler characteristic of the fixed locus X^T under the torus action:

$$e(X) = e(X^T).$$

More generally, the cohomology of X is determined by the cohomology of X^T together with the restriction maps on cohomology, given that the cohomology is taken *equivariantly* with respect to the torus T . This yields a practical and powerful tool to understand the cohomology of algebraic varieties with torus actions.

The seminar discusses the cohomology of symplectic varieties with torus actions and their relation to quantum groups. The key is to understand the equivariant restriction morphisms

$$H_T^*(X) \rightarrow H_T^*(X^T)$$

and to construct suitable maps in the other direction called *stable envelopes*. Understanding this in detail for Nakajima quiver varieties will lead to interesting R -matrices out of which one can construct a Yangian. The Yangian naturally act on the cohomology of these varieties, and one part of it acts by multiplication with Chern classes of tautological bundles.

A further enhancement of the theory is obtained by considering the quantum cohomology of the symplectic varieties. Quantum cohomology is a commutative and associate deformation of the cohomology of a smooth projective variety whose structure constants encode counts of rational curves on the underlying variety. Formally it is defined in terms of the Gromov–Witten theory of the variety. For Nakajima quiver varieties we will see that

the operators of quantum multiplication with tautological classes are also expressible through the action of the Yangian.

The main reference for the course is the book [10] by Maulik and Okounkov.

The theory presented here is best understood through concrete examples. Already the case of $T^*\mathbb{P}^1$, the cotangent bundle of \mathbb{P}^1 , showcases many of the properties of the general theory. Hence the $T^*\mathbb{P}^1$ case should be discussed in every talk. More general examples include:

- Cotangent bundle of the Grassmannian: $T^*\text{Gr}(k, n)$ or more generally of flag varieties.
- Hilbert scheme of points on the affine plane, $\text{Hilb}(\mathbb{C}^2)$, or on ADE surfaces
- Springer resolution $T^*(G/B)$
- Moduli space of framed rank r sheaves on the plane $\mathcal{M}(r, n)$

The schedule for the talks is as follows:

Oct. 8: Skipped.

Oct. 15: Equivariant Cohomology I (Speaker: Travis Schedler)

Basic definition of equivariant cohomology using the Borel construction. Approximation via algebraic spaces. Formality [1, Section 2.4]. Functoriality (pullback and pushforward).

The main focus should be to explain how it works practically in various examples such as the point, projective space, and the Grassmannian. In particular, describe explicitly the equivariant cohomology of \mathbb{P}^1 , including presentation of generators, chern classes of the line bundles $\mathcal{O}(n)$ and of the tangent bundle, and the classes of fixed points $[\mathbf{0}]$ and $[\infty]$. What does formality say in this case? Find the image of the restriction maps

$$H_T^*(X) \rightarrow H_T^*(X_T) \tag{1}$$

for $X = \mathbb{P}^1$, and show it is an isomorphism after inverting equivariant parameters. Discuss the restriction (1) and in particular the fixed points for \mathbb{P}^n and the Grassmannian. If time permits explain the second localization formula ([1, Thm.2.13]). Good reference: [1] Lecture 1, part of Lecture 2. See also [5] and the classical [2]. For algebraic geometers [7].

Oct. 22: Equivariant Cohomology II: Localization formula (Speaker: Johannes Schmitt)

This talk focuses on the Atiyah-Bott localization formula and its application. Discuss the statement and give examples. For example evaluate the basic integral

$$\int_{\mathbb{P}^1} c_1(\mathcal{O}_{\mathbb{P}^1}(1))$$

using the localization formula. More generally, perform an integral over \mathbb{P}^n , e.g. find some Chern numbers on \mathbb{P}^2 . Give also applications of the localization formula where the fixed locus has positive-dimension, e.g. integrate over $C \times \mathbb{P}^1$ where C is a higher genus curve. Define integration on non-compact spaces by the localization and perform some example calculations, e.g. over \mathbb{C}^2 . Explain the following crucial property: If the total space is compact or the cycle you integrate has compact support, then the value of the equivariant integral does not have poles in the equivariant cohomology of the point, or in other words it is a polynomial.

Sketch a proof of the Localization formula.

More generally, apply the localization formula to an integral over a Grassmannian, for example use the fixed point formula to determine the number of lines on a cubic surface along the lines of [11, Section 1]. This requires you to talk about the tangent weights at fixed points of the Grassmannian.

Good reference: [1] and see also the reference for the first talk.

Oct. 29: Stable envelopes I (Speaker: Mauri Mirko)

The main reference for this and the next talk is Section 3 of [10]. The general setup is Section 3.1.2. The chapter on Signs and adjoints 3.1.3 should be skipped. Attracting manifolds. Partial/Absolute order determined by choice of chamber, Introduce stable envelopes by their characterization (Theorem 3.3.4). Discuss the case of $T^*\mathbb{P}^1$ in all details, in particular give the map for both choices of chambers c_1, c_2 . Compare the corresponding maps and note that

$$\text{Stab}_{c_1}^{-1} \circ \text{Stab}_{c_2}$$

is non-trivial. If time permits, discuss the relationship between stable envelope for the cotangent bundle of a Grassmannian with classical Schubert cycles, compare in particular the positivity in [1, Section 3] with the conditions of Theorem 3.3.4. Sketch the uniqueness of stable envelopes.

Nov. 5: Stable envelopes II (Speaker: Isabell Hellmann)

Review stable envelopes. Introduce Lagrangian correspondences (3.2.5), Lagrangian residues (3.4) and prove the existence of stable envelopes (3.5). Discuss Symplectic resolutions and Steinberg correspondences (Section 3.6

and 3.7). In particular show that for symplectic resolutions the stable envelope is the specialization of the attractor manifold to the special fiber (Theorem 3.7.4). As an example discuss again $T^*\mathbb{P}^1$.

The name 'Steinberg correspondence' Z originates from Steinberg's study of the Springer resolution $T^*(G/B)$. Concretely we have

$$\begin{aligned} Z &= T^*(G/B) \times_N T^*(G/B) \\ &= \{(x, F_\bullet, F'_\bullet) \mid x \text{ nilpotent, } xF_i \subset F_{i+1} \text{ and } xF'_i \subset F'_{i+1} \text{ for all } i\} \end{aligned}$$

where N is the nilpotent cone, see the book Chris-Ginzburg [4] and also [3]. If time permits explain this case,

Nov. 12: Quantum groups and Yang-Baxter equations (Till Werhahn)

A quantum group is a Hopf algebra deformation of the universal enveloping algebra of a Lie algebra \mathfrak{g} . Explain the basics of quantum groups: Definition, R -matrices, and the Yang-Baxter equation. Explain how the quantum group can be reconstructed from the R -matrices using the $RTT = TTR$ construction. In this lecture use the Lie algebra $\mathfrak{g} = \mathfrak{gl}(2)$ as your leading example. In particular, perform the $RTT = TTR$ construction explicitly $U_q(\mathfrak{gl}(2))$. Reference is [8], in particular Prop. VIII.3.1, Thm. VIII.6.1 and Thm. VIII.6.4.

Nov. 19: The Yangian $Y(\mathfrak{gl}_N)$ (Speaker: Thorsten Heidersdorff)

Introduce and study the Yangian of $\mathfrak{gl}(n)$ explicitly in several ways (via the $RTT=TTR$ construction, via R -matrices ...).

Nov. 26: Nakajima quiver varieties and their fixed points (Speaker: Pieter Belmans)

Explain their basic construction as in Section 2.1 of [10]. See also [16] Section 4 for a more friendly introduction, see [15] for more details, and also [12, 13] for the original papers. Another good reference is [9]. The construction of the GIT quotient should be taken as a blackbox here. Explain how the various examples we discussed so far arise as a Nakajima variety. In particular describe the quiver for $T^*\mathbb{P}^1$, $T^*\mathbb{P}^n$ and the cotangent bundle of the Grassmannian and show it gives the desired varieties. Introduce the Hilbert scheme of points of \mathbb{C}^2 and its representation as a quiver variety, see [14] and the ADHM Construction.

In the second half of the talk focus on the torus fixed points of Nakajima quiver varieties. These are given as products of smaller Nakajima quiver varieties, see Section 2.3 and 2.4 of [10]. The general result is Proposition 2.3.1. For most of our application the presentation of equation (2.14) is

more relevant. Discuss a bit the general theory but spend most of the time to explain the fixed loci for the various examples discussed above: $T^*\mathbb{P}^1$, $T^*\mathbb{P}^n$, cotangent of Grassmannian. If time permits, another interesting case is the moduli of framed sheaves discussed in Section 12.1.

Dec. 17: The Yangian of the Grassmannian (Catharina Stroppel)

Explicit discussion of the Yangian in the Grassmannian case (quiver $Q = \text{pt}$).

Dec. 17: The Maulik-Okounkov Yangian in general (Georg Oberdieck)

Section 4 and 5 in [10]. First we recall that for every chamber c we have the stable envelope,

$$\text{Stab}_c : H^*(X^A) \rightarrow H^*(X)$$

which we can view as defining for us a natural basis of the cohomology $H^*(X)$ in terms of attracting cycles. The associated change-of-basis matrix is the R -matrix

$$R_{c'c} = \text{Stab}_{c'}^{-1} \circ \text{Stab}_c.$$

We follow Sections 4.1.4-9 to show that for Nakajima quiver varieties this naturally leads to a set of endomorphisms R_{ij} satisfying the Yang-Baxter equations. Intuitively, the stable envelopes are similar to charts of a vector bundle, and the Yang-Baxter equations are the cocycle condition for these charts. The key here is Lemma 3.6.1/Corollary 4.1.2 which says that the R -matrix for two adjacent chambers (with respect to the full torus A) is given by the R -matrix R_α of a single \mathbb{C}^* -action determined by the root α defining the wall between the chambers. In particular, every R -matrix is a combination of such elementary R -matrices satisfying the YBE equation.

We now jump forward to Section 5.2.6 and explain the construction of the Yangian. For the Grassmannian case ($Q = \text{pt}$) this reduced to the previous talk on the Yangian of \mathfrak{gl}_2 ; this can be done explicitly by matching the R -matrix in 4.1.2 with the R -matrix for \mathfrak{gl}_2 . Then we go through several properties of the Yangian (the proofs are mostly skipped): the TTR=RTT relation of Section 5.2.9; that the Yangian acts naturally on $H^*(\mathcal{M}(w))$ for all w (no direct reference; this is not immediately if we use $\{F_i\} = \{H(\delta_i)\}$ in the definition of the Yangian as in the book, but it follows by defining the action in a 'chart' $\text{Stab}_c : \otimes_j F_{i_j} \rightarrow H(w)$ and then using the YBE to show that this does not depend on the choice of chart); that we could have defined the Yangian with $\{F_i\} = \{H(w)\}_{w \in \mathbb{Z}^I}$ instead (Section 5.2.14); the definition of the coproduct as in Section 5.2.15 (if y lies in the Yangian then it comes naturally with an action on W , W' and $W \otimes W'$, and we define the action of Δy by the action y on $W \otimes W'$).

At the end we discuss the structure of the Yangian (Thm 5.5.1). The Lie Algebra \mathfrak{g}_Q is generated by $E(m_0)$ where m_0 are constants in u . The Cartan of \mathfrak{g}_Q is then generated by the operators v_i, w_i that act on $H(\mathcal{M}(v, w))$ by v_i and w_i respectively. The RTT=TTR relation shows that \mathfrak{g}_Q is a Lie algebra.

= Holidays break =

Jan. 7: Quantum cohomology (Speaker: Denis Nesterov)

Definition, basic properties, virtual dimension. Compute the quantum cohomology of \mathbb{P}^n . The reference for these topics is [6]. We further discuss the quantum cohomology of equivariant symplectic resolutions following the article [3]. In particular, we consider the reduced virtual class and we show that

$$(\text{ev}_1 \times \text{ev}_2)_* [\overline{M}_{0,2}(X, \beta)]^{\text{red}}$$

is a direct sum of components of the Steinberg varieties $X \times_{X_0} X$. State the main conjecture of [3] and give the Example of $T^*\mathbb{P}^1$.

Jan. 14: Quantum multiplication on Nakajima quiver varieties (Speaker: Georg Oberdieck)

We use virtual localization to prove explicitly

$$\int_{[\overline{M}_0(T^*\mathbb{P}^1, d)]^{\text{red}}} 1 = \frac{\hbar}{d^3}.$$

In doing so we learn about unbroken map and that only they contribute non-trivially in localization computations for quantum multiplication with a divisor. Reference is Section 7.3 and the references therein. Statement of Theorem 1.3.2/Theorem 10.2.1 of [10].

In the second half we return to the interpretation of Maulik-Okounkov's result in terms of their Yangian. Define the Baxter (or Bethe) subalgebra of Y following Section 6.5 and 1.2.2. We have two examples, the first one is given by $g \in \prod \text{End} H(\mathcal{M}(w))$ where the w -component g_w of g is given by the projection onto the vacuum vector $|w\rangle$, i.e. $g_w = |w\rangle\langle w|$. Indeed, using that v_i is primitive, one has that Δv_i commutes with R (Proposition 5.3.1, which follows from writing out $E(m_0)$ explicitly, and that the Yangian commutes with R in the sense that $y_{ij} R_{ij} = R_{ij} \circ \text{swap} \circ s_{ji} \circ \text{swap}$), hence the action of R on $H(w) \otimes H(w')$ preserve the grading by $v_i + v'_i$, which then implies that $[g_w \otimes g_{w'}, R_{ww'}] = 0$ as desired. Following Section 5.4.1 this example gives the algebra of classical multiplication by chern characters of tautological bundles (or at least it contains it). The second example is given by $g =$

$\exp(v_i)$ and yields the family of commuting Baxter subalgebras parametrized by $\mathcal{H} :=$ the torus with Lie algebra $\mathfrak{h} = \text{Span}(v_i)$. By Conjecture 1 this family of Baxter subalgebras should precisely correspond to the algebra of quantum multiplication by chern characters under the natural identification (or map?) between \mathcal{H} and $H^2(X, \mathbb{C})/2\pi i H^2(X, \mathbb{Z})$. (Of course, we need to work formally here as explained in Section 6.5). Finally we see in the particular case of the Grassmannian that under a certain limit the quantum algebra degenerates to the classical algebra (essentially, this is the calculation $\lim_{t \rightarrow -\infty} \begin{pmatrix} 1 & \\ & et \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$ on $\mathbb{C}^2 = H^*(\text{Gr}(0, 1) \sqcup \text{Gr}(1, 1)) = H(2)$).

Jan. 21: No seminar.

Jan. 28: A tale of two Yangians (Sjoerd Beentjes)

Comparison of the Maulik-Okounkov Yangian with the Yangian obtained from the Cohomological Hall Algebra (CoHA).

Further topics:

There are several direction one can go beyond [10]. For once, one can consider quantum cohomology of symplectic varieties which are not Nakajima quiver varieties. Basic examples are hyper-toric varieties [17, 18] or Springer resolutions [3]. Very interesting are also non-toric cases such as the Hilbert scheme of points of the cotangent bundle of a curve, moduli of Higgs bundles, or (compact) irreducible holomorphic-symplectic varieties. An another generalization is to consider K-theoretic quantum cohomology, see [16] for an introduction. Finally, the higher genus Gromov-Witten theory of $\text{Hilb}^n \mathbb{C}^2$ has recently been determined by Pandharipande and Tzeng [19].

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