CURVES ON THE HILBERT SCHEME OF A K3 SURFACE

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1. An enumerative problem

Let S be a smooth complex projective surface which we assume here for simplicity to be Fano (in particular, $p_g = q = 0$). Let L be a line bundle with no higher cohomology. We are interested in counting curves in the linear system $|L|$ of given geometric genus and *gonality*.

Definition 1. A smooth proper connected curve C is n-gonal if there exists a morphism $C \to \mathbb{P}^1$ of degree n.

Definition 2. Let $N_{g,n}(L)$ be the number of irreducible curves $C \in |L|$ such that:

(i) the normalization $\widetilde{C}$ is n-gonal of genus g

(ii) C passes through $\ell(g,n)$ generic points.

Here $\ell(g,n)$ is the number of points which makes the problem of expected dimension 0. To find it recall first that because the Brill-Noether number reads $\rho(g,a,d) = g-(a+1)(g-d+a)$, in a given family of genus g curves the loci of n-gonal curves has expected codimension $-\rho(g,1,n) = g+2-2n$. Second, the locus of geometric genus g curves in a family arithmetic genus p_a curves is of expected codimension $p_a - g$. Let $p_a(L)$ be the arithmetic genus of a curve in $|L|$. Hence

$$\begin{aligned}\ell = \ell(g,n) &= \dim |L| - (p_a(L) - g) + \rho(g,1,n) \\ &= \frac{1}{2} L\cdot (L-K) - \left(\frac{1}{2} L \cdot (K+L) + 1 - g \right) - (g+2-2n) \\ &= c_1(S)\cdot L - 1 + 2n - 2.\end{aligned}$$

2. Hilbert schemes

By a classical idea of Graber, the Hilbert scheme of n points $S^{[n]}$ can be used to approach the count $N_{g,n}(L)$. By definition a morphism $T \to S^{[n]}$ from a Noetherian scheme T corresponds to a closed subscheme $C \subset T \times S$ flat over T of degree n. Hence we find the natural bijection:

$$\left\{ \text{maps } f : \mathbb{P}^1 \to S^{[n]} \right\} \xrightarrow{\cong} \left\{ \begin{array}{l} \text{subcurves } C \subset \mathbb{P}^1 \times S \\ \text{flat over } \mathbb{P}^1 \text{ of degree } n \end{array} \right\}.$$

Moreover, as explained in [3, Sec. 1] the map f has class $\beta + kA$ under the natural isomorphism $H_2(S^{[n]}, \mathbb{Z}) \cong H_2(S,\mathbb{Z}) \oplus \mathbb{Z}A$ if and only if we have $[C] = \beta + n[\mathbb{P}^1] \in H_2(S \times \mathbb{P}^1, \mathbb{Z})$ and $\chi(\mathcal{O}_C) = k+n$. Similarly, the projection of C to S is incident to a point $P \in S$ if and only if $f(\mathbb{P}^1)$ is incident to the cycle $I(P) = \{ \xi \in S^{[n]} | P \in \xi \}$.

Define the genus g Gromov-Witten invariant of the Hilbert scheme:

$$\big\langle \alpha; \gamma_1, \ldots, \gamma_N \big\rangle^{S^{[n]}}_{g,\beta+kA} := \int_{[\overline{M}_{g,N}(S^{[n]}, \beta+kA)]^{\text{vir}}} \text{ev}_1^*(\gamma_1) \cdots \text{ev}_N^*(\gamma_N) \tau^*(\alpha)$$

where α is a tautological class on $\overline{M}_{g,N}$, which is the target of the forgetful morphism τ. A virtual count $H_{g,n}(\beta)$ of n-gonal genus g curves on S in class β passing through ℓ points is then defined by

$$\sum_{k\in\mathbb{Z}} \big\langle I(P)^{\ell} \big\rangle^{S^{[n]}}_{0,\beta+kA} p^k = \sum_g H_{g,n}(\beta)(p^{-1/2}+p^{1/2})^{2n+2g-2}.$$

The justification for this is that for an isolated genus g curves $C \subset S \times \mathbb{P}^1$, the corresponding map $f : \mathbb{P}^1 \to S^{[n]}$ meets the diagonal $\Delta_{S^{[n]}}$ in $2n+2g-2$ points, and by Graber each of these intersection points should contribute $p^{-1/2}+p^{1/2}$ to the left hand side. In particular Graber proves:

Theorem 1 ([1])**.** *For $S = \mathbb{P}^2$ the count $H_{g,2}(\beta)$ is enumerative, or in other words equal to $N_{g,2}(\beta)$. For an explicit recursion see* [1].

3. K3 surfaces

The above discussion motivates the study of the Gromov-Witten theory of the Hilbert scheme of points of a K3 surface. We state a triality of conjectures which governs the structure of the theory. Let $S \to \mathbb{P}^1$ be an elliptic K3 surface with section B and fiber class F. We define potential of reduced Gromov-Witten invariants:

$$F_{g,m}(\alpha;\gamma_1,\ldots,\gamma_N) = \sum_{d=-m}^{\infty} \sum_{r\in\mathbb{Z}} \langle \alpha;\gamma_1,\ldots,\gamma_N \rangle^{S^{[n]}}_{g,m(B+F)+dF+kA} q^d (-p)^k.$$

By deformation invariance these series determine all Gromov-Witten invariants of hyper-Kähler varieties of $K3^{[n]}$-type [4]. By convention, we assume $k=0$ for $n=1$. Recall the algebra QJac of quasi-Jacobi forms [3].

Conjecture A. *$F_{g,m}(\alpha;\gamma_1,\ldots,\gamma_N)$ is a quasi-Jacobi form of index $n-1$ and weight $n(2g-2)+\sum_i \underline{\deg}(\gamma_i)-10$ of the form*

$$F_{g,m}(\alpha;\gamma_1,\ldots,\gamma_N) \in \frac{1}{\Delta(q)}\mathsf{QJac}.$$

Here, if $\gamma \in H^*(S^{[n]})$ is written in terms of the action of Nakajima operators

$$\gamma = \prod_i \mathfrak{q}_{a_i}(\delta_i)1, \quad 1 \in H^*(S^{[0]})$$

where δ_i are elements of a fixed basis $\{W := B+F, F, \mathsf{p}, 1, e_3, \ldots, e_{22}\}$ with $e_i \in H^2(S)$ orthogonal to W, F, then the modified degree function $\underline{\deg}$ is defined by

$$\underline{\deg}(\gamma) = \deg(\gamma) + w(\gamma) - f(\gamma)$$

where $w(\gamma)$ and $f(\gamma)$ are the number of classes δ_i equal to W and F respectively.

Conjecture B. *We have the multiple cover conjecture:*

$$F_{g,m}(\alpha;\gamma_1,\ldots,\gamma_N) = m^{\sum_i \deg(\gamma_i) - \underline{\deg}(\gamma_i)} \cdot T_{m,\ell} F_{g,1}(\alpha;\gamma_1,\ldots,\gamma_N)$$

where $\ell = n(2g-2)+\sum_i \underline{\deg}(\gamma_i)$ and $T_{m,\ell}$ is the formal Hecke operator on Jacobi forms, see [4, 2.6].

Conjecture B implies that every $F_{g,m}$ is a quasi-Jacobi form (with poles at $q=0$) of index $m(n-1)$ for the congruence subgroup $\Gamma_0(n) \rtimes \mathbb{Z}^2$. The weight is as before.

Conjecture C. *We have the holomorphic anomaly equation:*

$$\begin{aligned}\frac{d}{dG_2}F_{g,m}(\alpha;\gamma_1,\ldots,\gamma_N) =& F_{g-1,m}(\alpha;\gamma_1,\ldots,\gamma_N,U)\\ &+2\sum_{\substack{g=g_1+g_2\\ \{1,\ldots,N\}=A\sqcup B}} F_{g_1,m}(\alpha_1;\gamma_A,U_1)F^{vir}_{g_2}(\alpha_2;\gamma_B,U_2)\\ &-2\sum_{i=1}^{N}F_{g,m}(\alpha\cdot q^*(\psi_i);\gamma_1,\ldots,\gamma_{i-1},U\gamma_i,\gamma_{i+1},\ldots,\gamma_N)\\ &-\frac{1}{m}\sum_{a,b}(G^{-1})_{ab}T_{e_a}T_{e_b}F_{g,m}(\alpha;\gamma_1,\ldots,\gamma_N)\end{aligned}$$

with the following notations:

- *by convention the last term vanishes in case* $m=0$,
- *the intersection matrix* G *of the* e_a *is defined by* $G_{ab}=\langle e_a,e_b\rangle$,
- *we let* $\rho:\wedge^2H^2(X)\cong\mathfrak{so}\left(H^2(X)\right)\to\mathrm{End}H^*(X)$ *be the Looijenga-Lunts-Verbitsky algebra action for* $X=S^{[n]}$ *with the conventions of* [2],
- $U=\tilde{f}_F=\rho(-f\wedge F)$,
- $T_\lambda F_{g,m}(\alpha;\gamma_1,\ldots,\gamma_N)=\sum_{i=1}^N F_{g,m}(\alpha;\gamma_1,\ldots,\gamma_{i-1},\rho(\lambda\wedge F)\gamma_i,\gamma_{i+1},\ldots,\gamma_N)$,
- $q:\overline{M}_{g,N}(S^{[n]},\beta)\to\overline{M}_{g,N}(\mathbb{P}^n,\pi_*\beta)$ *is induced by the Lagrangian fibration* $\pi:S^{[n]}\to\mathbb{P}^n$ *associated to* $S\to\mathbb{P}^1$,
- F_g^{vir} *is the potential of ordinary (non-reduced) Gromov-Witten invariants.*

The first two conjectures can be found in [3] and [4]. The last one generalizes the K3 surface case [5]. Example calculations will be discussed elsewhere.

References

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