

**LECTURES ON HOLOMORPHIC ANOMALY EQUATIONS
(FOR THE WORKSHOP GEOMETRIC ASPECTS OF THE
SWAMPLAND 2022, CFTMAT (MADRID))**

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1. QUASI-JACOBI FORMS

Jacobi forms are 2-variable generalizations of modular forms, which capture the properties of the Jacobi theta functions. Quasi-Jacobi forms, which are constant terms of almost-holomorphic Jacobi forms. In this part we introduce the basics of quasi-Jacobi forms. The theory of Jacobi forms can be found in the book [4] by Eichler and Zagier. For quasi-Jacobi forms basic references (at least for the form we consider it here) are [7], [10, Sec.1] and [5], and [8].

1.1. First Jacobi forms. Let $(z, \tau) \in \mathbb{C} \times \mathbb{H}$ be the standard variables, let $q = e^{2\pi i\tau}$ and $p = e^{2\pi iz}$.

Definition 1. A (weak) Jacobi form of weight k and index m is a function $\varphi : \mathbb{C} \times \mathbb{H} \rightarrow \mathbb{C}$ such that the following hold:

$$(1) \quad \begin{aligned} \Phi\left(\frac{z}{c\tau+d}, \frac{a\tau+b}{c\tau+d}\right) &= (c\tau+d)^k e\left(\frac{cmz^2}{c\tau+d}\right) \Phi(z, \tau) \\ \Phi(z + \lambda\tau + \mu, \tau) &= e(-m\lambda^2\tau - 2\lambda mz) \Phi(z, \tau) \end{aligned}$$

for all $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma$ and $\lambda, \mu \in \mathbb{Z}$, where $e(x) = e^{2\pi ix}$.

(2) $\phi(z, \tau) : \mathbb{C} \times \mathbb{H} \rightarrow \mathbb{C}$ is holomorphic and admits a Fourier expansion of the form $\sum_{n \geq 0} \sum_{r \in \mathbb{Z}} c(n, r) q^n p^r$ in the region $|q| < 1$.

1.2. Then the definition. Almost-holomorphic Jacobi forms are like weak Jacobi forms but we weaken the holomorphicity assumption in (2). Instead we allow polynomial dependence on the non-holomorphic functions

$$\nu = \frac{1}{8\pi\Im(\tau)}, \quad \alpha = \frac{\Im(z)}{\Im(\tau)}.$$

Definition 2. An almost-holomorphic Jacobi form of weight k and index m is a function $\Phi : \mathbb{C} \times \mathbb{H} \rightarrow \mathbb{C}$ such that:

- (1) As (1) above,
(2) $\Phi = \sum_{i,j \geq 0} \varphi_{i,j} \frac{\nu^i}{i!} \frac{\alpha^j}{j!}$ where only finitely many of the $\varphi_{i,j}$ are non-zero and each $\varphi_{i,j}$ satisfies (2).

Definition 3. A quasi-Jacobi form of weight k and index m is a function $\varphi : \mathbb{C} \times \mathbb{H} \rightarrow \mathbb{C}$ such that there exists an almost-holomorphic Jacobi form $\Phi = \sum_{i,j} \varphi_{i,j} \frac{\nu^i}{i!} \frac{\alpha^j}{j!}$ of weight k and index m with $\varphi = \varphi_{0,0}$.

Date: May 26, 2022.

We let $\text{Jac}_{k,m}$, $\text{QJac}_{k,m}$, $\text{AHJac}_{k,m}$ be the vector spaces of weak Jacobi, quasi-Jacobi and almost-holomorphic Jacobi forms of weight k and index m . We let $\text{Jac} = \bigoplus_{k,m} \text{Jac}_{k,m}$, etc. An index 0 quasi-Jacobi form is just a quasi-modular form: $\text{QJac}_{k,0} = \text{QMod}_k$.

1.3. **Examples.** There are really only two examples to consider:

1.3.1. *The second Eisenstein series.* Recall the discriminant form, which is a modular cusp form of weight 12,

$$\Delta(\tau) = q \prod_{n \geq 1} (1 - q^n)^{24} \in \text{Mod}_{12}.$$

Define

$$E_2(\tau) = D_\tau \log \Delta(\tau) = 1 - 24 \sum_{n \geq 1} \sigma(n) q^n$$

where $\sigma(n) = \sum_{d|n} d$ and

$$D_\tau = \frac{1}{2\pi i} \frac{d}{d\tau} = q \frac{d}{dq}.$$

We mostly use the renormalized series

$$G_2(\tau) = -\frac{1}{24} E_2(\tau).$$

By applying $D_\tau \log$ to the transformation law $\Delta(-1/\tau) = \tau^{12} \Delta(\tau)$ one finds

$$G_2(-1/\tau) = \tau^2 G_2(\tau) - \frac{1}{4\pi i} \tau.$$

A small calculation (exercise) shows that

$$\nu\left(-\frac{1}{\tau}\right) = \tau^2 \nu(\tau) + \frac{\tau}{4\pi i}.$$

We see that

$$\widehat{G}_2(\tau) = G_2(\tau) + \nu(\tau)$$

is an almost-holomorphic modular (hence Jacobi) form, and $G_2(\tau)$ is quasi-Jacobi: $G_2(\tau) \in \text{QJac}_{0,2}$.

1.3.2. *The log-derivative of the Theta function.* Consider the odd Jacobi-theta function

$$\vartheta(z, \tau) = \sum_{\nu \in \mathbb{Z} + \frac{1}{2}} (-1)^{\lfloor \nu \rfloor} p^\nu q^{\nu^2/2},$$

i.e. the unique section on the elliptic curve $\mathbb{C}_z/(\mathbb{Z} + \tau\mathbb{Z})$ which vanishes at the origin. We have the transformation properties (see [14, Prop.1.3])

$$\begin{aligned} \vartheta(z+1) &= -\vartheta(z), & \vartheta(z+\tau) &= -q^{-1/2} p^{-1} \vartheta(z) \\ \vartheta(z, \tau+1) &= e^{\pi i/4} \vartheta(z, \tau), & \vartheta(z/\tau, -1/\tau) &= -i\sqrt{\tau/i} e^{\pi i z^2/\tau} \vartheta(z, \tau). \end{aligned}$$

Let

$$D_z = p \frac{d}{dp} = \frac{1}{2\pi i} \frac{d}{dz}.$$

Define the series

$$A(z, \tau) = D_z \log \vartheta(z, \tau).$$

Then taking $D_z \log$ of the above transformation laws shows that:

$$A(z + \tau) = A(z) - 1, \quad A(z/\tau, -1/\tau) = \tau A(z, \tau) + z.$$

One also has (exercise)

$$\alpha(z + \tau, \tau) = \alpha(z, \tau) + 1, \quad \alpha(z/\tau, -1/\tau) = \tau \alpha(z, \tau) - z.$$

We find that $\widehat{A}(z, \tau) = A + \alpha$ is a (meromorphic) almost-holomorphic Jacobi form of weight 1 and index 0, and hence $A(z, \tau)$ is (meromorphic) quasi-Jacobi of weight 1 and index 0.

Remark 1. We will often use the renormalized Jacobi theta function

$$\Theta(z, \tau) = \vartheta_1(z, \tau) / \eta^3(\tau)$$

where $\eta(\tau) = q^{1/24} \prod_{n \geq 1} (1 - q^n)$ is the Dedekind eta function. It satisfies the triple product

$$\Theta(z, \tau) = (p^{1/2} - p^{-1/2}) \prod_{m \geq 1} \frac{(1 - pq^m)(1 - p^{-1}q^m)}{(1 - q^m)^2}.$$

This yields the Fourier-expansion

$$A(z, \tau) = -\frac{1}{2} - \sum_{m \neq 0} \frac{p^m}{1 - q^m}.$$

From the triple product, we also obtain the expansion

$$(2) \quad \Theta(z) = (2\pi iz) \exp \left(-2 \sum_{k \geq 2} G_k(\tau) \frac{(2\pi iz)^k}{k!} \right).$$

where $G_k(\tau) = -\frac{B_k}{2 \cdot k} + \sum_{n \geq 1} \sum_{d|n} d^{k-1} q^n$ for even positive k are the Eisenstein series, and with $G_k = 0$ for odd k .

1.3.3. *Elliptic genus.* We mention one natural place where quasi-Jacobi forms appear, namely as elliptic genera of complex compact manifolds.

Definition 4. The elliptic genus of a smooth compact complex manifold X is

$$\text{Ell}(X) = \int_X \prod_i x_i \frac{\Theta(\frac{x_i}{2\pi i} - z)}{\Theta(\frac{x_i}{2\pi i})}$$

where x_i are the Chern roots of the tangent bundle.

We refer to [7] for an introduction to the elliptic genus. By expressing the integrand as an exponential, the elliptic genus can be easily computed in terms of the Chern character numbers of X . Indeed, define functions for $k \geq 1$ by

$$F_k(z) = \frac{(-1)^k (k-1)!}{(2\pi iz)^k} + 2 \sum_{\ell > k} G_\ell(\tau) \frac{(2\pi iz)^{\ell-k}}{(\ell-k)!}$$

These function appear in the expansion (proven from the Jacobi triple product)

$$(3) \quad \log \left(\frac{x \Theta(x+z)}{\Theta(x)} \right) = \log(\Theta(z)) - \sum_{k \geq 1} \frac{x^k}{k!} F_k(z)$$

which immediately leads to:

$$\text{Ell}(X) = \Theta(-z)^{\dim X} \int_X \exp \left(\sum_{k \geq 1} (-1)^{k-1} F_k(z) \text{ch}_k(\text{Tan}) \right).$$

This shows the following:

Theorem 1 (Libgober [7]). *Ell(X) is a quasi-Jacobi form of weight 0 and index $\dim_{\mathbb{C}}(X)/2$ (where if the index is half-integral, we need to use quasi-Jacobi forms with the character defined by $\Theta(z)^{\dim(M)/2}$, see e.g. [5]). If X is Calabi-Yau, then Ell(X) is a weak Jacobi form.*

Proof. We claim that the F_k are (meromorphic) quasi-Jacobi forms of weight k , and that F_k for all $k \geq 2$ are Jacobi forms. To see this, consider the Zagier function [13]:

$$F_{\tau}(x, z) = \frac{\Theta(x+z)}{\Theta(x)\Theta(z)}.$$

By using the transformation properties of $\theta(z)$ one can derive its transformation properties under the Jacobi group. When taking the log of the transformation property only the linear term will lead to correction to the transformation properties of Jacobi forms in the F_k . This shows the claim. The statement follows since we integrate a series of cohomology classes with coefficients quasi-Jacobi forms. \square

1.4. Properties. We list 6 properties of quasi-Jacobi forms that we need (if no reference is given, they can be found in [10]):

1.4.1. Transformation property. Let $\Phi = \sum_{i,j} \varphi_{i,j} \nu^i \alpha^j / (i!j!) \in \text{AHJac}_{k,m}$. Then the transformation properties of the quasi-Jacobi form $\varphi := \varphi_{0,0}$ depends on all of the functions $\varphi_{i,j}$ as follows:

$$(4) \quad \begin{aligned} & \phi \left(\frac{z}{c\tau + d}, \frac{a\tau + b}{c\tau + d} \right) = (c\tau + d)^k e \left(\frac{cmz^2}{c\tau + d} \right) \\ & \times \sum_{i,j \geq 0} \frac{1}{i!} \frac{1}{j!} \left(-\frac{c}{4\pi i(c\tau + d)} \right)^i \left(\frac{cz}{c\tau + d} \right)^j \varphi_{i,j}(z, \tau) \\ & \phi(z + \lambda\tau + \mu, \tau) = e(-m\lambda^2\tau - 2\lambda mz) \sum_{j \geq 0} \frac{1}{j!} (-\lambda)^j \phi_{0,j}(z, \tau) \end{aligned}$$

for all $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma$ and $\lambda, \mu \in \mathbb{Z}$.

1.4.2. Constant-term map. The transformation property above shows that the constant-term $\varphi(z, \tau)$ of a almost-holomorphic quasi-Jacobi form Φ also knows something about all the higher coefficients $\varphi_{i,j}$. In fact, we have that the constant term map

$$\text{ct} : \text{AHJac}_{k,m} \xrightarrow{\cong} \text{QJac}_{k,m}, \quad \sum_{i,j} \varphi_{i,j} \nu^i \alpha^j / (i!j!) \mapsto \varphi_{0,0}$$

is an isomorphism.

(Idea of proof: Surjectivity is clear. Assume that $\varphi_{0,0} = 0$. Since ν and α are algebraically independent, we can take the $\nu^i \alpha^j$ -coefficients in (4). Solve for the equation involving $\varphi_{0,0}$. Since the $(c, d) \in \mathbb{C}^2$ for all $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \text{SL}_2(\mathbb{Z})$ have Zariski-dense image, we can treat the (c, d) as formal variables. Then by picking out the

correct coefficient of $c^i d^j$ we get the desired statement. See [1, Prop.3.4] for this argument in the quasi-modular case.)

1.4.3. *Derivative operators.* The algebra \mathbf{QJac} is preserved both by taking derivatives in τ and z :

$$\begin{aligned} D_\tau &: \mathbf{QJac}_{k,m} \rightarrow \mathbf{QJac}_{k+2,m} \\ D_z &: \mathbf{QJac}_{k,m} \rightarrow \mathbf{QJac}_{k+1,m}. \end{aligned}$$

1.4.4. *Anomaly operators.* One checks that $\frac{d}{d\nu}$ and $\frac{d}{d\alpha}$ preserves the algebra \mathbf{AHJac} and lowers the weight by 2 and 1 respectively. This means that we can define the following derivations, called 'anomaly operators':

$$\begin{aligned} \frac{d}{dG_2} &:= \text{ct} \circ \frac{d}{d\nu} \text{ct}^{-1} : \mathbf{QJac}_{k,m} \rightarrow \mathbf{QJac}_{k-2,m} \\ \frac{d}{dA} &:= \text{ct} \circ \frac{d}{d\alpha} \text{ct}^{-1} : \mathbf{QJac}_{k,m} \rightarrow \mathbf{QJac}_{k-1,m}. \end{aligned}$$

Recall that $\widehat{G}_2 = G_2 + \nu$ is the almost-holomorphic modular form with constant term G_2 , i.e. $\text{ct}(\widehat{G}_2) = G_2$. Hence we get

$$\frac{d}{dG_2} G_2 = \text{ct} \circ \frac{d}{d\nu} \text{ct}^{-1}(G_2) = \text{ct} \circ \frac{d}{d\nu}(G_2 + \nu) = \text{ct}(1) = 1,$$

in agreement with our notation. This is the reason for normalizing ν and α in the way we did it here.

More generally one can show that \mathbf{QJac} embeds into a free-polynomial algebra in \mathbf{A} and G_2 over the algebra of meromorphic Jacobi forms

$$\mathbf{QJac} \subset \mathbf{MJac}[\mathbf{A}(z, \tau), G_2(\tau)].$$

In this algebra the notation $\frac{d}{dG_2}$ also makes literally sense.

1.4.5. *Lie algebra relations.* Let wt and ind be the operators which act on $\mathbf{QJac}_{k,m}(\Gamma)$ by multiplication by the weight k and the index m respectively. By [10, (12)] we have the commutation relations:

$$(5) \quad \begin{aligned} \left[\frac{d}{dG_2}, D_\tau \right] &= -2\text{wt}, & \left[\frac{d}{dA}, D_x \right] &= 2\text{ind} \\ \left[\frac{d}{dG_2}, D_x \right] &= -2\frac{d}{dA}, & \left[\frac{d}{dA}, D_\tau \right] &= D_x. \end{aligned}$$

1.4.6. *Polynomial generators.* Recall the Weierstraß elliptic function

$$\wp(z, \tau) = \frac{1}{12} + \frac{p}{(1-p)^2} + \sum_{d \geq 1} \sum_{k|d} k(p^k - 2 + p^{-k})q^d.$$

and for even positive k the Eisenstein series

$$G_k(\tau) = -\frac{B_k}{2 \cdot k} + \sum_{n \geq 1} \sum_{d|n} d^{k-1} q^n.$$

Proposition 1 ([5, Prop.2.1]). *The algebra $\mathcal{R} = \mathbb{C}[\Theta, \mathbf{A}, G_2, \wp, \wp', G_4]$ is a free polynomial ring, and \mathbf{QJac} is equal to the subring of all polynomials which define holomorphic functions $\mathbb{C} \times \mathbb{H} \rightarrow \mathbb{H}$.*

2. HOLOMORPHIC ANOMALY EQUATIONS

2.1. Setup. Let X, B be smooth projective varieties of dimension n and $n-1$, and let

$$\pi : X \rightarrow B$$

be an elliptic fibration, that is a morphism whose fibers are reduced curves of arithmetic genus 1. In particular, the generic fiber of π is a smooth elliptic curve. We also assume that π has a section $\iota : B \rightarrow X$.

The following divisor plays a crucial role in the anomaly equation:

$$W = [\iota(B)] - \frac{1}{2}\pi^*c_1(N_{B/X}) \in H^2(X, \mathbb{Z})$$

Consider the weight endomorphism:

$$\text{Wt} = [W \cup -, \pi^*\pi_*] : H^*(X) \rightarrow H^*(X).$$

As shown in [10] it is diagonalizable with eigenvalues $1, -1, 0$ and corresponding eigenspaces

$$H_+(X) = W \cup \pi^*H^*(B), \quad H_-(X) = \pi^*H^*(B), \quad H_\perp(X).$$

We let

$$D \in H_\perp^2(X)$$

be any fixed divisor (which may be zero).

2.2. Gromov-Witten invariants. Gromov-Witten invariants are rational numbers which are virtual counts genus g degree $\beta \in H_2(X, \mathbb{Z})$ stable maps to a target X incident to cycles Poincaré dual to cohomology classes $\gamma_1, \dots, \gamma_n \in H^*(X)$.

Here we will use the following special convention: Let $\overline{M}_{g,n}(X, \beta)$ be the moduli space of n -marked genus g degree β stable maps to X . Assume that $2g-2+n > 0$ or $\pi_*\beta \neq 0$. Then there exists an induced morphism

$$\pi : \overline{M}_{g,n}(X, \beta) \rightarrow \overline{M}_{g,n}(B, \pi_*\beta).$$

Given $k_1, \dots, k_n \geq 0$ we define the descendent invariants

$$\langle \tau_{k_1}(\gamma_1) \cdots \tau_{k_n}(\gamma_n) \rangle_{g,\beta}^X := \int_{[\overline{M}_{g,n}(X,\beta)]^{\text{vir}}} \prod_{i=1}^n \text{ev}_i^*(\gamma_i) \cdot \pi^*(\psi_i)^{k_i}.$$

Here $\psi_i = c_1(\mathcal{L}_i)$ where $\mathcal{L}_i \rightarrow \overline{M}_{g,n}(B, \pi_*\beta)$ are the cotangent line bundles with fibers over a point $[f : C \rightarrow B, p_1, \dots, p_n]$ given by $T_{p_i}^*C$.

Remark 2. The usual descendent Gromov-Witten invariants are defined by integrating over $\psi_i^{k_i}$ instead of $\pi^*(\psi_i)^{k_i}$. The usual invariants can be expressed in terms of the above ones, and vice versa, by rewriting $\pi^*(\psi)$ as ψ plus boundary correction terms. The above convention is needed for the correct formulation of the holomorphic anomaly equation.

2.3. Gromov-Witten potential. Let $\beta \in H_2(B, \mathbb{Z})$ be a class in the base, and let $g, n \geq 0$ be the genus and number of markings. Let $\gamma_i \in H^*(X)$ and $k_i \geq 0$ for $i \in \{1, \dots, n\}$.

Definition 5. If $2g - 2 + n > 0$ or $\beta \neq 0$ define

$$F_{g,\beta}(\tau_{k_1}(\gamma_1) \cdots \tau_{k_n}(\gamma_n)) := \sum_{\substack{\tilde{\beta} \in H_2(X, \mathbb{Z}) \\ \pi_* \tilde{\beta} = \beta}} q^{W \cdot \tilde{\beta}} p^{D \cdot \tilde{\beta}} \langle \tau_{k_1}(\gamma_1) \cdots \tau_{k_n}(\gamma_n) \rangle_{g, \tilde{\beta}}^X.$$

If $k_i = 0$ for all i , we write $F_{g,\beta}(\gamma_1, \dots, \gamma_n)$ for the above series.

Remark 3. If $\beta = 0$ and $2g - 2 + n \leq 0$ then $F_{g,\beta}(\cdot)$ is NOT defined. In particular, whenever we sum over coefficients involving $F_{g,\beta}(\cdot)$ we always assume that g, β, n satisfies these conditions.

2.4. Holomorphic anomaly equation conjectures. We state a special case of the general conjecture proposed in [10], see also [9].

Conjecture A (Quasi-Jacobi form).

$$F_{g,\beta}(\tau_{k_1}(\gamma_1) \cdots \tau_{k_n}(\gamma_n)) \in \frac{1}{\Delta(q)^{-c_1(N_{B/X}) \cdot \beta/2}} \text{QJac}_{*,m}$$

where the index is $m = -\frac{1}{2} \int_X \pi^*(\beta) \cdot D^2$.

Conjecture B (G_2 -Holomorphic anomaly equation). *Assuming the above conjecture, we have:*

$$\begin{aligned} \frac{d}{dG_2} F_{g,\beta}(\tau_{k_1}(\gamma_1) \cdots \tau_{k_n}(\gamma_n)) &= F_{g-1,\beta}(\tau_{k_1}(\gamma_1) \cdots \tau_{k_n}(\gamma_n)) \tau_0 \tau_0(\Delta_B) \\ &+ \sum_{\substack{g=g_1+g_2 \\ \beta=\beta_1+\beta_2 \\ \{1,\dots,n\}=A \sqcup B}} F_{g_1,\beta_1}(\tau_{k_A}(\gamma_A)) \tau_0(\Delta_{B,1}) F_{g_2,\beta_2}(\tau_{k_B}(\gamma_B)) \tau_0(\Delta_{B,2}) \\ &- 2 \sum_{i=1}^n F_{g,\beta}(\cdots \tau_{k_i+1}(\pi^* \pi_* \gamma_i) \cdots). \end{aligned}$$

Here $\Delta_B \in H^*(B \times B)$ is the class of the diagonal. If $\Delta_B = \sum_i \delta_i \boxtimes \delta_i^\vee$ then $\tau_0 \tau_0(\Delta_B)$ stands for $\sum_i \tau_0(\delta_i) \tau_0(\delta_i^\vee)$. Similarly, $\Delta_{B,1}, \Delta_{B,2}$ stands for summing over the components of the Künneth decomposition. We have also suppressed the pullback by π , implicitly viewing $H^*(X)$ as a module over $H^*(B)$ via π^* .

Conjecture C (A-Holomorphic anomaly equation).

$$\frac{d}{dA} F_{g,\beta}(\tau_{k_1}(\gamma_1) \cdots \tau_{k_n}(\gamma_n)) = \sum_{i=1}^n F_{g,\beta}(\cdots \tau_{k_i}(f_D(\gamma_i)) \cdots)$$

where we used the endomorphism $f_D = [D \cup -, \pi^* \pi_*] : H^*(X) \rightarrow H^*(X)$.

To describe the weight of the quasi-Jacobi form, recall the weight operator $\text{Wt} = [W \cup -, \pi^* \pi_*]$. It is semisimple and defines a weight grading on $H^*(X)$. Concretely, if $\gamma \in H^*(X)$ is an eigenvector, we define $\text{wt}(\gamma) \in \mathbb{Z}$ by $\text{Wt}\gamma = \text{wt}(\gamma)\gamma$.

Conjecture D (Weight). *If γ_i are wt-homogeneous classes, then $F_{g,\beta}(\tau_{k_1}(\gamma_1) \cdots \tau_{k_n}(\gamma_n))$ is of weight $2g - 2 + n + \sum_i \text{wt}(\gamma_i)$.*

3. EXAMPLES: ELLIPTIC CALABI-YAU n -FOLDS

Let $\pi : X \rightarrow B$ be an elliptic fibration with section $\iota : B \rightarrow X$. Assume that X is a Calabi-Yau manifold, so $K_X = 0$. Since $T_X|_B$ has trivial determinant, we find that

$$N_{B/X} \cong \omega_X.$$

Hence we get

$$W = [\iota(B)] + \frac{1}{2}\pi^*c_1(T_B).$$

Below we consider the anomaly equation of X in the following cases:

- (1) $\dim X = 3$, all genus
- (2) $\dim X = 4$, genus 0 and 1
- (3) any dimension, genus 0 and 1 (but using descendent classes).

3.1. Calabi–Yau threefolds. Let X be a Calabi–Yau 3-fold. We have that B is a surface. We write $\langle \gamma_1, \gamma_2 \rangle = \int_B \gamma_1 \cup \gamma_2$ for the intersection pairing. Since $\overline{M}_g(X, \beta)$ is of virtual dimension zero, we do not need insertions. We simply set

$$F_{g,\beta} := F_{g,\beta}(\emptyset).$$

Conjecture A implies

$$F_{g,\beta}(q) \in \frac{1}{\Delta(q)^{\frac{1}{2}c_1(B)\cdot\beta}} \mathbf{QJac}$$

where $F_{g,\beta}$ of weight $2g - 2$ and index $m = -\frac{1}{2}(\pi^*\beta \cdot D^2)$.

Conjecture B implies (see [10, Sec.3.6] for details on this computation) the following anomaly equation:

$$\frac{d}{dG_2} F_{g,\beta} = \langle \beta + K_S, \beta \rangle F_{g-1,\beta} + \sum_{\substack{g=g_1+g_2 \\ \beta=\beta_1+\beta_2}} \langle \beta_1, \beta_2 \rangle F_{g_1,\beta_1} F_{g_2,\beta_2} + \frac{1}{4} \delta_{g,2} \delta_{k,0} \langle K_B, K_B \rangle.$$

Remark 4. Using $e(X) = -60\langle K_B, K_B \rangle$ we obtain the holomorphic-anomaly equation as proposed in string theory by Bershadsky-Cecotti-Ooguri-Vafa, Klemm et al. and others.

3.2. Calabi–Yau 4-folds. Let X be a Calabi–Yau 4-fold. Then $\overline{M}_{g,n}(X, \beta)$ is of virtual dimension $1 - g + n$. Hence we have non-zero Gromov-Witten invariants only for $g \in \{0, 1\}$. Moreover, in genus 0 we need our curves to pass through a 4-cycle $\gamma \in H^4(X)$. In genus 1 we do not need any insertions:

$$F_{0,\beta}(\gamma), \quad F_{1,\beta}.$$

Conjecture A implies

$$F_{0,\beta}(\gamma), F_{1,\beta} \in \frac{1}{\Delta(q)^{\frac{1}{2}c_1(B)\cdot\beta}} \mathbf{QJac}.$$

of weight $-1 + \text{wt}(\gamma)$ and 0 respectively. Conjecture B yields the genus 1 holomorphic anomaly equation:

$$\frac{d}{dG_2} F_{1,\beta} = F_{0,\beta} \left(2\pi^*\beta - \frac{1}{12}\pi^*\pi_*c_3(X) \right) + 2 \sum_{\substack{\beta=\beta_1+\beta_2 \\ \beta_1, \beta_2 > 0}} F_{1,\beta_1} \cdot F_{0,\beta_2}(\pi^*\beta_1).$$

In genus 0 we obtain the equation:

$$\frac{d}{dG_2} F_{0,\beta}(\gamma) = 2 \left(\sum_{\substack{\beta=\beta_1+\beta_2 \\ \beta_1, \beta_2 > 0}} F_{0,\beta_1}(\gamma) \cdot F_{0,\beta_2}(\pi^* \beta_1) \right) - 2F_{0,\beta}(\tau_1(\pi^* \pi_* \gamma))$$

The descendent invariant can be removed by Lemma 1 below. We find the following: Let $\ell \in H^2(B)$ be any element with $\beta \cdot \ell \neq 0$. Then¹

$$\boxed{\begin{aligned} \frac{d}{dG_2} F_{0,\beta}(\gamma) &= F_{0,\beta} \left(4 \frac{\pi^* \pi_*(\gamma) \cup \ell}{\langle \beta, \ell \rangle} - 2 \frac{\langle \pi_*(\gamma), \beta \rangle}{\langle \beta, \ell \rangle^2} \ell^2 \right) \\ &+ 2 \sum_{\substack{\beta=\beta_1+\beta_2 \\ \beta_1, \beta_2 > 0}} \left\{ F_{0,\beta_1}(\gamma) F_{0,\beta_2}(\pi^* \beta_1) - \left[\frac{\langle \beta_1, \ell \rangle^2 \langle \beta_2, \pi_* \gamma \rangle}{\langle \beta, \ell \rangle^2} \right] \sum_a F_{0,\beta_1}(\Delta_{X,a}) F_{0,\beta_2}(\Delta_{X,a}^\vee) \right\} \end{aligned}}$$

where $\Delta_X = \sum_a \Delta_{X,a} \otimes \Delta_{X,a}^\vee$ is the Künneth decomposition of the diagonal of X .

If $\gamma = h_1 h_2$ for some $h_1, h_2 \in H^2(B)$ the genus 0 expression simplifies significantly. In this case we have

$$\frac{d}{dG_2} F_{0,\beta}(h_1 h_2) = 2 \sum_{\substack{\beta=\beta_1+\beta_2 \\ \beta_1, \beta_2 > 0}} F_{0,\beta_1}(h_1 h_2) F_{0,\beta_2}(\pi^* \beta_1).$$

The holomorphic anomaly equations above are in agreement with results of [3] and [11, 12].

Lemma 1. *For $\alpha \in H^2(B)$ and non-zero $\beta \in H_2(B, \mathbb{Z})$ let $\ell \in H^2(B)$ be any class such that $\beta \cdot \ell \neq 0$. Then*

$$\begin{aligned} F_{0,\beta}(\tau_1(\pi^* \alpha)) &= F_0 \left(-\frac{2}{\langle \beta, \ell \rangle} \pi^*(\alpha \cup \ell) + \frac{\langle \beta, \alpha \rangle}{\langle \beta, \ell \rangle^2} \pi^*(\ell)^2 \right) \\ &+ \sum_{\substack{\beta=\beta_1+\beta_2 \\ \beta_1, \beta_2 > 0}} \frac{\langle \beta_1, \alpha \rangle \langle \beta_2, \ell \rangle^2}{\langle \beta, \ell \rangle^2} F_{0,\beta_1}(\Delta_{X,1}) F_{0,\beta_2}(\Delta_{X,2}) \end{aligned}$$

where $\Delta_{X,1}, \Delta_{X,2}$ stands for summing over the Künneth decomposition of the diagonal $\Delta_X \in H^*(X^2)$.

Proof. This follows from a standard geometric recursions argument, see e.g. [2, Lemma 1.1]. \square

3.3. The elliptic fibration over \mathbb{P}^3 . Let $\pi : X \rightarrow \mathbb{P}^3$ be the elliptic fibration over $B = \mathbb{P}^3$. We let $h \in H^2(\mathbb{P}^3)$ be the hyperplane class. By [6, Sec.6.3] we have $\pi_* c_3(X) = -960h^2$. Every integer $k \in \mathbb{Z}$ determines a class in $H_2(\mathbb{P}^3, \mathbb{Z})$ via kh^2 . We write $F_{1,k}$ instead of F_{1, kh^2} , etc. The genus 1 holomorphic anomaly equation reads

$$\frac{d}{dG_2} F_{1,k} = (2k + 80) F_{0,k}(h^2) + 2 \sum_{\substack{k=k_1+k_2 \\ k_1, k_2 > 0}} k_1 F_{1,k_1} F_{0,k_2}(h^2).$$

¹We consider here $H^*(X)$ as a module over $H^*(B)$ via pullback along π . The pullback maps π^* are suppressed.

The genus 0 holomorphic anomaly equations read (with $W = B_0 + 2h$):

$$\begin{aligned} \frac{d}{dG_2} F_{0,k}(h^2) &= 2 \sum_{\substack{k=k_1+k_2 \\ k_1, k_2 > 0}} k_1 F_{0,k_1}(h^2) F_{0,k_2}(h^2) \\ \frac{d}{dG_2} F_{0,k}(Wh) &= \frac{2}{k} \left[F_{0,k}(h^2) + \sum_{\substack{k=k_1+k_2 \\ k_1, k_2 > 0}} k_1^2 F_{0,k_1}(Wh) F_{0,k_2}(h^2) \right] \end{aligned}$$

We refer also to [3] for an independent derivation of these equations on the physics side.

3.4. Calabi–Yau n -folds. Let X be a Calabi–Yau n -fold, let $\beta \in H_2(B, \mathbb{Z})$ and $\gamma \in H^{n-2, n-2}(X, \mathbb{C})$. Conjecture A in [9, 10] implies

$$F_{0,\beta}(\gamma), F_{1,\beta} \in \frac{1}{\Delta(q)^{\frac{1}{2}c_1(B)\cdot\beta}} \mathbf{QJac}$$

Conjecture B in turn yields the genus 1 holomorphic anomaly equation:

$$\boxed{\frac{d}{dG_2} F_{1,\beta} = F_{0,\beta} \left(2\pi^* \beta - \frac{1}{12} \pi^* \pi_* c_{n-1}(X) \right) + 2 \sum_{\substack{\beta=\beta_1+\beta_2 \\ \beta_1, \beta_2 > 0}} F_{1,\beta_1} \cdot F_{0,\beta_2}(\pi^* \beta_1).}$$

The genus 0 holomorphic anomaly equation reads:

$$\boxed{\begin{aligned} \frac{d}{dG_2} F_{0,\beta}(\gamma) &= 2 \sum_{\substack{\beta=\beta_1+\beta_2 \\ \beta_1, \beta_2 > 0}} F_{0,\beta_1}(\gamma) \cdot F_{0,\beta_2}(\pi^* \beta_1) \\ &\quad - 2F_{0,\beta}(\tau_1(\pi^* \pi_* \gamma)). \end{aligned}}$$

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