NOTES ON MONODROMY AND PARALLEL TRANSPORT OF HOLOMORPHIC-SYMPLECTIC VARIETIES OF $K3^{[n]}$-TYPE

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There are three natural sets of actions on the cohomology of holomorphic-symplectic varieties of $K3^{[n]}$-type:

- the action of the Looijenga-Lunts-Verbitsky (LLV) algebra,
- the action of $O(H^2(X,\mathbb{C}))$ via Markman's operator, and
- the action of the monodromy group.

In this note we describe each of these actions and relate them to each other. We also discuss the case of parallel transport operators.

0.1. **LLV algebra.** We begin our discussion with the Looijenga-Lunts-Verbitsky (LLV) Lie algebra [2, 10], for which we can work with an arbitrary holomorphic symplectic variety X. We let $\dim_{\mathbb{C}}(X) = 2n$.

For any $a \in H^2(X,\mathbb{Q})$ such that $(a,a) \neq 0$, consider the operator of multiplication by a,

$$e_a : H^*(X,\mathbb{Q}) \to H^*(X,\mathbb{Q}), x \mapsto a \cup x$$

Let also h be the Lefschetz grading operator acting on $H^{2i}(X,\mathbb{Z})$ by $i-n$. Then there exists a unique operator $f_a : H^*(X,\mathbb{Z}) \to H^*(X,\mathbb{Z})$ such that the $\mathfrak{sl}_2$ commutation relations are satisfied:

$$[e_a, f_a] = h, \quad [h, e_a] = e_a, \quad [h, f_a] = -f_a.$$

The LLV Lie algebra $\mathfrak{g}(X)$ is defined as the Lie subalgebra of $\operatorname{End} H^*(X,\mathbb{Q})$ generated by e_a, f_a, h for all $a \in H^2(X,\mathbb{Q})$ as above. One has

$$\mathfrak{g}(X) = \mathfrak{so}(H^2(X,\mathbb{Q}) \oplus U_{\mathbb{Q}})$$

where $U = \binom{0\ 1}{1\ 0}$ is the hyperbolic plane.

The degree zero part of $\mathfrak{g}(X)$ decomposes as

$$\mathfrak{g}(X)_0 = \mathfrak{so}(H^2(X,\mathbb{Q})) \oplus \mathbb{Q}h.$$

The summand $\mathfrak{so}(H^2(X,\mathbb{Q}))$ is also called the reduced LLV algebra. Base changing to $\mathbb{C}$ and integrating then yields the LLV representation

$$\rho_{\mathrm{LLV}} : \mathrm{SO}(H^2(X,\mathbb{C})) \to \mathrm{GL}(H^*(X,\mathbb{C})).$$

which acts by degree-preserving orthogonal ring isomorphisms [2, Prop. 4.4(ii)], where we endow $H^*(X,\mathbb{C})$ with the Poincaré pairing. In the case of the Hilbert scheme of a K3 surfaces, explicit formulas for the action of the Lie algebras in the Nakajima basis are given in [9].

Consider a parallel transport operator $P : H^*(X_1, \mathbb{Z}) \to H^*(X_2, \mathbb{Z})$ induced by a deformation from a holomorphic-symplectic variety X_1 to X_2. Since P is a ring isomorphism, for $a \in H^2(X_1, \mathbb{Q})$ we have that

$$P \circ e_a \circ P^{-1} = e_{P(a)}.$$

Since clearly $P \circ h \circ P^{-1} = h$ we also find $P \circ f_a \circ P^{-1} = f_{Pa}$ whenever $(a, a) \neq 0$. Under the identification of elements in $\mathfrak{so}(H^2(X, \mathbb{Q}))$ with endomorphisms of $H^2(X, \mathbb{Q})$ we have $Pa \wedge Pb = P \circ (a \wedge b) \circ P^{-1}$. Hence for any $g \in \mathrm{SO}(H^2(X, \mathbb{C}))$ we find that

$$P \circ \rho_{\mathrm{LLV}}(g) \circ P^{-1} = \rho_{\mathrm{LLV}}(\psi \circ g \circ \psi^{-1})$$

where $\psi = P|_{H^2(X_1, \mathbb{Z})}$, that is the LLV algebra action is intertwined by parallel transport.

0.2. **Basic definitions.** Let S be a K3 surface and consider the lattice $\Lambda = H^*(S, \mathbb{Z})$ endowed with the Mukai pairing

$$(x \cdot y) := -\int_S x^\vee y,$$

where, if we decompose an element $x \in \Lambda$ according to degree as (r, D, n), we have written $x^\vee = (r, -D, n)$. Given a sheaf or complex E on S the Mukai vector of E is defined by

$$v(E) = \sqrt{\mathrm{td}_S} \cdot \mathrm{ch}(E) \in \Lambda.$$

Let $v \in \Lambda$ be an effective vector, H be an ample divisor on S and let $M_H(v)$ be a proper smooth moduli space of H-stable sheaves with Mukai vector v.[1] For simplicity we assume that there exists an universal sheaf $\mathcal{F}$ on $M_H(v) \times S$. By definition of the moduli problem, $\mathcal{F}$ is unique only up to tensoring of a line bundle from the base.

The results we state below also hold in the general case where there exists only a universal twisted sheaf. By this we mean that all statements below can be formulated in terms of the Chern character $\mathrm{ch}(\mathcal{F})$ alone and this class can be defined in the twisted case as well, see [3, Sec.3]. The proofs carry over likewise using that the ingredients hold in the twisted case as well.

Consider the morphism $\theta_{\mathcal{F}} : \Lambda \to H^2(M, \mathbb{Z})$ defined by

$$\theta_{\mathcal{F}}(x) = \left[\pi_* \left(\mathrm{ch}(\mathcal{F})\sqrt{\mathrm{td}_S} \cdot x^\vee\right)\right]_{\deg=1}. \tag{1}$$

Then $\theta_{\mathcal{F}}$ restricts to an isomorphism

$$\theta = \theta_{\mathcal{F}}|_{v^\perp} : v^\perp \xrightarrow{\cong} H^2(M, \mathbb{Z}) \tag{2}$$

[1] More generally, we can also work with σ-stable objects for a Bridgeland stability condition in the distinguished component.

which does not depend on the choice of universal family (use that the degree 0 component of the pushforward (1) vanishes) and for which we hence have dropped the subscript $\mathcal{F}$. The isomorphism θ preserves the Mukai pairing on the left, and the pairing given by the Beauville-Bogomolov-Fujiki form on the right. We will identify $v^\perp \subset \Lambda$ with $H^2(M_H(v), \mathbb{Z})$ under this isomorphism.

The universal sheaf $\mathcal{F}$ and hence its Chern character $\mathrm{ch}(\mathcal{F})$ is unique only up to pullback of a line bundle from the base. We can pick a canonical normalization as follows:

$$u_v := \exp\left(\frac{\theta_{\mathcal{F}}(v)}{(v,v)}\right) \cdot \mathrm{ch}(\mathcal{F}) \cdot \sqrt{\mathrm{td}_S}$$

where we have suppressed the pullback morphisms from M and S in the first and last term on the right. The invariance is a short check (replace $\mathcal{F}$ by $\mathcal{F} \otimes \mathcal{L}$ and calculate). The class u_v is characterized among the classes $\mathrm{ch}(\mathcal{F}) \cdot \sqrt{\mathrm{td}_S}$ by the property that $\theta_{u_v}(v) = 0$ (Use that $\pi_*(\mathrm{ch}(\mathcal{F})\sqrt{\mathrm{td}_S} \cdot v^\vee) = -(v \cdot v) + \theta_{\mathcal{F}}(v) + \ldots$ for a universal family $\mathcal{F}$.).

Example 1. Let $M = \mathsf{Hilb}_n(S)$ the Hilbert scheme of n points on S. We have $v = 1 - (n-1)c$, and take $\mathcal{F} = I_{\mathcal{Z}}$ the ideal sheaf of the universal subscheme. For $\alpha \in H^2(S)$ we have

$$\theta(\alpha) = \pi_*(\mathrm{ch}_2(\mathcal{O}_Z)\pi_S^*(\alpha)).$$

If α is the class of a divisor A, then this is the class of subschemes incident to A. Similarly, define[2]

$$\delta = -\frac{1}{2}\Delta_{\mathsf{Hilb}_n(S)} = c_1(\pi_*\mathcal{O}_Z) = \pi_*\mathrm{ch}_3(\mathcal{O}_Z).$$

Then $\theta\big(-(1 + (n-1)c)\big) = \delta$, that is under the identification (2) we have

$$\delta = -\big(1 + (n-1)c\big).$$

Since $\theta_{\mathcal{F}}(v) = -\delta$ the canonical normalization of $\mathrm{ch}(\mathcal{F})$ takes the form

$$u_v = \exp\left(\frac{-\delta}{2n-2}\right)\mathrm{ch}(\mathcal{F})\sqrt{\mathrm{td}_S}.$$

0.3. **Markman's operator.** For $i = 1, 2$ let (S_i, H_i, v_i) be the data defining the moduli space $M_i = M_{H_i}(S_i, v_i)$, and let $\mathcal{F}_i$ be the universal family on $M_i \times S$. Let

$$g : H^*(S_1, \mathbb{Z}) \to H^*(S_2, \mathbb{Z})$$

be an isomorphism of Mukai lattice such that $g(v_1) = v_2$. We will identify g also with an isomorphism of topological K-groups

$$g : K_{\mathrm{top}}(S_1) \to K_{\mathrm{top}}(S_2)$$

[2] Beware, in some circles it is common to define $\delta = \frac{1}{2}\Delta_{\mathsf{Hilb}}$.

using the lattice isomorphism $K_{\mathrm{top}}(S) \xrightarrow{\cong} H^*(S,\mathbb{Z})$ given by $E \mapsto v(E)$. Here $K_{\mathrm{top}}(S)$ carries the Euler pairing $(E \cdot F) = -\chi(E^\vee \otimes F)$. Hence the following diagram commutes

$$\begin{array}{ccc} K_{\mathrm{top}}(S_1) & \xrightarrow{g} & K_{\mathrm{top}}(S_2) \\ \downarrow v & & \downarrow v \\ H^*(S_1,\mathbb{Z}) & \xrightarrow{g} & H^*(S_2,\mathbb{Z}). \end{array}$$

Similar identification will apply to morphisms g defined over $\mathbb{C}$. The Markman operator associated to g is given by the following result:

Theorem 1. *(Markman, [4]) For any isometry $g : H^*(S_1,\mathbb{C}) \to H^*(S_2,\mathbb{C})$ such that $g(v_1) = v_2$ there exists a unique operator*

$$\gamma(g) : H^*(M_1,\mathbb{C}) \to H^*(M_2,\mathbb{C})$$

such that

(a) $\gamma(g)$ is degree-preserving orthogonal ring-isomorphism
(b) $\gamma(g) \otimes g(u_{v_1}) = u_{v_2}$.

The operator is called the Markman operator *and given by*

$$\gamma(g) = c_m \left[-\pi_* \Big(((1 \otimes g) u_{v_1})^\vee \cdot u_{v_2} \Big) \right]. \tag{3}$$

Moreover, we have

(c) $\gamma(g_1) \circ \gamma(g_2) = \gamma(g_1 g_2)$ and $\gamma(g)^{-1} = \gamma(g^{-1})$ if it makes sense.
(d) $\gamma(g) c_k(T_{M_1}) = c_k(T_{M_2})$.

We refer to [4] for further properties of the operator $\gamma(g)$. For example, it is an Hodge isometry whenever g is.

We should explain the notation for the Chern class c_m in (3). Let $\ell : \oplus_i H^{2i}(M,\mathbb{Q}) \to \oplus_i H^{2i}(M,\mathbb{Q})$ be the universal map that takes the exponential Chern character to Chern classes, so in particular $c(E) = \ell(\mathrm{ch}(E))$ for any vector bundle. Then given $\alpha \in H^*(M)$ we write $c_m(\alpha)$ for $[\ell(\alpha)]_{\mathrm{deg}=m}$.

We want to explain a bit what goes into the proof of the result. The main ingredient of the discussion is the following uniqueness statement:

Lemma 1. *Let $f : H^*(M_1,\mathbb{Q}) \to H^*(M_2,\mathbb{Q})$ be a morphism such that:*

(i) f is a degree-preserving orthogonal ring isomorphism.
(ii) There exists universal families $\mathcal{F}$ on $M_1 \times S_1$ and $\mathcal{F}'$ on $M_2 \times S_2$ such that

$$(f \otimes g)\left(\mathrm{ch}(\mathcal{F})\sqrt{\mathrm{td}_S}\right) = \mathrm{ch}(\mathcal{F}')\sqrt{\mathrm{td}_{S_2}} \cdot \exp(\ell)$$

for some $\ell \in H^2(M_2,\mathbb{Q})$.

Then we have

$$f = c_m\left(-\operatorname{Ext}_\pi((1\otimes g)\mathcal{F},\mathcal{F}')\right). \tag{4}$$

Moreover, in (ii) it is enough to assume that $\mathcal{F}$, $\mathcal{F}'$ are elements in $K_{\mathrm{top}}(M_i\times S)_{\mathbb{Q}}$, i.e. differ from a universal family by tensor product by a fractional line bundle from the base (see the proof). In particular, we have

$$f = c_m\left[-\pi_*\Big(((1\otimes g)u_{v_1})^\vee\cdot u_{v_2}\Big)\right]. \tag{5}$$

Proof of Lemma 1. Assume that f satisfies (i) and (ii). Note that, since f is a ring isomorphism, the equality in (ii) is equivalent to the parallel equality where we replace $\mathrm{ch}(F)$ by $\mathrm{ch}(F)\exp(\mu)$ for any $\mu\in H^2(M_1,\mathbb{Q})$, and similarly for $\mathrm{ch}(F')$. Hence we may have also assumed (i) with $\mathrm{ch}(\mathcal{F})$ replaced by $\mathrm{ch}(F)\exp(\mu)$ instead.

We will prove that for any $\ell_i\in H^2(M_i,\mathbb{Q})$ we have:

(6)

$$f = c_m\left[-\pi_*\Big(((1\otimes g)(\exp(\ell_1)\mathrm{ch}(\mathcal{F})\sqrt{\mathrm{td}_S}))^\vee\cdot(\exp(\ell_2)\mathrm{ch}(\mathcal{F}')\sqrt{\mathrm{td}_S})\Big)\right].$$

Taking ℓ_i both to be trivial then gives (4), and taking ℓ_i to be as in the definition of u_v gives (5).

The main input of the lemma is the following theorem which we state for an arbitrary moduli space of stable sheaves M on a K3 surface:

Theorem 2 (Markman [3])**.** *For any universal families $\mathcal{F},\mathcal{F}'$ on $M\times S$,*

$$\Delta_M = c_m(-\operatorname{Ext}^\bullet_\pi(\mathcal{F},\mathcal{F}')).$$

More generally, for any $\gamma,\gamma'\in H^2(M,\mathbb{Q})$ we have

$$\Delta_M = c_m\left[-\pi_*\Big((\exp(\gamma)\mathrm{ch}(\mathcal{F})\sqrt{\mathrm{td}_S})^\vee\cdot\exp(\gamma')\mathrm{ch}(\mathcal{F})\sqrt{\mathrm{td}_S}\Big)\right]$$

By the theorem, for any $\gamma\in H^2(M_2,\mathbb{Q})$ we thus have:

$$\Delta_{M_2} = c_m\left[-\pi_*\left((\mathrm{ch}(\mathcal{F}')\sqrt{\mathrm{td}_S}\exp(\gamma))^\vee\cdot\mathrm{ch}(\mathcal{F}')\sqrt{\mathrm{td}_S}\exp(\ell_2)\right)\right]$$

Inserting

$$\mathrm{ch}(\mathcal{F}')\sqrt{\mathrm{td}_S} = (f\otimes g)\left(\exp(f^{-1}(\ell))\mathrm{ch}(\mathcal{F})\sqrt{\mathrm{td}_{S_1}}\right)$$

iin the first term, then using that f is degree-preserving (so commutes with $\vee$), and a ring isomorphism (so commutes with c_m), we get that Δ_{M_2} is equal to

$$(f\otimes 1)c_m\left[-\pi_*\left(((1\otimes g)(\mathrm{ch}(\mathcal{F})\sqrt{\mathrm{td}_S}\exp(\gamma+f^{-1}(\ell))))^\vee\mathrm{ch}(\mathcal{F}')\sqrt{\mathrm{td}}\exp(\ell_2)\right)\right]$$

Setting $\gamma=-f^{-1}(\ell)+\ell_1$, and taking Q to be the right hand side of (6) we find

$$\mathrm{id}_{H^*(M_2)} = \Delta_{M_2} = (f\otimes 1)(Q) = Q\circ f^t$$

where f is the transpose with respect to the standard cup product (or as a correspondence, identical with f up to swapping the factors). Since f is orthogonal, we conclude $\mathrm{id}_{H^*(M_2)} = Q \circ f^{-1}$, so $f = Q$. ☐

After the uniquness, we prove two basic results on the operators satisfying the condition of the previous lemma.

Lemma 2. *Assume f satisfies (i) and (ii) of Lemma 1. Then*

$$(f \otimes g)(u_{v_1}) = u_{v_2}.$$

Proof. Assuming (i) and (ii) we have

$$(f \otimes g)(u_{v_1}) = \exp\left(\frac{f(\theta_{\mathcal{F}}(v_1))}{(v_1, v_1)}\right) \mathrm{ch}(\mathcal{F}')\sqrt{\mathrm{td}_S}\exp(\ell).$$

Hence the claim follows from the following calculation:

$$\begin{aligned} f(\theta_{\mathcal{F}}(v_1)) &= [\deg 1]\pi_*\Big((1 \otimes g^{-1})(f \otimes g)(\mathrm{ch}(\mathcal{F})\sqrt{\mathrm{td}_S}) \cdot v_1^\vee\Big) \\ &= [\deg 1]\pi_*((1 \otimes g^{-1})(\mathrm{ch}(\mathcal{F}')\sqrt{\mathrm{td}_S}) \cdot v_1^\vee)\exp(\ell) \\ &\overset{(*)}{=} [\deg 1]\pi_*(\mathrm{ch}(\mathcal{F}')\sqrt{\mathrm{td}_S} \cdot g(v_1)^\vee)\exp(\ell) \\ &= [\deg 1](-(v_2, v_2) + \theta_{\mathcal{F}'}(v_2))\exp(\ell) \\ &= -(v_2, v_2)\ell + \theta_{\mathcal{F}'}(v_2), \end{aligned}$$

where (*) follows by Künneth decomposition and that g is an isometry of Mukai lattice. Indeed, if we assume $\mathrm{ch}(\mathcal{F})\sqrt{\mathrm{td}_S} = a \otimes b$ for simplicity, then

$$\begin{aligned} \pi_*((1 \otimes g^{-1})(\mathrm{ch}(\mathcal{F}')\sqrt{\mathrm{td}_S}) \cdot v_1^\vee) &= a\int_S g^{-1}(b)v_1^\vee \\ &= -a \cdot (g^{-1}(b) \cdot v_1) \\ &= -a \cdot (b \cdot g(v_1)) \\ &= a\int_S bg(v_1)^\vee \\ &= \pi_*(\mathrm{ch}(\mathcal{F}')\sqrt{\mathrm{td}_S} \cdot g(v_1)^\vee). \end{aligned} \tag{7}$$

☐

We reinterpret the condition $(f \otimes g)(u_{v_1}) = u_{v_2}$ in terms of generators of the cohomology ring. Consider the canonical morphism $B : H^*(S, \mathbb{Q}) \to H^*(M, \mathbb{Q})$ defind by

$$B(x) = \pi_*(u_v \cdot x^\vee).$$

Lemma 3. *Let $f : H^*(M_1, \mathbb{Q}) \to H^*(M_2, \mathbb{Q})$ be a degree-preserving orthogonal ring isomorphism. Then the following are equivalent:*

(a) $(f \otimes g)(u_{v_1}) = u_{v_2}$
(b) $f(B(x)) = B(gx)$ *for all* $x \in H^*(S_1, \mathbb{Q})$.

Proof. By the same argument as in (7) we have

$$\pi_*(u_{v_2} \cdot (gx)^\vee) = \pi_*((1 \otimes g^{-1})u_{v_2} \cdot x^\vee).$$

Hence we see that:

$$\begin{aligned}
\text{(b)} &\iff \forall x \in H^*(S_1, \mathbb{Z}) : f\pi_*(u_{v_1} \cdot x^\vee) = \pi_*(u_{v_2} \cdot (gx)^\vee) \\
&\iff \forall x \in H^*(S_1, \mathbb{Z}) : \pi_*((f \otimes 1)u_{v_1} \cdot x^\vee) = \pi_*((1 \otimes g^{-1})u_{v_2} \cdot x^\vee) \\
&\iff (f \otimes 1)(u_{v_1}) = (1 \otimes g^{-1})(u_{v_2}) \\
&\iff (a).
\end{aligned}$$

□

We come now to the question of existence of the operator in Theorem 1. One may be tempted to define these operators directly using the closed formula (3) and then derive their properties from it. However, (3) is unfortunately hard to work with in practice. It is even not clear how to use it to prove $\gamma(g_1) \circ \gamma(g_2) = \gamma(g_1 g_2)$. Nevertheless, it can be used for the following: if we know the statements of the theorem for a Zariski dense subset of all operators g (e.g. the integral isometries), then we can define $\gamma(g)$ by (3) for arbitrary isometries and then conclude Theorem 1 in general using Zariski density.

Hence it remains to consider the case of integral isometries. For this one considers the set S of triples

$$((S_1, H_1, v_1), (S_2, H_2, v_2), g : H^*(S_1, \mathbb{Z}) \to H^*(S_2, \mathbb{Z})),$$

where g is a isometry such that $g(v_1) = v_2$, for which the result of the theorem holds. Since the elements for which the statements of the theorem hold are closed under composition and inverse, we can think about S as the set of arrows in a groupoid. Then elements of the groupoid can be constructed in three different ways:

- For any deformation $(S_1, v_1, H_1) \rightsquigarrow (S_2, v_2, H_2)$ which keeps v_1 and H_1 of Hodge type type, we have an associated deformation of moduli spaces $M_{H_1}(S_1, v_1) \rightsquigarrow M_{H_2}(S_2, v_2)$. The associated parallel transport operator P is a degree-preserving orthogonal ring isomorphism. Moreover, since u_v is defined in terms of the universal family which deforms along the family, the classes u_{v_t} of the individual fibers are the parallel transports of u_{v_1}. Hence if g is the parallel transport operator associated to $S_1 \rightsquigarrow S_2$, then $(P \otimes g)(u_{v_1}) = u_{v_2}$. We conclude that P satisfies the theorem.
- Assume that $\Phi : D^b(S_1) \to D^b(S_2)$ is a derived equivalence that takes H_1-stable sheaves of Mukai vector v_1 to H_2-stable sheaves of Mukai vector v_2. Then Φ induces an isomorphism of moduli spaces

$$\varphi : M_{H_1}(S_1, v_1) \to M_{H_2}(S_2, v_2)$$

such that by its definition we have

$$(\mathrm{Id} \boxtimes \Phi)(\mathcal{F}) = (\varphi \times \mathrm{id})^*(\mathcal{F}')$$

hence

$$(\varphi_* \boxtimes \Phi_*)(\mathrm{ch}(\mathcal{F})\sqrt{\mathrm{td}_S}) = \mathrm{ch}(\mathcal{F}')\sqrt{\mathrm{td}_S}.$$

Hence with $g = \Phi_*$ we have $\gamma(g) = \varphi_*$ by the main lemma.

- Assume in a slight modification, that $(\Phi E)^\vee$ is H_2-stable of Mukai vector v_2 for any H_1-stable sheaf $E \in M_1$. Then we have an induced morphism

$$\varphi : M_{H_1}(S_1, v_1) \to M_{H_2}(S_2, v_2)$$

such that by its definition we have

$$(\mathrm{Id} \boxtimes \Phi)(\mathcal{F})^\vee = (\varphi \times \mathrm{id})^*(\mathcal{F}')$$

and hence

$$(\varphi \times \mathrm{id})_*(\mathrm{Id} \boxtimes \Phi)(\mathcal{F})^\vee = \mathcal{F}'.$$

Since φ commutes with dualizing, we get

$$(\varphi_* \boxtimes \Phi)(\mathcal{F})^\vee = \mathcal{F}'$$

and hence

$$\Big((\varphi_* \boxtimes \Phi_*)(\mathrm{ch}(\mathcal{F})\sqrt{\mathrm{td}_S})\Big)^\vee = \mathrm{ch}(\mathcal{F}')\sqrt{\mathrm{td}_S}.$$

Going through the argument of the proof of Lemma 1 then shows

$$\varphi_* = D \circ \gamma(D\Phi_*)$$

where D is the operator that acts on H^{2i} by $(-1)^i$. We note that $\Phi_* v_1 = v_2^\vee$, so $g = D\Phi_*$ sends v_1 to v_2 as required. Since D is a degree-preserving orthogonal ring isomorphism, we see that $\gamma(g)$ satisfies the statements of the Theorem.

This shows the existence of Markman operators for g of these form. Markman then (roughly) shows that any integral g can be written as a composition of these three operations. This concludes the proof.

In the next section we will see that part of the existence of the Markman operator can also be seen directly from the LLV algebra action. The proof above will naturally relate the operators $\gamma(g)$ to the monodromy action.

0.4. **Monodromy representation.** Let X be a holomorphic-symplectic variety of $K3^{[n]}$-type. Let $\mathsf{Mon}(X)$ be the subgroup of $O(H^*(X,\mathbb{Z}))$ generated by all monodromy operators, and let $\mathsf{Mon}^2(X)$ be its restriction to $H^2(X,\mathbb{Z})$. By Markman and the global Torelli theorem [7, Lemma 2.1], [5], [6, Thm.1.3] we have

$$\mathsf{Mon}(X) = \mathsf{Mon}^2(X) = \widetilde{O}^+(H^2(X,\mathbb{Z})) \tag{8}$$

where $\widetilde{O}^+(H^2(X,\mathbb{Z}))$ is the subgroup of $O(H^2(X,\mathbb{Z}))$ of orientation preserving lattice automorphisms which act by ± 1 on the discriminant.

The orientation is defined as follows: the lattice $H^2(X,\mathbb{Z})$ is of signature $(3,20)$, hence the positive cone $\mathcal{C} = \{x \in H^2(M,\mathbb{R}) | \langle x, x\rangle\}$ is homotopy equivalent to the 2-sphere S^2. An orthogonal transformation of $H^2(M,\mathbb{Z})$ is called orientation preserving if it acts as the identity on $H^2(\mathcal{C},\mathbb{Z}) \cong H^2(S^2,\mathbb{Z}) = \mathbb{Z}$. For example, multiplication by (-1) is orientation reversing. (Similar definitions hold for orthogonal transformations on any lattice of signature (m,n) for $m>0$; the multiplication by (-1) then acts by $(-1)^m$ on $H^{m-1}(S^{m-1})$.) We let $O^+(H^2(M,\mathbb{Z})) \subset O(H^2(M,\mathbb{Z}))$ be the subgroup of orientation preserving transformation. Since any holomorphic-symplectic variety defines a canonical class in $H^2(\mathcal{C},\mathbb{Z})$ (determined by $\mathrm{Re}([\sigma])$, $\mathrm{Im}([\sigma])$ and a Kähler class), and any small deformation preserves this choice, any monodromy operator is orientation preserving.

We will write

$$\mu : \mathsf{Mon}(X) \to O(H^*(X,\mathbb{Z}))$$

for the monodromy representation.

We will also consider the case of parallel transport operators. An operator

$$P : H^*(X_1,\mathbb{Z}) \to H^*(X_2,\mathbb{Z})$$

is a parallel transport operator if there exists a family of holomorphic-symplectic varieties over a base with two special fibers X_1, X_2 such that P is the parallel transport along a path in the base. We write $\mathsf{PT}(X_1,X_2)$ for the subspace of parallel transport operators, and $\mathsf{PT}^2(X_1,X_2)$ for its projection to $H^2(X_1,\mathbb{Z})$. By the global Torelli Theorem ([6, Thm.1.3], [7, Lemma 2.1]) we have again that

$$\mathsf{PT}(X_1,X_2) \cong \mathsf{PT}^2(X_1,X_2)$$

where the isomorphism is by restriction. The right hand group is described as follows: For any holomorphic-symplectic variety of $K3^{[n]}$ is equipped with a natural embedding

$$\iota_X : H^2(X,\mathbb{Z}) \to \Lambda = U^4 \oplus E_8(-1)^2$$

unique up to composing with an element of $O(\Lambda)$. For $X = M_H(S,v)$ one has (up to composition) $\iota_X = \theta^{-1}$ where θ is given by (2).

Theorem 3. *(*[6, Thm.9.8]*) An isometry $\psi : H^2(X_1,\mathbb{Z}) \to H^2(X_1,\mathbb{Z})$ is a parallel transport operator if and only of if it is orientation preserving and there exists an $\eta \in O(\Lambda)$ such that*

$$\eta \circ \iota_{X_1} = \iota_{X_2} \circ \psi,$$

Orientation preserving is here defined with respect to the canonical choice of orientation. In the case $X_1 = X_2$ the theorem reduces to (8).

0.5. **Markman's operator and monodromy.** We start to relate the three operations on chomology we have defined above. The natural one to start with is to relate Markman's operator to the monodromy, since the latter played the crucial role in proving the existence of the former.

We first consider the case of monodromy. Let as before $M = M_H(S, v)$ be a fixed moduli space of stable sheaves on a K3 surface S and let $\Lambda = H^*(S, \mathbb{Z})$. Recall that we have a natural isomorphism

$$\theta : v^{\perp} \xrightarrow{\cong} H^2(M, \mathbb{Z}).$$

Consider the isomorphism

$$\alpha : O(H^2(M, \mathbb{C})) \to O(\Lambda_{\mathbb{C}})_v, \;\; \varphi \mapsto \alpha(\varphi)$$

where we define $\alpha(\varphi)$ by $\alpha(\phi)|_{v^{\perp}} = \varphi$ and $\alpha(\phi)(v) = v$. Define the Markman representation by

$$\rho : O(H^2(M, \mathbb{C})) \to O(H^*(M, \mathbb{C})), g \mapsto \gamma(\alpha(g)).$$

Recall also the monodromy representation:

$$\mu : \widetilde{O}^+(H^2(M, \mathbb{Z})) \to O(H^*(M, \mathbb{Z})).$$

Consider the character

$$\tau : \widetilde{O}^+(H^2(M, \mathbb{Z})) \to \mathbb{Z}/2\mathbb{Z} = \{0, 1\} \tag{9}$$

such that g acts on the discriminant of $H^2(X, \mathbb{Z})$ by $(-1)^{\tau(g)}$. Let also $D : H^*(X, \mathbb{Q}) \to H^*(X, \mathbb{Q})$ be the dualizing automorphism given by

$$D|_{H^{2i}(X,\mathbb{Q})} = (-1)^i \mathrm{id}_{H^{2i}(X,\mathbb{Q})}.$$

Theorem 4. *(Markman, [4]) For any $g \in \widetilde{O}^+(H^2(M, \mathbb{Z}))$ we have*

$$\mu(g) = D^{\tau(g)} \circ \rho((-1)^{\tau(g)} g)$$

We discuss a basic example that we will use for Gromov-Witten theory.

Example 2. Let $M = \mathsf{Hilb}_n(S)$ and consider $g \in \widetilde{O}^+(H^2(M, \mathbb{Z}))$ given by

$$g|_{H^2(S,\mathbb{Z})} = \mathrm{id}, \quad g(\delta) = -\delta.$$

This is orientation preserving (it fixes a slice of the positive cone) and acts by -1 on the discriminant lattice, hence is a monodromy operator.

By Theorem we obtain that the full monodromy assigned to it is:

$$\mu(g) = D \circ \rho(-g) = D \circ \gamma(\tilde{g})$$

where we let

$$\tilde{g} = \mathrm{id}_{H^0 \oplus H^4} \oplus -\mathrm{id}_{H^2(S,\mathbb{Z})}$$

be -1 times the unique lift of g to $H^*(S, \mathbb{Z})$.

We can give a more concrete description. Since $\tilde{g}$ acts trivially on $H^0 \oplus H^4$, the operator $\rho(\tilde{g})$ intertwines the Nakajima operators $\mathfrak{q}_i(\alpha)$.[3] In particular, if all α_i are homogeneous, then

$$\rho(\tilde{g})(\mathfrak{q}_{a_1}(\alpha_1)\cdots\mathfrak{q}_{a_k}(\alpha_k)1) = (-1)^{k_1}\mathfrak{q}_{a_1}(\alpha_1)\cdots\mathfrak{q}_{a_k}(\alpha_k)1$$

where $k_1 = |\{i : \alpha_i \in H^2(S,\mathbb{Q})\}|$. Since

$$\deg\big(\mathfrak{q}_{a_1}(\alpha_1)\cdots\mathfrak{q}_{a_k}(\alpha_k)1\big) = n - k + \sum_i \deg(\alpha_i).$$

we conclude that

$$\mathrm{mon}(g)\big(\mathfrak{q}_{a_1}(\alpha_1)\cdots\mathfrak{q}_{a_k}(\alpha_k)1\big) = (-1)^{n+k}\big(\mathfrak{q}_{a_1}(\alpha_1)\cdots\mathfrak{q}_{a_k}(\alpha_k)1\big).$$

□

We state also the version for parallel transport operators. Let

$$\psi : H^2(M_1,\mathbb{Z}) \to H^2(M_2,\mathbb{Z})$$

be a parallel transport operator. By the characterization of parallel transport operators in Theorem 3 there exists a (unique) lattice isomorphism

$$\eta : H^*(S_1,\mathbb{Z}) \to H^*(S_2,\mathbb{Z})$$

which extends ψ, i.e. such that the following diagram commutes:

$$\begin{array}{ccc} H^2(M_1,\mathbb{Z}) & \xrightarrow{\psi} & H^2(M_2,\mathbb{Z}) \\ \downarrow{\iota_{M_1}} & & \downarrow{\iota_{M_2}} \\ H^*(S_1,\mathbb{Z}) & \xrightarrow{\eta} & H^*(S_2,\mathbb{Z}). \end{array}$$

The isometry η sends v_1 to $\pm v_2$. Let $\tau(\psi) \in \mathbb{Z}/2\mathbb{Z}$ such that $\eta(v_1) = (-1)^{\tau(\psi)}v_2$. This defines a map[4]

$$\tau : \mathsf{PT}(M_1, M_2) \to \mathbb{Z}/2\mathbb{Z} = \{0,1\}.$$

Theorem 5. *(Markman, [4]) For any parallel transport operator $\psi : H^2(M_1,\mathbb{Z}) \to H^2(M_2,\mathbb{Z})$ we have that:*

$$P(\psi) = D^{\tau(\psi)} \circ \gamma((-1)^{\tau(\psi)}\eta)$$

where $P(\psi) : H^*(M_1,\mathbb{Z}) \to H^*(M_2,\mathbb{Z})$ is the unique parallel transport operator that restricts to ψ on $H^2(M_1,\mathbb{Z})$.

---

[3]This follows since the Nakajima operators are equivariant with respect to the monodromy group $O(H^2(S,\mathbb{Z}))^+$ which is Zariski dense in $O(H^2(S,\mathbb{C}))$.

[4]For an alternative definition, use that there exists a natural isomorphism ([1, Sec.14])

$$H^2(M_i,\mathbb{Z})^\vee / H^2(M_i,\mathbb{Z}) \to L^\vee/L$$

were $L = \mathbb{Z}v$ and hence we obtain a canonical element in $H^2(M_i,\mathbb{Z})^\vee/H^2(M_i,\mathbb{Z})$ corresponding to $v/(2n-2)$. We have $\tau(\psi) = 0$ if and only if ψ preserves this distinguished element. This makes the connection to the definition of the character (9).

Of course, $(-1)^{\tau(\psi)}\eta$ is just the map $H^*(S_1,\mathbb{Z}) \to H^*(S_2,\mathbb{Z})$ defined by sending v_1 to v_2 and acting by $(-1)^{\tau(\psi)}$ on $H^2(M_1,\mathbb{Z})$.

Proof. We just need to check that this is compatible with the three bulleted operations in Section 0.3. The first two are immediate. We check the third. In the notation there we have

$$\varphi_* = D \circ \gamma(D\Phi_*) : H^*(M_1,\mathbb{Z}) \to H^*(M_2,\mathbb{Z}).$$

Let ψ be the restriction of the parallel transport operator φ_* to $H^2(M_1,\mathbb{Z})$. Then under the identification (2) we have

$$\psi = -(D\Phi_*)|_{v_1^\perp}.$$

The natural extension of ψ is then $\eta = -D\Phi_*$ which sends v_1 to $-D\Phi_*(v_1) = -Dv_2^\vee = -v_2$, so $\tau(\psi) = 1$. We conclude that

$$D \circ \gamma(-\eta) = D\gamma(D\Phi_*) = \varphi_*. \qquad \square$$

We can reformulate the above discussion slightly. On $M = M_H(S,v)$ consider the involution

$$\widetilde{D} = D \circ \rho(-\mathrm{id}_{H^2(M,\mathbb{Z})}) = D \circ \gamma(\mathrm{id}_{\mathbb{Q}v} \oplus -\mathrm{id}_{v^\perp}).$$

We define the extended Markman representation

$$\widetilde{\rho} : O(H^2(M,\mathbb{C})) \times \mathbb{Z}_2 \to O(H^*(M,\mathbb{C}))$$

by

$$\widetilde{\rho}(g,\tau) = \widetilde{D}^\tau \circ \rho(g).$$

Then we have a natural embedding

$$\mathsf{Mon}(M) \to O(H^2(M,\mathbb{C})) \times \mathbb{Z}_2,\ g \mapsto (g,\tau(g))$$

which fits into the commutative diagram

$$\begin{array}{ccc} \mathsf{Mon}(M) & \longrightarrow & O(H^2(M,\mathbb{C})) \times \mathbb{Z}_2 \\ & \searrow^{\mu} & \downarrow{\scriptstyle \widetilde{\rho}} \\ & & O(H^*(M,\mathbb{C})). \end{array}$$

Lemma 4. *The subgroup* $\mathsf{Mon}(M) = \widetilde{O}^+(H^2(M,\mathbb{Z}))$ *is Zariski dense in* $O(H^2(M,\mathbb{C})) \times \mathbb{Z}_2$.

Proof. Since $\mathsf{Mon}(M)$ contains both elements with $\tau(g) = 1$ and with $\tau(g) = 0$ and $\det = -1$, it is enough to show that the subgroup of $\mathsf{Mon}(M)$ of elements of determinant 1 and $\tau(g) = 0$ is Zariski dense. But this is an arithmetic subgroup of $\mathrm{SO}(H^2(M,\mathbb{C}))$ so it is dense. $\square$

The last question we investigate is how does the extended Markman representation behave under parallel transport. Let M_1, M_2 be two moduli spaces of stable sheaves on a K3 surface. Let $P_\psi : H^*(M_1,\mathbb{Z}) \to H^*(M_2,\mathbb{Z})$ be a parallel transport operator, and let ψ be the restriction to $H^2(M_1,\mathbb{Z})$, and let $\eta : H^*(S_1,\mathbb{Z}) \to H^*(S_2,\mathbb{Z})$ be the unique extension of ψ.

Lemma 5. *For any $h \in O(H^2(M_1,\mathbb{C}))$ and $\tau \in \mathbb{Z}_2$ we have that*

$$P_\psi \circ \widetilde{\rho}(h,\tau) \circ P_\psi^{-1} = \widetilde{\rho}(\psi \circ h \circ \psi^{-1}, \tau).$$

Proof. Clearly we have $P_\psi \circ D \circ P_\psi^{-1} = D$ so we can assume $\tau = 0$. Recall the map $B : H^*(S_i,\mathbb{Z}) \to H^*(M_i,\mathbb{Z})$ from Lemma 3. We will show that for all $x \in H^*(S_2,\mathbb{Z})$ we have

$$(P_\psi \circ \rho(h) \circ P_\psi^{-1})B(x) = B\big((\eta \circ \alpha(h) \circ \eta^{-1})x\big)$$

which will imply by Lemma 1 that

$$(P_\psi \circ \rho(h) \circ P_\psi^{-1}) = \gamma(\eta \circ \alpha(h) \circ \eta^{-1}) = \rho(\psi \circ h \circ \psi^{-1})$$

as desired.

By Theorem 5 we have that

$$P_\psi B(x) = \begin{cases} B(\eta x) & \text{if } \tau(\psi) = 0 \\ DB(-\eta x) & \text{if } \tau(\psi) = 1. \end{cases}$$

Thus we get that

$$(P_\psi \circ \rho(h) \circ P_\psi^{-1})B(x) = P_\psi \gamma(\alpha(h))B(\eta^{-1}) = B(\eta\alpha(h)\eta^{-1})$$

if $\tau(\psi) = 0$ and

$$\begin{aligned}(P_\psi \circ \rho(h) \circ P_\psi^{-1})B(x) &= P_\psi \gamma(\alpha(h))DB(-\eta^{-1}x) \\ &= DP_\psi B(-\alpha(h)\eta^{-1}x) \\ &= B(\eta\alpha(h)\eta^{-1}x)\end{aligned}$$

if $\tau(\psi) = 1$. □

We hence can define the extended Markman representation

$$\rho : O(H^2(X,\mathbb{C})) \times \mathbb{Z}_2 \to O(H^*(X,\mathbb{C}))$$

for any holomorphic-symplectic variety of $K3^{[n]}$-type as follows: Let $P_\psi : H^*(X,\mathbb{Z}) \to H^*(M,\mathbb{Z})$ be a parallel transport operator, where M is a moduli space of sheaves on a K3 surface, and let $\psi = P_\psi|_{H^2(X,\mathbb{C})}$ be its restriction. Then for any $(g,\tau) \in O(H^2(X,\mathbb{C})) \times \mathbb{Z}_2$ we set

$$\rho(\psi,\tau) = P_\psi^{-1} \circ \widetilde{\rho}(\psi g \psi^{-1}, \tau) \circ P_\psi.$$

By the previous lemma this is independent of the choice of M and parallel transport operator P_ψ.

The monodromy group embeds then $\mathsf{Mon}(X) \to O(H^2(X,\mathbb{C})) \times \mathbb{Z}_2$ via $g \mapsto (g, \tau(g))$, and the monodromy representation is the restriction of ρ. Another way to characterize the representation ρ is that the image

$$\rho(O(H^2(X,\mathbb{C})) \times \mathbb{Z}_2) \subset O(H^*(X,\mathbb{C}))$$

is the Zariski closure of the Monodromy group (see also [4, Lemma 4.11]). If $n \geq 3$, the representation ρ is faithful (kernel is trivial).

0.6. **The Markman and LLV algebra representation.** Recall the representation of the LLV algebra

$$\rho_{\mathrm{LLV}} : \mathrm{SO}(H^2(X,\mathbb{C})) \to \mathrm{SO}(H^*(X,\mathbb{C})).$$

and the extended Markman representation

$$\rho : O(H^2(X,\mathbb{C})) \times \mathbb{Z}_2 \to O(H^*(X,\mathbb{C})).$$

Proposition 1. *We have*

$$\rho|_{\mathrm{SO}(H^2(X,\mathbb{C}))\times\{0\}} = \rho_{\mathrm{LLV}}.$$

Proof. Both actions are intertwined by the parallel transport operators, hence it is enough to prove this in the case of a moduli space of sheaves on a K3 surface M. Also it is enough to prove this on a Zariski-dense subset. Markman does this in [4, Lemma 4.13] using that the LLV algebra action and ρ both are intertwined by the monodromy action and the later is dense in $O(H^2(X,\mathbb{C})) \times \mathbb{Z}$, as well as that the cohomology is generated by the degree components of $B(x)$ for all x.

For the Hilbert scheme one can also give a direct argument as follows. It gives also the existence of the Markman operator in this special case.

Claim: Let $M = \mathsf{Hilb}_n(S)$. Then for any $g \in SO(H^2(M,\mathbb{C}))$,

$$(\rho_{\mathrm{LLV}}(g) \otimes \alpha(g))u_v = u_v.$$

Hence by Lemma 1 we have $\rho_{\mathrm{LLV}}(g) = \rho(g)$.

Proof of Claim: For $\alpha, \beta \in H^2(S,\mathbb{Q})$ let

$$h_{\alpha\beta} = \rho_{\mathrm{LLV}}(\alpha \wedge \beta)$$
$$h_{\alpha\delta} = \rho_{\mathrm{LLV}}(\alpha \wedge \delta)$$

where we identify $\wedge^2 H^2(M,\mathbb{Z}) = \mathfrak{so}(H^2(M,\mathbb{Z})) \subset \mathrm{so}(H^*(S,\mathbb{C}))$ in the usual way, that is $a \wedge b$ acts by

$$(a \wedge b)(x) = a(b \cdot x) - b(a \cdot x).$$

Since $h_{\alpha\beta}, h_{\alpha\delta}$ are degree-preserving orthogonal derivations, by Lemma 3 it suffices to prove that $H(B(x)) = B(H(x))$ for all $x \in H^*(S,\mathbb{Q})$ and $H \in \{h_{\alpha\beta}, h_{\alpha\delta}\}$. Since H is a derivation this is equivalent to proving

$$[H, \widehat{G}(x)] = \widehat{G}(Hx)$$

where $\widehat{G}(x) : H^*(M,\mathbb{Q}) \to H^*(M,\mathbb{Q})$ is the operator of multiplication by $B(x)$.

To do so, we begin with introducing some notation. Let $Z \subset \mathsf{Hilb}_n \times S$ be the universal subscheme. For $x \in H^*(X,\mathbb{Z})$ consider the class

$$\mathrm{univ}(x) = \pi_*(\mathrm{ch}(\mathcal{O}_Z) \cup \pi_S^*(x)).$$

A subscript d of these classes will denote their degree d component. For $\alpha \in H^2(S)$ we have

$$\mathrm{univ}_2(\alpha) = \theta_{\mathcal{F}}(\alpha), \quad \mathrm{univ}_3(1) = \delta.$$

Let $G(x) : H^*(M,\mathbb{Q}) \to H^*(M,\mathbb{Q})$ be the operator of multiplication by $\mathrm{univ}(x)$. By Example 1 we then have

$$\widehat{G}(x) = \exp\left(\frac{-G_3(1)}{2n-2}\right)\left\{-(v(\mathcal{O}_S)\cdot x)\mathrm{id} - G(\sqrt{\mathrm{td}}\cdot x^\vee)\right\}.$$

In [8, Prop 4.3 and 4.4] the following was proven:

$$\begin{aligned}
[h_{\alpha\beta}, G(x)] &= G(h_{\alpha\beta}(x)) \\
[h_{\alpha\delta}, G(x)] &= -G_2(\alpha)G(x) - G\left(-n\alpha x - \alpha\int_S x - \int_S \alpha x + 2\alpha\int_S xc\right) \\
&\quad + G_2\left(\alpha\int_S x + \int_S \alpha x\right)
\end{aligned}$$

where the last term comes from the fact that we need $[h_{\alpha\delta}, G_1(x)] = 0$. With this formula it is then straightforward that

$$[h_{\alpha\beta}, \widehat{G}(x)] = \widehat{G}(h_{\alpha\beta}(x)).$$

For $h_{\alpha\delta}$ one shows

$$\left[h_{\alpha\delta}, \exp\left(-\frac{G_3(1)}{2n-2}\right)\right] = G_2(\alpha)\exp\left(-\frac{G_3(1)}{2n-2}\right)$$

and

$$\begin{aligned}
&[h_{\alpha\delta}, G(\sqrt{\mathrm{td}}\cdot x^\vee)] = -G_2(\alpha)G(\sqrt{\mathrm{td}}\cdot x^\vee) \\
&\qquad + G\left(\sqrt{\mathrm{td}_S}\cdot((\delta\cdot x)\alpha - (\alpha\cdot x)\delta)^\vee\right) - (\sqrt{\mathrm{td}_S}\cdot x)G_2(\alpha) - n(x\cdot\alpha)\mathrm{id}
\end{aligned}$$

(use $((\delta\cdot x)\alpha - (\alpha\cdot x)\delta)^\vee = -(\delta\cdot x)\alpha - (\alpha\cdot x)\delta$ and that $\delta = -(1+(n-1)c)$). These two formulas then give $[h_{\alpha\delta}, \widehat{G}(x)] = \widehat{G}(h_{\alpha\delta}(x))$. □

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