**Noether-Lefschetz theory of hyper-Kähler varieties via Gromov-Witten invariants**

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## 1. Hassett-Looijenga-Shah divisors

Let $\mathcal{M}_2$ be the moduli space of quasipolarized K3 surfaces of genus 2. It is birational to the GIT quotient $\mathbb{P}(H^0(\mathbb{P}^2, \mathcal{O}(6)))//\mathrm{SL}_6$ parametrizing double covers of $\mathbb{P}^2$ branched along a sextic curve. Hwoever, there are differences in moduli. For once, the Noether-Lefschetz divisor in $\mathcal{M}_2$ of elliptic K3 surfaces with section,

$$\mathsf{NL}_{\begin{pmatrix} 2 & 1 \\ 1 & 0 \end{pmatrix}} \in \mathrm{Div}(\mathcal{M}_2),$$

can *not* correspond to any divisor in the GIT moduli space.[1] The divisor $\mathsf{NL}_{\begin{pmatrix} 2 & 1 \\ 1 & 0 \end{pmatrix}}$ is the simplest example of an *Hassett-Looijenga-Shah (HLS) divisor*.

Let $V_{10}$ be a 10-dimensional vector space and $\sigma \in \wedge^3 V_{10}^*$ a non-zero tri-linear form. The Debarre-Voisin variety [4] associated to $\sigma$ is defined by

$$X_\sigma = \{ V \in \mathrm{Gr}(6,10) \,|\, \sigma|_V = 0 \}$$

and is hyperkähler if it is smooth of dimension 4. The period map gives rise to a birational [7] morphism from the GIT moduli space to the moduli space of hyperkähler fourfolds of $K3^{[2]}$-type of degree 22 and divisibility 2:

$$\mathcal{P} \colon \mathbb{P}(\wedge^3 V_{10}^*)//\mathrm{SL}_{10} \dashrightarrow \mathcal{M}_{22}^{(2)}$$

The rational map $\mathcal{P}$ restricts to a regular morphism $\mathcal{P} : U \to \mathcal{M}_{22}^{(2)}$ on the open subset parametrizing 4-dimensional Debarre-Voisin varieties which are smooth or at most nodal along a smooth K3 surface. The complement of $U$ has codimension $\geq 2$, see [4, 1, 6].

**Definition 1.** *An irreducible divisor $D \subset \mathcal{M}_{22}^{(2)}$ is HLS if $\mathcal{P}^*(D) = 0$ in $\mathrm{Div}(U)$.*

**Remark 1.** *This definition was first given in* [2] *although in slightly different but equivalent form (see* [6, Sec.6.2] *why they are equivalent).*

Following [3], let $\mathcal{C}_{2e} \in \mathrm{Div}(\mathcal{M}_{22}^{(2)})$ be the Noether-Lefschetz divisor of first kind of discriminant $e$. We have that $e$ is a square mod 11.

**Theorem 1** ([2, 6])**.** *$\mathcal{C}_2, \mathcal{C}_6, \mathcal{C}_8, \mathcal{C}_{10}, \mathcal{C}_{18}$ are HLS divisors on $\mathcal{M}_{22}^{(2)}$.*

**Remark 2.** *The HLS-ness of $\mathcal{C}_2, \mathcal{C}_6, \mathcal{C}_{10}, \mathcal{C}_{18}$ was first proven geometrically in* [2]*. An independent proof of these cases plus an extension to $\mathcal{C}_8$ was later given in* [6] *using Gromov-Witten theory (the intersection theory on the moduli space of stable maps).*

[1] If such K3 were realized as double cover of $\mathbb{P}^2$, the algebraic class of square zero and degree 1 against the quasi-polarization would give rise to an elliptic curve mapping with degree 1 to $\mathbb{P}^2$, which is absurd.

## 2. Proof

The approach of [6] to the theorem is as follows: Let $\widetilde{U} \subset \mathbb{P}(\wedge^3 V_{10}^*)$ be the preimage of $U$ under the quotient map. The complement of $\widetilde{U}$ is of codimension 2. Since $\widetilde{U}$ is contained in the stable locus, the quotient map $\pi : \widetilde{U} \to U$ is a $\mathrm{SL}_{10}$-torsor. Let $\ell \subset \widetilde{U}$ be a fixed line, and $\iota_\ell \to \mathcal{M}_{22}^{(2)}$ the classifying map of the associated pencil of Debarre-Voisin varieties. Then we have:

$$\begin{aligned} \mathcal{P}^*(\mathcal{C}_{2e}) = 0 \quad &\Longleftrightarrow \quad \pi^*\mathcal{P}^*(\mathcal{C}_{2e}) = 0 \\ &\Longleftrightarrow \quad (\pi^*\mathcal{P}^*\mathcal{C}_{2e}) \cdot \ell = 0 \\ &\Longleftrightarrow \quad \int_\ell \iota_\ell^* \mathcal{C}_{2e} = 0. \end{aligned}$$

The numbers $\int_\ell \iota_\ell^* \mathcal{C}_{2e}$ are the Noether-Lefschetz numbers of first kind of the pencil $\ell$. They are related by an explicit invertible upper-triangular relation to the Noether-Lefschetz numbers of second kind $\mathsf{NL}^\ell(e)$ [6, Prop.5]. By a result of Borcherds the generating series of Noether-Lefschetz numbers of second kind

$$\varphi(q) = \sum_e \mathsf{NL}^\ell(e) q^{e/11}$$

is a modular form of weight 11 for a specific congruence subgroup. In particular, it only depends on finitely many data. In fact, when working with the corresponding vector-valued modular form, one finds that 6 Fourier coefficients are enough to fix the modular form completely. These coefficients are obtained by from the following mostly formal input:

(i) the Gromov-Witten/Noether-Lefschetz relation [5],
(ii) known cases of a new multiple-cover conjecture for $K3^{[n]}$-type hyperkähler varieties [6] (the original motivation for this work),
(iii) the abelian/non-abelian correspondence [8] (which allows to compute genus 0 Gromov-Witten invariants for zero loci of homogeneous vector bundles in GIT quotients by passing to the abelian quotient).

This yields a closed evaluation of the Noether-Lefschetz series which we state next.

## 3. Generating series

Define the weight 1, 2, 3 modular forms

$$E_1(\tau) = 1 + 2\sum_{n\geq 1} q^n \sum_{d|n} \chi_p\left(\frac{n}{d}\right), \qquad \Delta_{11}(\tau) = \eta(\tau)^2\eta(11\tau)^2$$
$$E_3(\tau) = \sum_{n\geq 1} q^n \sum_{d|n} d^2 \chi_p\left(\frac{n}{d}\right)$$

where $\eta(\tau) = q^{1/24}\prod_{n\geq 1}(1-q^n)$ is the Dirichlet eta function, $q = e^{2\pi i \tau}$ and $\chi_{11}$ is the Dirichlet character given by the Legendre symbol $\left(\frac{\cdot}{11}\right)$. Consider the following weight 11 modular forms for $\Gamma_0(11)$ and character $\chi_{11}$:

$$\begin{aligned}\varphi_0(q) = {} & -5\,E_1^{11} + 430\,E_1^8 E_3 + \frac{5199920}{9}\,\Delta_{11}^3 E_1^5 - \frac{35407490}{27}\,\Delta_{11}^4 E_1^3 \\ & + \frac{49194440}{9}\,\Delta_{11}^2 E_1^4 E_3 + 248350\,E_1^5 E_3^2 - \frac{596661440}{27}\,\Delta_{11}^3 E_1^2 E_3 \\ & - \frac{306631760}{9}\,\Delta_{11} E_1^3 E_3^2 + \frac{51243500}{3}\,\Delta_{11}^4 E_3 + \frac{1331452540}{27}\,\Delta_{11}^2 E_1 E_3^2 \\ & + \frac{349019440}{9}\,E_1^2 E_3^3 \\ \varphi_1(q) = {} & -5\,E_1^{11} + 110\,E_1^8 E_3 + \frac{722740}{3993}\,\Delta_{11}^3 E_1^5 - \frac{1805750}{3993}\,\Delta_{11}^4 E_1^3 \\ & - \frac{12660620}{11979}\Delta_{11}^2 E_1^4 E_3 - 990 E_1^5 E_3^2 + \frac{118940}{363}\Delta_{11}^5 E_1 + \frac{5609180}{3993}\Delta_{11}^3 E_1^2 E_3 \\ & + \frac{29208460}{11979}\,\Delta_{11} E_1^3 E_3^2 + \frac{3500}{33}\,\Delta_{11}^4 E_3 + \frac{2610980}{1089}\,E_1^2 E_3^3\end{aligned}$$

**Theorem 2.** *The generating series of Noether-Lefschetz numbers of a generic pencil of Debarre-Voisin varieties is:*

$$\begin{aligned}\varphi(q^{11}) = \varphi_0(q^{11}) + \varphi_1(q) = -10 + 640q^{11} + 990q^{12} + 5500q^{14} + 11440q^{15} \\ +21450q^{16} + 198770q^{20} + 510840q^{22} + \dots\end{aligned}$$

The vanishing of the coefficients $q^1, q^3, q^4, q^5, q^9$ implies that the corresponding divisors are HLS. The number 640 is the number of singular fibers of the pencil. The numbers 990 and 5500 have been independently computed by J. Song, see [1] for a study of the geometry of the associated divisors.

## References


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