STABLE PAIRS AND GOPAKUMAR-VAFA TYPE INVARIANTS ON HOLOMORPHIC SYMPLECTIC 4-FOLDS

YALONG CAO, GEORG OBERDIECK, AND YUKINOBU TODA

ABSTRACT. As an analogy to Gopakumar-Vafa conjecture on Calabi-Yau 3-folds, Klemm-Pandharipande defined Gopakumar-Vafa type invariants of a Calabi-Yau 4-fold X using Gromov-Witten theory. When X is holomorphic symplectic, Gromov-Witten invariants vanish and one can consider the corresponding reduced theory. In a companion work, we propose a definition of Gopakumar-Vafa type invariants for such a reduced theory. In this paper, we give them a sheaf theoretic interpretation via moduli spaces of stable pairs.

0. INTRODUCTION

0.1. Gopakumar-Vafa invariants.

A smooth complex projective 4-fold X is *holomorphic symplectic* if it is equipped with a non-degenerate holomorphic 2-form $\sigma \in H^0(X, \Omega_X^2)$. The ordinary Gromov-Witten invariants of X always vanish for non-zero curve classes. Instead a reduced Gromov-Witten theory is defined by Kiem-Li's cosection localization [KiL].

Given cohomology classes $\gamma_i \in H^*(X, \mathbb{Z})$, the (reduced) Gromov-Witten invariants of X in a non-zero curve class $\beta \in H_2(X, \mathbb{Z})$ are defined by

$$(0.1) \qquad \mathrm{GW}_{g,\beta}(\gamma_1, \ldots, \gamma_l) = \int_{[\overline{M}_{g,l}(X,\beta)]^{\mathrm{vir}}} \prod_{i=1}^{l} \mathrm{ev}_i^*(\gamma_i),$$

where

$$[\overline{M}_{g,l}(X,\beta)]^{\mathrm{vir}} \in A_{2-g+l}(\overline{M}_{g,l}(X,\beta))$$

is the (reduced) virtual class and $\mathrm{ev}_i \colon \overline{M}_{g,l}(X,\beta) \to X$ is the evaluation map at the i-th marking. We refer to [O18a, O21, OSY] for some references on computations for (0.1). Gromov-Witten invariants are in general rational numbers because the moduli space $\overline{M}_{g,l}(X,\beta)$ of stable maps is a Deligne-Mumford stack. It is an interesting question to find out integer-valued invariants which underlie them.

In [COT22], we studied this question and defined *genus 0 Gopakumar-Vafa invariants*

$$(0.2) \qquad n_{0,\beta}(\gamma_1, \ldots, \gamma_l) \in \mathbb{Q}$$

for any non-zero curve class β and *genus 1 and 2 Gopakumar-Vafa invariants*

$$(0.3) \qquad n_{1,\beta}(\gamma) \in \mathbb{Q}, \ \forall \ \gamma \in H^4(X, \mathbb{Z}); \quad n_{2,\beta} \in \mathbb{Q}$$

for any primitive curve class β (i.e. it is not a multiple of a non-zero curve class in $H_2(X, \mathbb{Z})$) from Gromov-Witten invariants (0.1) (see §1.1 for details). This may be compared with the previous works of Gopakumar and Vafa [GV] on Calabi-Yau 3-folds, Klemm and Pandharipande [KP] on Calabi-Yau 4-folds and Pandharipande and Zinger [PZ] on Calabi-Yau 5-folds.

In loc. cit., we conjectured the integrality of (0.2), (0.3) and provided substantial evidence for it. The aim of this paper is to give a sheaf theoretic interpretation of these Gopakumar-Vafa invariants using moduli spaces of stable pairs, in analogy with the discussion of [CMT19, CT19] on ordinary Calabi-Yau 4-folds.

0.2. GV/Pairs correspondence.

Let F be a one dimensional coherent sheaf on X and $s \in H^0(F)$ be a section. For an ample divisor ω on X, we denote the slope function by $\mu(F) = \chi(F)/(\omega \cdot [F])$. The pair (F, s) is called Z_t-*stable* ($t \in \mathbb{R}$) if

(i) for any subsheaf $0 \neq F' \subseteq F$, we have $\mu(F') < t$,
(ii) for any subsheaf $F' \subsetneq F$ such that s factors through F', we have $\mu(F/F') > t$.

For a non-zero curve class $\beta \in H_2(X, \mathbb{Z})$ and $n \in \mathbb{Z}$, we denote by

$$P_n^t(X, \beta)$$

the moduli space of Z_t-stable pairs (F, s) with $([F], \chi(F)) = (\beta, n)$. It has a wall-chamber structure and for a *general* $t \in \mathbb{R}$ (i.e. outside a finite subset of rational numbers in $\mathbb{R}$), it is a projective scheme.

When $t < \frac{n}{\omega \cdot \beta}$, $P_n^t(X, \beta)$ is empty. The first nontrivial chamber appears when $t = \frac{n}{\omega \cdot \beta} + 0^+$, which we call *Joyce-Song (JS) chamber* (here 0^+ denotes a sufficiently small positive number with respect to the fixed ω, β, n). When $t \gg 1$, it recovers the moduli space of *Pandharipande-Thomas (PT) stable pairs* [PT09] (Proposition 1.6).

For general $t \in \mathbb{R}$, by Theorem 1.7, we can define its DT_4 virtual class following [BJ, OT] (see also [CL14]). However, by a cosection argument the virtual class vanishes, see [KiP, Sav]. Using Kiem-Park's cosection localization [KiP], we have a (reduced) virtual class

$$[P_n^t(X, \beta)]^{\mathrm{vir}} \in A_{n+1}(P_n^t(X, \beta), \mathbb{Q}),$$

depending on the choice of orientation [CGJ, CL17]. More precisely, for each connected component of $P_n^t(X, \beta)$, there are two choices of orientation which affect the virtual class by a sign (component-wise). To define its counting invariants, let

$$\tau : H^m(X, \mathbb{Z}) \to H^{m-2}(P_n^t(X, \beta), \mathbb{Z}),$$

$$\tau(\gamma) := \pi_{P*}\left(\pi_X^* \gamma \cup \mathrm{ch}_3(\mathbb{F})\right),$$

where $\mathbb{I} = (\mathcal{O} \to \mathbb{F})$ is the universal Z_t-stable pair and π_P, π_X are projections from $P_n^t(X, \beta) \times X$ onto its factors. For $\gamma_i \in H^{m_i}(X, \mathbb{Z})$, the Z_t-*stable pair invariants* are defined by

$$P_{n,\beta}^t(\gamma_1, \ldots, \gamma_l) := \int_{[P_n^t(X,\beta)]^{\mathrm{vir}}} \prod_{i=1}^{l} \tau(\gamma_i) \in \mathbb{Q}.$$

When $n = -1$, we also write

$$P_{-1,\beta}^t := \int_{[P_{-1}^t(X,\beta)]^{\mathrm{vir}}} 1.$$

Here is the main conjecture of this paper, which gives a sheaf theoretic interpretation of all genus Gopakumar-Vafa invariants using Z_t-stable pair invariants.

Conjecture 0.1. (Conjecture 1.10) *Fix* $n \in \mathbb{Z}$, $\beta \in H_2(X, \mathbb{Z})$ *and let* $t > \frac{n}{\omega \cdot \beta}$ *be generic. For certain choice of orientation, we have*

(1) *If* $n \geqslant 2$, *then*
$$P_{n,\beta}^t(\gamma_1, \ldots, \gamma_l) = 0.$$

(2) *If* $n = 1$, *then*
$$P_{1,\beta}^t(\gamma_1, \ldots, \gamma_l) = n_{0,\beta}(\gamma_1, \ldots, \gamma_l).$$

(3) *If* $n = 0$ *and* β *is primitive, then*
$$P_{0,\beta}^t(\gamma) = n_{1,\beta}(\gamma).$$

(4) *If* $n = -1$ *and* β *is primitive, then*
$$P_{-1,\beta}^t = n_{2,\beta}.$$

We verify this conjecture by a computation in an ideal geometry where curves deform in families of expected dimensions and have expected generic properties (see §1.4). Besides this, we study several examples and prove our conjecture in those cases.

0.3. **Verification of conjectures I:** $K3 \times K3$. Let $X = S \times T$ be the product of two $K3$ surfaces. When the curve class $\beta \in H_2(S \times T, \mathbb{Z})$ is of non-trivial degree over both S and T, then one can construct two linearly independent cosections for moduli spaces of stable maps, which imply that the (reduced) Gromov-Witten invariants of X in this class vanish. Therefore we always restrict to consider curve classes of form

$$\beta \in H_2(S, \mathbb{Z}) \subseteq H_2(X, \mathbb{Z}).$$

Theorem 0.2. (Theorem 2.11, 2.14, 2.15, Remark 2.16) *Let* $X = S \times T$ *be as above. Then Conjecture 0.1 holds for any primitive curve class* $\beta \in H_2(S, \mathbb{Z}) \subseteq H_2(X, \mathbb{Z})$.

In fact, by the global Torelli theorem (see e.g. [Ver, Huy]), primitive curve classes on $K3$ surfaces can be deformed to irreducible curve classes. By deformation invariance, we only need to deal with an irreducible curve class β, in which case we have an isomorphism (Proposition 2.3):

$$P_n^t(X, \beta) \cong P_n^t(S, \beta) \times T,$$

and a forgetful map

(0.4) $$P_n^t(S, \beta) \to M_n(S, \beta),$$

where $M_n(S, \beta)$ is the coarse moduli space of one dimensional stable sheaves F on S with $[F] = \beta$, $\chi(F) = n$. Both $P_n^t(S, \beta)$ and $M_n(S, \beta)$ are smooth schemes. We can then determine the DT_4 virtual class of $P_n^t(X, \beta)$ (Theorem 2.8) and its pushforward (under the forgetful map) by the Thom-Porteus formula (Proposition 2.9). This enables us to reduce the computation of Z_t-stable pair invariants to certain tautological integrals on $M_n(S, \beta)$. By Markman's framework of monodromy operators [M08], we relate such integrals to certain tautological integrals on Hilbert schemes of points on S (see §2.6 for details), which we explicitly determine using [COT22] (see the proof of Theorem 2.14, 2.15 for details).

0.4. **Verification of conjectures II:** $T^* \mathbb{P}^2$. Let $H \in H^2(T^* \mathbb{P}^2)$ be the pullback of the hyper-plane class and let us identify $H_2(T^* \mathbb{P}^2, \mathbb{Z}) \equiv \mathbb{Z}$ by taking the degree against H.

By explicitly describing the moduli spaces and virtual classes, we obtain:

Proposition 0.3. (Proposition 3.3) *For certain choice of orientation, we have*

$$P_{1,1}(H^2, H^2) = 1, \quad P_{1,2}(H^2, H^2) = -1, \quad P_{1,3}(H^2, H^2) = 0,$$

$$P_{0,1}(H^2) = P_{0,2}(H^2) = 0, \quad P_{0,3}(H^2) = 1, \quad P_{-1,1} = P_{-1,2} = P_{-1,3} = 0.$$

Moreover, $P_n^t(X, d)$ is independent of the choice of $t > n/d$ in the listed cases above.

In particular, for $X = T^* \mathbb{P}^2$, we have

- *Conjecture 0.1 (2) holds when $d \leqslant 3$.*
- *Conjecture 0.1 (3), (4) hold.*

0.5. **Verification of conjectures III: exceptional curves on** $\mathrm{Hilb}^2(K3)$. Let S be a $K3$ surface and $\mathrm{Hilb}^2(S)$ be the Hilbert scheme of two points on S. Consider the Hilbert-Chow map

$$\pi : \mathrm{Hilb}^2(S) \to \mathrm{Sym}^2(S)$$

to the symmetric product of S. Let D be the exceptional divisor fitting into Cartesian diagram:

$$
\begin{array}{ccc}
D & \xrightarrow{\;i\;} & \mathrm{Hilb}^2(S) \\
\pi \downarrow & & \downarrow \pi \\
S & \xrightarrow{\;\Delta\;} & \mathrm{Sym}^2(S),
\end{array}
$$

where Δ is the diagonal embedding and $\pi : D \to S$ is a $\mathbb{P}^1$-bundle. The following provides a verification of our (genus 0) conjecture for imprimitive curve classes.

Theorem 0.4. (Theorem 3.1) *In the JS chamber, Conjecture 0.1 (1), (2) hold for multiple fiber classes $\beta = r[\mathbb{P}^1]$ $(r \geqslant 1)$ of π as above.*

In fact, by the Jordan-Hölder filtration and a dimension counting, the JS pair invariants of $P_n^{\mathrm{JS}}(X, r[\mathbb{P}^1])$ are zero unless $n = r$ and in which case we have

$$P_n^{\mathrm{JS}}(X, n[\mathbb{P}^1]) \cong \mathrm{Hilb}^n(S).$$

Then the proof makes use of the Chern class operator of tautological bundles by Lehn [Lehn].

0.6. **Multiple fiber classes of elliptic fibrations.** Let $p : S \to \mathbb{P}^1$ be an elliptic $K3$ surface and consider the elliptic fibration:

$$\bar{p} := p \times \mathrm{id}_T : X := S \times T \to \mathbb{P}^1 \times T =: Y,$$

where T is a $K3$ surface. Denote f to be a generic fiber of $\bar{p}$ and $\mathsf{p} \in H_0(T)$ be the point class. The following gives a closed formula of Z_t-stable pair invariants for multiple fiber classes.

Theorem 0.5. (Theorem 2.19) *Let $t > 0$. Then for certain choice of orientation, we have*

$$\sum_{r \geqslant 0} P_{0, r[f]}^t(\gamma) \, q^r = 24 \left(\int_{S \times \mathsf{p}} \gamma \right) \cdot \sum_{m \geqslant 1} \sum_{n | m} n^2 q^m.$$

As for the proof, we note that there is an isomorphism

$$\bar{p}^* : \mathrm{Hilb}^r(Y) \cong P_0^t(X, r[f]), \quad I_Z \mapsto \bar{p}^* I_Z,$$

under which the (reduced) virtual classes

$$(-1)^{n+1} [\mathrm{Hilb}^r(Y)]^{\mathrm{vir}} = [P_0^t(X, r[f])]^{\mathrm{vir}} \in A_1(\mathrm{Hilb}^r(Y))$$

can be identified for certain choice of orientation on the right hand side. Then we are left to evaluate an integral on $[\mathrm{Hilb}^r(Y)]^{\mathrm{vir}}$ which can be done via the degeneration method and a

Behrend function argument [B, OS]. We refer to Theorem 2.17 for a similar result for trivial elliptic fibration $E \times E \times T \to E \times T$ and the proof therein for details.

The formula in Theorem 0.5 seems to support our speculation of a GV/Pairs correspondence in genus 1 for imprimivite curve classes (see §1.5 for details).

0.7. **A conjectural virtual pushforward formula.** Finally we remark that for a general holomorphic symplectic 4-fold X and an irreducible curve class $\beta \in H_2(X, \mathbb{Z})$, we have a forgetful map as in (0.4):

$$P_n^t(X, \beta) \to M_n(X, \beta),$$

where $M_n(S, \beta)$ is the coarse moduli space of one dimensional stable sheaves F on X with $[F] = \beta$, $\chi(F) = n$. In Appendix §A, we conjecture a virtual pushforward formula for this map (which we verify for the product of $K3$ surfaces, see Proposition A.5). Together with Conjecture 0.1 (4), this formula implies a conjectural relation between genus 2 Gopakumar-Vafa invariants and certain descendent invariants on $M_1(X, \beta)$ (Proposition A.3), which appears as [COT22, Conj. 2.2 (iii)].

0.8. **Notation and convention.** In this paper, all varieties and schemes are defined over $\mathbb{C}$. For a morphism $\pi \colon X \to Y$ of schemes, and for $\mathcal{F}, \mathcal{G} \in \mathrm{D}^{\mathrm{b}}(\mathrm{Coh}(X))$, we denote by $\mathbf{R}\mathcal{H}om_\pi(\mathcal{F}, \mathcal{G})$ the functor $\mathbf{R}\pi_* \mathbf{R}\mathcal{H}om_X(\mathcal{F}, \mathcal{G})$.

A class $\beta \in H_2(X, \mathbb{Z})$ is called *effective* if there exists a non-empty curve $C \subset X$ with class $[C] = \beta$. An effective class β is called *irreducible* if it is not the sum of two effective classes, and it is called *primitive* if it is not a positive integer multiple of an effective class.

A holomorphic-symplectic variety is a smooth projective variety with a non-degenerate holomorphic two form $\sigma \in H^0(X, \Omega_X^2)$. A holomorphic-symplectic variety is irreducible *hyperkähler* if X is simply connected and $H^0(X, \Omega_X^2)$ is generated by a symplectic form. A $K3$ surface is an (irreducible) hyperkähler variety of dimension 2.

1. Definitions and conjectures

1.1. **Gopakumar-Vafa invariants.** Let X be a holomorphic symplectic 4-fold and $\overline{M}_{g,l}(X, \beta)$ be the moduli stack of genus g, l-pointed stable maps to X with non-zero curve class β. Its virtual class [BF, LT] vanishes due to a trivial factor in the obstruction sheaf. By Kiem-Li's theory of cosection localization [KiL], one can define a (reduced) virtual class [1]

$$[\overline{M}_{g,l}(X, \beta)]^{\mathrm{vir}} \in A_{2-g+l}(\overline{M}_{g,l}(X, \beta)).$$

For integral classes

(1.1) $$\gamma_i \in H^{m_i}(X, \mathbb{Z}), \ 1 \leqslant i \leqslant l,$$

the (primary) Gromov-Witten invariant is defined by

(1.2) $$\mathrm{GW}_{g,\beta}(\gamma_1, \ldots, \gamma_l) = \int_{[\overline{M}_{g,l}(X,\beta)]^{\mathrm{vir}}} \prod_{i=1}^l \mathrm{ev}_i^*(\gamma_i) \in \mathbb{Q},$$

where $\mathrm{ev}_i \colon \overline{M}_{g,l}(X, \beta) \to X$ is the i-th evaluation map.

When $g = 0$, the virtual dimension of $\overline{M}_{0,l}(X, \beta)$ is $l + 2$, and (1.2) is zero unless

(1.3) $$\sum_{i=1}^l (m_i - 2) = 4.$$

Similar to the case of Calabi-Yau 4-folds and 5-folds [KP, PZ], we make the following definition:

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[1]The virtual class mentioned in this paper is always assumed to be the reduced one.

Definition 1.1. ([COT22, Def. 1.5]) *For any $\gamma_1, \ldots, \gamma_l \in H^*(X, \mathbb{Z})$, we define the genus 0 Gopakumar-Vafa invariant $n_{0,\beta}(\gamma_1, \ldots, \gamma_l) \in \mathbb{Q}$ recursively by the multiple cover formula:*

$$\mathrm{GW}_{0,\beta}(\gamma_1, \ldots, \gamma_l) = \sum_{k \geqslant 1, k | \beta} k^{n-3} n_{0,\beta/k}(\gamma_1, \ldots, \gamma_l).$$

When $g = 1$, the virtual dimension of $\overline{M}_{1,l}(X, \beta)$ is $l + 1$, and (1.2) is zero unless

$$(1.4) \qquad\qquad \sum_{i=1}^{l} (m_i - 2) = 2.$$

In this paper, we concentrate on the case when $l = 1$ and $m_1 = 4$. Because curves in imprimitive curve classes are very difficult to control, we restrict hereby to the case of a primitive curve class.

Definition 1.2. ([COT22, Def. 1.6]) *Assume that $\beta \in H_2(X, \mathbb{Z})$ is primitive. For any $\gamma \in H^4(X, \mathbb{Z})$, we define the genus 1 Gopakumar-Vafa invariant $n_{1,\beta}(\gamma) \in \mathbb{Q}$ by*

$$\mathrm{GW}_{1,\beta}(\gamma) = n_{1,\beta}(\gamma) - \frac{1}{24} \mathrm{GW}_{0,\beta}(\gamma, c_2(X)),$$

where $c_2(X)$ is the second Chern class of T_X.

When $g = 2$, the virtual dimension of $\overline{M}_{2,0}(X, \beta)$ is zero, so we can consider (1.2) without insertions:

$$\mathrm{GW}_{2,\beta} := \int_{[\overline{M}_{2,0}(X,\beta)]^{\mathrm{vir}}} 1 \in \mathbb{Q}.$$

Definition 1.3. ([COT22, Def. 1.7]) *Assume that $\beta \in H_2(X, \mathbb{Z})$ is primitive. We define the genus 2 Gopakumar-Vafa invariant $n_{2,\beta} \in \mathbb{Q}$ by*

$$\mathrm{GW}_{2,\beta} = n_{2,\beta} - \frac{1}{24} n_{1,\beta}(c_2(X)) + \frac{1}{2 \cdot 24^2} \mathrm{GW}_{0,\beta}(c_2(X), c_2(X)) + \frac{1}{24} N_{\mathrm{nodal},\beta}.$$

Here $n_{1,\beta}(-)$ is given in Definition 1.2 and $N_{\mathrm{nodal},\beta} \in \mathbb{Q}$ is the virtual count of rational nodal curves [NO] as defined by

$$(1.5) \qquad N_{\mathrm{nodal},\beta} := \frac{1}{2} \left[\int_{[\overline{M}_{0,2}(X,\beta)]^{\mathrm{vir}}} (\mathrm{ev}_1 \times \mathrm{ev}_2)^*(\Delta_X) - \int_{[\overline{M}_{0,1}(X,\beta)]^{\mathrm{vir}}} \frac{\mathrm{ev}_1^*(c(X))}{1 - \psi_1} \right],$$

where

- *$\Delta_X \in H^8(X \times X)$ is the class of the diagonal, and*
- *$c(X) = 1 + c_2(X) + c_4(X)$ is the total Chern class of T_X.*

1.2. **Z_t-stable pair invariants.** Let ω be an ample divisor on X and $t \in \mathbb{R}$, we recall the following notion of Z_t-stable pairs.

Definition 1.4. ([CT19, Lem 1.7]) *Let F be a one dimensional coherent sheaf and $s : \mathcal{O}_X \to F$ be a section. For an ample divisor ω, we denote the slope function by $\mu(F) = \chi(F)/(\omega \cdot [F])$.*
We say (F, s) is a Z_t-(semi)stable pair ($t \in \mathbb{R}$) if

(i) *for any subsheaf $0 \neq F' \subseteq F$, we have $\mu(F') < (\leqslant)t$,*
(ii) *for any subsheaf $F' \subsetneq F$ such that s factors through F', we have $\mu(F/F') > (\geqslant)t$.*

There are two distinguished stability conditions appearing as special cases of Z_t-stability.

Definition 1.5. ([PT09], [CT19, Def. 1.10])
(i) A pair (F, s) is a PT stable pair if F is a pure one dimensional sheaf and s is surjective in dimension one.
(ii) A pair (F, s) is a JS stable pair if s is a non-zero morphism, F is μ-semistable and for any subsheaf $0 \neq F' \subsetneq F$ such that s factors through F' we have $\mu(F') < \mu(F)$.

Proposition 1.6. ([CT19, Prop. 1.11]) *For a pair (F, s) with $[F] = \beta$ and $\chi(F) = n$, its*
(i) Z_t-stability with $t \to \infty$ is exactly PT stability,
(ii) Z_t-stability with $t = \frac{n}{\omega \cdot \beta} + 0^+$ is exactly JS stability.

For $\beta \in H_2(X, \mathbb{Z})$ and $n \in \mathbb{Z}$, we denote by

$$P_n^t(X, \beta) \subseteq \mathcal{P}_n^t(X, \beta)$$

the moduli stack of Z_t-stable (semistable) pairs (F, s) with $[F] = \beta$ and $\chi(F) = n$.

By Proposition 1.6, there are two disinguished moduli spaces, PT moduli spaces and JS moduli spaces, by specializing $t \to \infty$ and $t = \frac{n}{\omega \cdot \beta} + 0^+$ respectively:

$$P_n(X, \beta) := P_n^{t \to \infty}(X, \beta), \quad P_n^{\mathrm{JS}}(X, \beta) := P_n^{t = \frac{n}{\omega \cdot \beta} + 0^+}(X, \beta).$$

By a GIT construction, $P_n^t(X, \beta)$ is a quasi-projective scheme, and $\mathcal{P}_n^t(X, \beta)$ admits a good moduli space

$$\mathcal{P}_n^t(X, \beta) \to \overline{P}_n^t(X, \beta),$$

where $\overline{P}_n^t(X, \beta)$ is a projective scheme which parametrizes Z_t-polystable objects. The following result shows that moduli stacks of Z_t-stable pairs are indeed open substacks of moduli stacks of objects in the derived categories of coherent sheaves.

Theorem 1.7. ([CT19, Thm. 0.1]) $P_n^t(X, \beta)$ admits an open immersion

$$P_n^t(X, \beta) \to \mathcal{M}_0, \quad (F, s) \mapsto (\mathcal{O}_X \xrightarrow{s} F)$$

to the moduli stack $\mathcal{M}_0$ of $E \in D^b \operatorname{Coh}(X)$ with $\operatorname{Ext}^{<0}(E, E) = 0$ and $\det(E) \cong \mathcal{O}_X$.

Therefore for a general choice of t (i.e. outside a finite subset of rational numbers in $\mathbb{R}$), $P_n^t(X, \beta)$ is a projective scheme which can be given a (-2)-shifted symplectic derived scheme structure [PTVV] and has a virtual class [BJ, OT] (see also [CL14]).

Parallel to GW theory, the virtual class of $P_n^t(X, \beta)$ vanishes [KiP, Sav]. One can define a reduced virtual class due to Kiem-Park [KiP, Def. 8.7, Lem. 9.4]:

$$(1.6) \qquad [P_n^t(X, \beta)]^{\mathrm{vir}} \in A_{n+1}(P_n^t(X, \beta), \mathbb{Q}),$$

depending on the choice of orientation [CGJ, CL17]. To define its counting invariants, let

$$(1.7) \qquad \tau : H^m(X, \mathbb{Z}) \to H^{m-2}(P_n^t(X, \beta), \mathbb{Z}),$$

$$\tau(\gamma) := \pi_{P*}\left(\pi_X^* \gamma \cup \operatorname{ch}_3(\mathbb{F})\right),$$

where $\mathbb{I} = (\mathcal{O} \to \mathbb{F})$ is the universal Z_t-stable pair and π_P, π_X are projections from $P_n^t(X, \beta) \times X$ onto its factors.

Definition 1.8. Let $t \in \mathbb{R}$ be generic and $\gamma_i \in H^{m_i}(X, \mathbb{Z})$ $(1 \leqslant i \leqslant l)$. The Z_t-stable pair invariants are

$$P_{n, \beta}^t(\gamma_1, \ldots, \gamma_l) := \int_{[P_n^t(X, \beta)]^{\mathrm{vir}}} \prod_{i=1}^l \tau(\gamma_i) \in \mathbb{Q}.$$

When $n = -1$, we write

$$P_{-1, \beta}^t := \int_{[P_{-1}^t(X, \beta)]^{\mathrm{vir}}} 1.$$

In PT and JS stabilites, we also write

$$P_{n, \beta}(\gamma_1, \ldots, \gamma_l) := P_{n, \beta}^{t \to \infty}(\gamma_1, \ldots, \gamma_l), \quad P_{n, \beta}^{\mathrm{JS}}(\gamma_1, \ldots, \gamma_l) := P_{n, \beta}^{t = \frac{n}{\omega \cdot \beta} + 0^+}(\gamma_1, \ldots, \gamma_l).$$

Remark 1.9. By Definition 1.4 and a dimension counting, Z_t-stable pair invariants are non-zero only if both of the following conditions hold:

$$t > \frac{n}{\omega \cdot \beta}, \quad \sum_{i=1}^l (m_i - 2) = 2n + 2.$$

In [CMT19, CT19], similar invariants are used to give sheaf theoretic interpretations of Gopakumar-Vafa type invariants for ordinary Calabi-Yau 4-folds [KP]. Below, we give a parallel proposal for holomorphic symplectic 4-folds using Definition 1.8.

1.3. **Conjecture.** We state the main conjecture of this paper.

Conjecture 1.10. Let X be a holomorphic symplectic 4-fold with an ample divisor ω. Fix $n \in \mathbb{Z}$ and $\beta \in H_2(X, \mathbb{Z})$ and let $t > \frac{n}{\omega \cdot \beta}$ be generic. For certain choice of orientation, we have

(1) If $n \geqslant 2$, then

$$P_{n, \beta}^t(\gamma_1, \ldots, \gamma_l) = 0.$$

(2) If $n = 1$, then

$$P_{1, \beta}^t(\gamma_1, \ldots, \gamma_l) = n_{0, \beta}(\gamma_1, \ldots, \gamma_l) \in \mathbb{Z}.$$

(3) *If $n = 0$ and β is primitive, then*

$$P_{0,\beta}^t(\gamma) = n_{1,\beta}(\gamma) \in \mathbb{Z}.$$

(4) *If $n = -1$ and β is primitive, then*

$$P_{-1,\beta}^t = n_{2,\beta} \in \mathbb{Z}.$$

Remark 1.11. *By the global Torelli theorem* [Ver, Huy], *primitive curve classes on irreducible hyperkähler varieties can be deformed to irreducible curve classes. Therefore Z_t-stable pair invariants are independent of the choice of $t > \frac{n}{\omega \cdot \beta}$ for such cases by* [CT19, Prop. 1.12].

Remark 1.12. *Our conjecture implies that there is no nontrivial wall-crossing for Z_t-stable pairs invariants when $t > \frac{n}{\omega \cdot \beta}$, contrary to the ordinary CY_4 case* [CT19, CT20b, CT20c].

Remark 1.13. *Similarly to* [CK19, Conj. 0.3], *we may use counting invariants on Hilbert schemes $I_n(X, \beta)$ of curves to give a sheaf theoretic interpretation of Gopakumar-Vafa invariants in which case zero dimensional subschemes* [CK18] *(conjecturally) will not contribute, i.e. "DT = PT". It is curious whether one can do a K-theoretic refinement as* [CKM19].

1.4. **Heuristic argument.** In this section, we verify Conjecture 1.10 using heuristic argument in an ideal geometry (ref. [COT22, §1.4, §1.5]). To be specific, as the virtual dimension of $\overline{M}_{g,0}(X, \beta)$ is $2 - g$, we assume that:

Any genus g curve moves in a smooth compact $(2 - g)$-dimensional family.

In particular, there are no curves of genus $g \geqslant 3$. Unfortunately, complicated phenomena still arise even in the ideal case, for example, one can have two (resp. one) dimensional families of reducible rational (resp. elliptic) curves, and any member of a rational curve family is expected to intersect nontrivially with some member in the same family (see [COT22, §1.4] for details).

However, things will be simplified if we make the following additional assumptions:

- X is irreducible hyperkähler,
- the effective curve class $\beta \in H_2(X, \mathbb{Z})$ is primitive,

By the global Torelli for (irreducible) hyperkähler varieties [Ver, Huy], the pair (X, β) is deformation equivalent (through a deformation with keeps β of Hodge type) to a pair (X', β'), where $\beta' \in H_2(X, \mathbb{Z})$ is irreducible, so we may without loss of generality assume:

- the effective curve class $\beta \in H_2(X, \mathbb{Z})$ is irreducible.

Under these assumptions, our ideal geometry of curves simplifies to the following form:

(1) The rational curves in X of class β move in a proper 2-dimensional smooth family of embedded irreducible rational curves. Except for a finite number of rational nodal curves, the rational curves are smooth, with normal bundle $\mathcal{O}_{\mathbb{P}^1} \oplus \mathcal{O}_{\mathbb{P}^1} \oplus \mathcal{O}_{\mathbb{P}^1}(-2)$.

(2) The arithmetic genus 1 curves in X of class β move in a proper 1-dimensional smooth family of embedded irreducible genus 1 curves. Except for a finite number of rational nodal curves, the genus one curves are smooth elliptic curves with normal bundle $L \oplus L^{-1} \oplus \mathcal{O}$, where L is a generic degree zero line bundle.

(3) All genus two curves are smooth and rigid.

(4) There are no curves of genus $g \geqslant 3$.

We need to compute Z_t-stable pair invariants in this ideal setting. The key heuristic we use is that only Z_t-stable pairs with *connected support* will 'contribute' to our invariants.

The observation is that for a Z_t-stable pair $I = (\mathcal{O}_X \to F)$ which is supported on a disconnected curve $C = C_1 \sqcup C_2$, we may write

$$I = I_1 \oplus I_2, \quad I_1 = (\mathcal{O}_X \to F_1), \ I_2 = (\mathcal{O}_X \to F_2),$$

where I_i is supported on C_i $(i = 1, 2)$. Then the obstruction space satisfies

$$\mathrm{Ext}^2(I, I)_0 = \mathrm{Ext}^2(I_1, I_1)_0 \oplus \mathrm{Ext}^2(I_2, I_2)_0.$$

Therefore the surjective isotropic cosections (see [KiP, Lem. 9.4]) of obstruction spaces in the RHS give rise to a (mutually orthogonal) two dimensional isotropic cosection in the LHS. Heuristically speaking, such Z_t-stable pairs will not 'contribute' to the reduced virtual class as the reduced obstruction space still have a surjective isotropic cosection.

By Definition 1.8 and above discussion, Z_t-stable pair invariants

$$P_{n,\beta}^t(\gamma_1, \dots, \gamma_l) = \int_{[P_n^t(X,\beta)]^{\mathrm{vir}}} \prod_{i=1}^l \tau(\gamma_i)$$

count Z_t-stable pairs whose support are connected and incident to cycles dual to $\gamma_1, \ldots, \gamma_l$. Say such an incident Z_t-stable pair is supported on a $(2-g)$-dimensional family:

$$p : \mathcal{C}_\beta^g \to S_\beta^g$$

of genus g curves ($g = 0, 1, 2$), where $\mathcal{C}_\beta^g$ is the total space of this family. Each cycle γ_i will cut down real dimension of S_β^g by $\deg(\gamma_i) - 2$. As we have

$$\sum_{i=1}^l (\deg(\gamma_i) - 2) = 2n + 2,$$

so all insertions in total cut down real dimension of S_β^g by $2n + 2$.

The case $n \geqslant 1$. When $n \geqslant 2$, the dimension cut down by insertions is bigger than the largest possible dimension of S_β^g, so there can not be such incident stable pairs and

$$P_{n \geqslant 2, \beta}^t(\gamma_1, \ldots, \gamma_l) = 0.$$

This confirms Conjecture 1.8 (1).

When $n = 1$, insertions cut down real dimension of S_β^g by 4, so any incident Z_t-stable pair $I = (\mathcal{O}_X \to F)$ can only be supported on genus 0 family. As in [CT19, §4.1], by Harder-Narasimhan and Jordan-Hölder filtration, we know

$$F \cong \mathcal{O}_C,$$

for some rational curve C in S_β^0. Therefore incident Z_t-stable pairs (with $\chi = 1$) are in one to one correspondence with intersection points of $\mathcal{C}_\beta^0$ with cycles dual to $\gamma_1, \ldots, \gamma_l$ and

$$P_{1,\beta}^t(\gamma_1, \ldots, \gamma_l) = \int_{S_\beta^0} \prod_{i=1}^l p_*(f^*\gamma_i),$$

where $f : \mathcal{C}_\beta^0 \to X$ is the evaluation map. Therefore Conjecture 1.8 (2) is confirmed in this ideal case as both sides are (virtually) enumerating rational curves of class β incident to cycles dual to $\gamma_1, \ldots, \gamma_l$.

The case $n = 0$. Since Z_t-stable pairs $I = (\mathcal{O}_X \to F)$ supported on genus 0 curves satisfy $\chi(F) > 0$ and a 4-cycle $\gamma \in H^4(X)$ misses genus 2 curves in general position, so when $[F] = \beta$ is irreducible, the pair must be scheme theoretically supported on an elliptic curve C and

$$I = (\mathcal{O}_X \twoheadrightarrow \mathcal{O}_C \xrightarrow{s} L),$$

where L is a line bundle on C with $\chi(C, L) = 0$. By Z_t-stability, s is non-trivial, so s must be an isomorphism by the stability of line bundles. Therefore incident Z_t-stable pairs (with $\chi = 0$) are in one to one correspondence with intersection points of 4-cycle γ with genus 1 curve family $\mathcal{C}_\beta^1$ of class β and

$$P_{0,\beta}^t(\gamma) = \int_{\mathcal{C}_\beta^1} f^*\gamma,$$

where $f : \mathcal{C}_\beta^1 \to X$ is the evaluation map. Therefore Conjecture 1.8 (3) is confirmed in this ideal setting as both sides are (virtually) enumerating elliptic curves of class β incident to γ.

The case $n = -1$. Any Z_t-stable pair $I = (\mathcal{O}_X \to F)$ with irreducible curve class $[F] = \beta$ is scheme theoretically supported on a smooth rigid genus 2 curve C:

$$I = (\mathcal{O}_X \twoheadrightarrow \mathcal{O}_C \xrightarrow{s} L),$$

where L is a line bundle on C with $\chi(C, L) = -1$. As above, by Z_t-stability, s must be an isomorphism. Hence $P_{-1}^t(X, \beta)$ is identified with the set of all rigid genus 2 curves in class β in the ideal geometry, whose count gives exactly genus 2 Gopakumar-Vafa invariant $n_{2,\beta}$. Therefore Conjecture 1.8 (4) is confirmed in the ideal setting.

1.5. **Speculations for general curve classes.** For a smooth projective Calabi-Yau 4-fold X and $\gamma \in H^4(X, \mathbb{Z})$, we have genus 0, 1 Gopakumar-Vafa type invariants $n_{0,\beta}(\gamma), n_{1,\beta} \in \mathbb{Q}$ defined by Klemm and Pandharipande from Gromov-Witten theory [KP] and stable pair invariants $P_{n,\beta}(\gamma) \in \mathbb{Z}$ [CMT19]. They are related by the following conjectural formula [CMT19, §1.7]:

$$(1.8) \qquad \sum_{n,\beta} \frac{P_{n,\beta}(\gamma)}{n!} y^n q^\beta = \prod_{\beta>0} \left(\exp(yq^\beta)^{n_{0,\beta}(\gamma)} \cdot M(q^\beta)^{n_{1,\beta}} \right),$$

where $M(q) = \prod_{k \geqslant 1}(1-q^k)^{-k}$ is the MacMahon function.

By taking logarithmic differentiation with respect to y, we obtain

$$y\frac{d}{dy} \log \left(\sum_{n,\beta} \frac{P_{n,\beta}(\gamma)}{n!} y^n q^\beta \right) = \sum_{\beta>0} y\frac{d}{dy} n_{0,\beta}(\gamma) yq^\beta = \sum_{\beta>0} n_{0,\beta}(\gamma) yq^\beta.$$

If we view it as an equality for corresponding reduced invariants on holomorphic symplectic 4-folds, it surprisingly recovers Conjecture 1.10 (i), (ii) (i.e. the genus zero part).

We do similar manipulations for genus one invariants. Note that $y^0 q^\beta$ parts of (1.8) are

$$\sum_\beta P_{0,\beta} q^\beta = \prod_{\beta>0} M(q^\beta)^{n_{1,\beta}}.$$

This equality is written down by a computation in the "CY$_4$ ideal geometry" (ref. [CMT19, §2.5]), where rational curves contribute zero and each super-rigid elliptic curve (on an ideal CY$_4$) in class β contributes by $M(q^\beta)$ (ref. [CMT19, Thm. 5.10]). Taking logarithmic differentiation with respect to q:

$$q\frac{d}{dq} \log \left(M(q) \right) = \sum_{d \geqslant 1} q^d \sum_{i \geqslant 1, i|d} i^2.$$

We then wonder whether in the holomorphic symplectic 4-folds setting, each ideal elliptic curve family in class β contributes to $P_{0,d\beta}(\gamma)$ by

$$\sum_{i \geqslant 1, i|d} i^2.$$

Summing over all elliptic curve families, this would imply

$$(1.9) \qquad P_{0,\beta}(\gamma) = \sum_{d \geqslant 1, d|\beta} n_{1,\beta/d}(\gamma) \sum_{i \geqslant 1, i|d} i^2.$$

It is quite curious whether the above formula gives the correct PT/GV correspondence. For multiple fiber classes of elliptic fibrations, our computations show the formula seems correct (see Theorem 2.17, 2.19, Remark 2.20). As for $P_{-1,\beta}$ and genus 2 Gopakumar-Vafa invariants, we haven't found analogous formula for general curve classes.

2. PRODUCT OF $K3$ SURFACES

In this section, we consider the product of two $K3$ surfaces:

$$X = S \times T, \quad \text{with } \beta \in H_2(S, \mathbb{Z}) \subseteq H_2(X, \mathbb{Z}).$$

As observed in [COT22, §5], this contains all interesting curve classes on X because if $\beta \in H_2(X, \mathbb{Z})$ is of non-trivial degree over both S and T, one can construct two linearly independent cosections, which imply that reduced Gromov-Witten invariants of X in this class vanish.

2.1. **Gopakumar-Vafa invariants.** Recall Gopakumar-Vafa invariants specified in Definitions 1.1, 1.2, 1.3. They are computed in [COT22, Prop. 5.1] as follows: write $\gamma, \gamma' \in H^4(X)$ as

$$\gamma = A_1 \cdot 1 \otimes \mathsf{p} + D_1 \otimes D_2 + A_2 \cdot \mathsf{p} \otimes 1,$$
$$\gamma' = A_1' \cdot 1 \otimes \mathsf{p} + D_1' \otimes D_2' + A_2' \cdot \mathsf{p} \otimes 1,$$

based on Künneth decomposition:

$$H^4(X) \cong (H^0(S) \otimes H^4(T)) \oplus (H^2(S) \otimes H^2(T)) \oplus (H^4(S) \otimes H^0(T)).$$

Fix also a curve class

$$\alpha = \theta_1 \otimes \mathsf{p} + \mathsf{p} \otimes \theta_2 \in (H^6(X) \cong H^2(S)) \otimes (H^4(T) \oplus H^4(S) \otimes H^2(T)).$$

Proposition 2.1. ([COT22, Prop. 5.1]) *For $\beta \in H_2(S, \mathbb{Z}) \subseteq H_2(X, \mathbb{Z})$, we have*

$$n_{0,\beta}(\gamma, \gamma') = (D_1 \cdot \beta) \cdot (D_1' \cdot \beta) \cdot \int_T (D_2 \cdot D_2') \cdot N_0 \left(\frac{\beta^2}{2} \right),$$

$$n_{0,\beta}(\alpha) = (\theta_1 \cdot \beta) N_0 \left(\frac{\beta^2}{2} \right).$$

If β is primitive, we have

$$n_{1,\beta}(\gamma) = 24 A_2 N_1 \left(\frac{\beta^2}{2} \right), \quad n_{2,\beta} = N_2 \left(\frac{\beta^2}{2} \right),$$

where

$$(2.1) \qquad \sum_{l \in \mathbb{Z}} N_0(l) q^l = \frac{1}{q} \prod_{n \geqslant 1} \frac{1}{(1 - q^n)^{24}},$$

$$(2.2) \qquad \sum_{l \in \mathbb{Z}} N_1(l) q^l = \left(\frac{1}{q} \prod_{n \geqslant 1} \frac{1}{(1 - q^n)^{24}} \right) \left(q \frac{d}{dq} G_2(q) \right),$$

$$(2.3) \qquad \sum_{l \in \mathbb{Z}} N_2(l) q^l = \left(\frac{1}{q} \prod_{n \geqslant 1} \frac{1}{(1 - q^n)^{24}} \right) \left(24 q \frac{d}{dq} G_2 - 24 G_2 - 1 \right),$$

with Eisenstein series:

$$G_2(q) = -\frac{1}{24} + \sum_{n \geqslant 1} \sum_{d | n} d q^n.$$

2.2. **Moduli spaces of Z_t-stable pairs.** For a point $t \in T$, let $i_t \colon S \to S \times \{t\} \hookrightarrow X$ be the inclusion. Consider the pushforward map

$$(2.4) \qquad i_* : P_n^t(S, \beta) \times T \to P_n^t(X, \beta),$$

$$(\mathcal{O}_S \xrightarrow{s} F, t) \mapsto (\mathcal{O}_X \twoheadrightarrow i_{t*}\mathcal{O}_S \xrightarrow{i_{t*}s} i_{t*}F),$$

where $P_n^t(S, \beta)$ is the moduli space of Z_t-stable pairs (F, s) on S with $[F] = \beta$ and $\chi(F) = n$.
   We restrict to the following setting.

Setting 2.2. *We consider the case when the following conditions are satisfied:*

(1) *The map (2.4) is an isomorphism and $P_n^t(S, \beta)$ is smooth of dimension $\beta^2 + n + 1$.*
(2) *There is a well-defined forgetful map*

$$f : P_n^t(S, \beta) \to M_n(S, \beta), \quad (\mathcal{O}_S \to F) \mapsto F,$$

to the coarse moduli scheme $M_n(S, \beta)$ of one dimensional stable sheaves F on S with $[F] = \beta$ and $\chi(F) = n$.

Proposition 2.3. *Setting 2.2 is satisfied when β is irreducible.*

Proof. When β is irreducible, $P_n^t(X, \beta)$ is independent of the choice of $t > \frac{n}{\omega \cdot \beta}$ [CT19, Prop. 1.12], so we can set $t \to \infty$ and work with PT stability. The isomorphism follows from similar argument as [CMT19, Prop. 3.11]. The key point is that for any such Z_t-stable pair (F, s), F is stable and therefore scheme theoretically supported on $S \times \{t\}$ for some $t \in T$ ([CMT18, Lem. 2.2]). The smoothness of $P_n^t(S, \beta)$ follows from [KY], [PT10, Prop. C.2]. $\qquad \square$

2.3. **Virtual classes.** We determine the virtual class of $P_n^t(X, \beta)$ in Setting 2.2. Firstly recall:

Definition 2.4. ([Sw, Ex. 16.52, pp. 410], [EG, Lem. 5]) *Let E be a $\mathrm{SO}(2n, \mathbb{C})$-bundle with a non-degenerate symmetric bilinear form Q on a connected scheme M. Denote E_+ to be its positive real form[2]. The half Euler class of (E, Q) is*

$$e^{\frac{1}{2}}(E, Q) := \pm e(E_+) \in H^{2n}(M, \mathbb{Z}),$$

where the sign depends on the choice of orientation of E_+.

---

[2]This means a real half dimensional subbundle such that Q is real and positive definite on it. By homotopy equivalence $\mathrm{SO}(m, \mathbb{C}) \sim \mathrm{SO}(m, \mathbb{R})$, it exists and is unique up to isomorphisms.

Definition 2.5. ([EG], [KiP, Def. 8.7]) *Let E be a $SO(2n, \mathbb{C})$-bundle with a non-degenerate symmetric bilinear form Q on a connected scheme M. An isotropic cosection of (E, Q) is a map*

$$\phi : E \to \mathcal{O}_M,$$

such that the composition

$$\phi \circ \phi^{\vee} : \mathcal{O}_M \to E^{\vee} \overset{Q}{\cong} E \to \mathcal{O}_M$$

is zero. If ϕ is furthermore surjective, we define the (reduced) half Euler class:

$$e_{\mathrm{red}}^{\frac{1}{2}}(E, Q) := e^{\frac{1}{2}}\left((\phi^{\vee}\mathcal{O}_M)^{\perp}/(\phi^{\vee}\mathcal{O}_M), \bar{Q}\right) \in H^{2n-2}(M, \mathbb{Z}),$$

as the half Euler class of the isotropic reduction. Here $\bar{Q}$ denotes the induced non-degenerate symmetric bilinear form on $(\phi^{\vee}\mathcal{O}_M)^{\perp}/(\phi^{\vee}\mathcal{O}_M)$.

We show reduced half Euler classes are independent of the choice of surjective isotropic cosection.

Lemma 2.6. ([COT22, Lem. 5.5]) *Let E be a $SO(2n, \mathbb{C})$-bundle with a non-degenerate symmetric bilinear form Q on a connected scheme M and*

$$\phi : E \to \mathcal{O}_M$$

be a surjective isotropic cosection. Then we can write the positive real form E_+ of E as

$$E_+ = \mathcal{E}_+ \oplus \underline{\mathbb{R}}^2$$

such that

$$e_{\mathrm{red}}^{\frac{1}{2}}(E, Q) = \pm e(\mathcal{E}_+).$$

Moreover, it is independent of the choice of surjective cosection.

In particular, when $E = \mathcal{O}^{\oplus 2} \oplus V$ such that $Q = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \oplus Q|_V$, we have

$$e_{\mathrm{red}}^{\frac{1}{2}}(E, Q) = \pm e^{\frac{1}{2}}(V, Q|_V).$$

Recall a $Sp(2r, \mathbb{C})$-bundle (or symplectic vector bundle) is a complex vector bundle of rank $2r$ with a non-degenerate anti-symmetric bilinear form. One class of quadratic vector bundles is given by tensor product of two symplectic vector bundles V_1, V_2. Their half Euler classes can be computed using Chern classes of V_1, V_2. For our purpose, we restrict to the following case.

Lemma 2.7. ([COT22, Lem. 5.6]) *Let (V_1, ω_1), (V_2, ω_2) be a $Sp(2r, \mathbb{C})$ (resp. $Sp(2, \mathbb{C})$-bundle) on a connected scheme M. Then*

$$(V_1 \otimes V_2, \omega_1 \otimes \omega_2)$$

defines a $SO(4r, \mathbb{C})$-bundle whose half Euler class satisfies

$$e^{\frac{1}{2}}(V_1 \otimes V_2, \omega_1 \otimes \omega_2) = \pm \left(e(V_1) - c_{2r-2}(V_1) \cdot e(V_2)\right).$$

We determine the (reduced) virtual class of $P_n^t(X, \beta)$.

Theorem 2.8. *In Setting 2.2, for certain choice of orientation, we have*

$$(2.5) \quad [P_n^t(X, \beta)]^{\mathrm{vir}} = \left([P_n^t(S, \beta)] \cap f^* e(T_{M_n(S, \beta)})\right) \times [T] - e(T) \left([P_n^t(S, \beta)] \cap f^* c_{\beta^2}(T_{M_n(S, \beta)})\right),$$

where $f : P_n^t(S, \beta) \to M_n(S, \beta)$ is the map as in Setting 2.2.

Proof. The proof is similar as [CMT19, Prop. 4.7]. Under the isomorphism (2.4):

$$P_n^t(S, \beta) \times T \cong P_n^t(X, \beta),$$

the universal stable pair $\mathbb{I}_X = (\mathcal{O} \to \mathbb{F}_X)$ of $P_n^t(X, \beta)$ satisfies

$$(2.6) \quad\quad\quad\quad\quad\quad \mathbb{F}_X = \mathbb{F}_S \boxtimes \mathcal{O}_{\Delta_T},$$

where $\mathbb{I}_S = (\mathcal{O} \to \mathbb{F}_S)$ is the universal stable pair of $P_n^t(S, \beta)$ and Δ_T denotes the diagonal.

As in [CMT19, Eqn. (29)], we have a distinguished triangle

$$(2.7) \quad\quad \mathbf{R}\mathcal{H}om_{\pi_{P_X}}(\mathbb{I}_X, \mathbb{F}_X) \to \mathbf{R}\mathcal{H}om_{\pi_{P_X}}(\mathbb{I}_X, \mathbb{I}_X)_0[1] \to \mathbf{R}\mathcal{H}om_{\pi_{P_X}}(\mathbb{F}_X, \mathcal{O})[2],$$

where $\pi_{P_X} : P_n^t(X, \beta) \times X \to P_n^t(X, \beta)$ is the projection.

From stable pair $\mathbb{I}_X = (\mathcal{O} \to \mathbb{F}_X)$ and Eqn. (2.6), we get a distinguished triangle

$$(2.8) \quad \mathbf{R}\mathcal{H}om_{\pi_{P_X}}(\mathbb{F}_S \boxtimes \mathcal{O}_{\Delta_T}, \mathbb{F}_S \boxtimes \mathcal{O}_{\Delta_T}) \to \mathbf{R}\mathcal{H}om_{\pi_{P_X}}(\mathcal{O}, \mathbb{F}_S \boxtimes \mathcal{O}_{\Delta_T}) \to \mathbf{R}\mathcal{H}om_{\pi_{P_X}}(\mathbb{I}_X, \mathbb{F}_X).$$

By adjunction, we get an isomorphism

$$(2.9) \quad\quad \mathbf{R}\mathcal{H}om_{\pi_{P_X}}(\mathbb{F}_S \boxtimes \mathcal{O}_{\Delta_T}, \mathbb{F}_S \boxtimes \mathcal{O}_{\Delta_T}) \cong \mathbf{R}\mathcal{H}om_{\pi_{P_S}}(\mathbb{F}_S, \mathbb{F}_S) \boxtimes \wedge^i T_T[-i],$$

where $\pi_{P_S} \colon P_n^t(S, \beta) \times S \to P_n^t(S, \beta)$ is the projection.

Combining (2.8) and (2.9), we obtain

$$(2.10) \qquad \mathbf{R}\mathcal{H}om_{\pi_{P_X}}(\mathbb{I}_X, \mathbb{F}_X) \cong \mathbf{R}\mathcal{H}om_{\pi_{P_S}}(\mathbb{I}_S, \mathbb{F}_S) \oplus \mathbf{R}\mathcal{H}om_{\pi_{P_S}}(\mathbb{F}_S, \mathbb{F}_S) \boxtimes (T_T \oplus \mathcal{O}_T[-1]).$$

Combining (2.7) and (2.10), we obtain

$$\mathcal{E}xt^1_{\pi_{P_X}}(\mathbb{I}_X, \mathbb{I}_X)_0 \cong \mathcal{E}xt^0_{\pi_{P_S}}(\mathbb{I}_S, \mathbb{F}_S) \oplus T_T,$$

and an exact sequence

$$(2.11)$$
$$0 \to \mathcal{E}xt^1_{\pi_{P_S}}(\mathbb{I}_S, \mathbb{F}_S) \oplus \mathcal{E}xt^1_{\pi_{P_S}}(\mathbb{F}_S, \mathbb{F}_S) \boxtimes T_T \oplus \mathcal{E}xt^0_{\pi_{P_S}}(\mathbb{F}_S, \mathbb{F}_S) \boxtimes \mathcal{O}_T \to \mathcal{E}xt^2_{\pi_{P_X}}(\mathbb{I}_X, \mathbb{I}_X)_0 \to \cdots.$$

We claim that the second arrow above is an isomorphism, which can be done by a dimension counting. In fact, let $I = (\mathcal{O}_S \to F) \in P_n^t(S, \beta)$, the cohomology of the distinguished triangle

$$\mathbf{R}\mathrm{Hom}_S(F, F) \to \mathbf{R}\mathrm{Hom}_S(\mathcal{O}_S, F) \to \mathbf{R}\mathrm{Hom}_S(I, F)$$

implies that $\mathrm{Ext}_S^i(I, F) = 0$ for $i \geqslant 2$. In Setting 2.2, we know $\mathrm{ext}_S^0(I, F) = \beta^2 + n + 1$, therefore

$$\mathrm{ext}_S^1(I, F) = 1.$$

As F is stable, we have

$$\mathrm{ext}_S^2(F, F) = \mathrm{ext}_S^0(F, F) = 1, \quad \mathrm{ext}_S^1(F, F) = \beta^2 + 2.$$

So the rank of the second term of (2.11) is $2\beta^2 + 6$. One can easily check the rank of the third term in (2.11) is also $2\beta^2 + 6$ by Riemann-Roch formula and first condition of Setting 2.2. To sum up, we get an isomorphism:

$$\mathcal{E}xt^1_{\pi_{P_S}}(\mathbb{I}_S, \mathbb{F}_S) \oplus \mathcal{E}xt^1_{\pi_{P_S}}(\mathbb{F}_S, \mathbb{F}_S) \boxtimes T_T \oplus \mathcal{E}xt^0_{\pi_{P_S}}(\mathbb{F}_S, \mathbb{F}_S) \boxtimes \mathcal{O}_T \cong \mathcal{E}xt^2_{\pi_{P_X}}(\mathbb{I}_X, \mathbb{I}_X)_0.$$

As in [CMT19, Prop. 4.7], one can show the decomposition in the LHS is also with respect to the Serre duality pairing on $\mathcal{E}xt^2_{\pi_{P_X}}(\mathbb{I}_X, \mathbb{I}_X)_0$. The our claim follows from Lemmata 2.6 and 2.7. $\quad\square$

2.4. **Thom-Porteus formula.** As our insertion (1.7) depends only on the fundamental cycle of the universal sheaf, it is useful to know the pushforward of the virtual class (2.5) under the forgetful map. In this section, let $\beta \in H_2(S, \mathbb{Z})$ be an irreducible curve class, then $P_n^t(X, \beta)$ is independent of the choice of $t > \frac{n}{\omega \cdot \beta}$ [CT19, Prop. 1.12], so we can set $t \to \infty$ and work with PT stability. Consider the forgetful map

$$f : P_n(S, \beta) \to M_n(S, \beta), \quad (\mathcal{O}_S \to F) \mapsto F.$$

Recall that $P_n(S, \beta)$ is smooth of dimension $\beta^2 + n + 1$ and $M_n(S, \beta)$ is smooth of dimension $\beta^2 + 2$. The image of f in $M_n(S, \beta)$ is the locus

$$(2.12) \qquad \{F \in M_n(S, \beta) \,|\, h^0(F) \geqslant 1\},$$

where surjectivity follows since β is irreducible and F is pure, so any non-zero section $s \in H^0(S, F)$ must have zero-dimensional cokernel. The expected dimension of sections is $\chi(F) = n$, so the image is everything if $n = 1$, a divisor if $n = 0$ and a codimension 2 cycle if $n = -1$.

Let $\mathbb{F}_S$ be a (twisted) universal sheaf on $M_n(S, \beta) \times S$. If $n = 1$ (or more generally, there exists a K-theory class pairing with 1 with a sheaf parametrized by $M_n(S, \beta)$) the twisted sheaf can be taken to be an actual sheaf. For us here the difference will not matter, since we are only interested in the Chern character of the universal sheaf, which can also be easily defined in the twisted case. We refer to [M08] for a discussion.

Let $\pi_M : M_n(S, \beta) \times S \to M_n(S, \beta)$ be the projection. We resolve the complex $\mathbf{R}\pi_{M*}(\mathbb{F}_S)$ by a 2-term complex of vector bundles:

$$\mathbf{R}\pi_{M*}(\mathbb{F}_S) \cong (E_0 \xrightarrow{\sigma} E_1).$$

Then (2.12) is the *degeneracy locus*

$$D_1(\sigma) = \{x \in M_n(S, \beta) \,|\, \dim_{\mathbb{C}} \ker(\sigma(x)) \geqslant 1\}.$$

By the *Thom-Porteus formula* [Ful, §14.4] (see [GT, Prop. 1] for a modern treatment and observe that $P_n(S, \beta)$ is precisely what is called $\tilde{D}_1(\sigma)$ there), we get the following:

Proposition 2.9.
$$f_*[P_n(S, \beta)] = c_{1-n}(-\mathbf{R}\pi_{M*}(\mathbb{F}_S)) \cap [M_n(S, \beta)].$$

We can calculate the right hand side above by Grothendieck-Riemann-Roch formula

$$\operatorname{ch}(-\mathbf{R}\pi_{M*}(\mathbb{F}_S)) = -\pi_{M*}(\operatorname{ch}(\mathbb{F}_S) \cdot \pi_S^* \operatorname{td}(S)).$$

We obtain the following:

$$(2.13) \quad \begin{aligned} f_*[P_1(S,\beta)] &= 1, \\ f_*[P_0(S,\beta)] &= -\pi_{M*}(\operatorname{ch}_3(\mathbb{F}_S)) - 2\pi_{M*}(\operatorname{ch}_1(\mathbb{F}_S)\pi_S^*\mathsf{p}), \\ f_*[P_{-1}(S,\beta)] &= \frac{1}{2}\left(c_1(-\mathbf{R}\pi_{M*}(\mathbb{F}_S))\right)^2 + \pi_{M*}(\operatorname{ch}_4(\mathbb{F}_S)) + 2\pi_{M*}(\operatorname{ch}_2(\mathbb{F}_S)\pi_S^*\mathsf{p}), \end{aligned}$$

where we used Poincaré duality on the right to identify homology and cohomology and $\mathsf{p} \in H^4(S)$ denotes the point class. A small calculation shows that the right hand side is indeed independent of the choice of universal family $\mathbb{F}$ (i.e. the formulae stay invariant under replacing $\mathbb{F}$ by $\mathbb{F}\otimes\pi_M^*\mathcal{L}$ for $\mathcal{L} \in \operatorname{Pic}(M_n(S,\beta))$). This will be useful later on.

2.5. **Genus 0 in irreducible classes.** In this section, we prove Conjecture 1.10 (1), (2) for irreducible curve classes. We first recall a result of Fujiki [Fuji] and its generalization in [GHJ, Cor. 23.17].

Theorem 2.10. ([Fuji], [GHJ, Cor. 23.17]) *Let M be a hyperkähler variety of dimension $2n$. Assume $\alpha \in H^{4j}(M,\mathbb{C})$ is of type $(2j,2j)$ on all small deformation of M. Then there exists a constant $C(\alpha) \in \mathbb{C}$ depending only on α (called Fujiki constant of α) such that*

$$\int_M \alpha \cdot \beta^{2n-2j} = C(\alpha) \cdot q_M(\beta)^{n-j}, \quad \forall\, \beta \in H^2(M,\mathbb{C}),$$

where $q_M : H^2(M,\mathbb{C}) \to \mathbb{C}$ denotes the Beauville-Bogomolov-Fujiki form.

Theorem 2.11. *Let $X = S \times T$ and $\beta \in H_2(S,\mathbb{Z}) \subseteq H_2(X,\mathbb{Z})$ be an irreducible curve class. Then Conjecture 1.10 (1), (2) hold.*

Proof. By Proposition 2.3, we have a forgetful map

$$\bar{f} = (f, \operatorname{id}_T) : P_n(X,\beta) = P_n(S,\beta) \times T \to M_n(S,\beta) \times T.$$

As our insertion (1.7) only involves fundamental cycle of the universal one dimensional sheaf, so it is the pullback $\bar{f}^*$ of a cohomology class from $M_n(S,\beta) \times T$.

When $n > 1$, we have

$$\dim_{\mathbb{C}} P_n(S,\beta) = \beta^2 + n + 1 > \beta^2 + 2 = \dim_{\mathbb{C}} M_n(S,\beta).$$

By Theorem 2.8 and Proposition 2.9, it is easy to see

$$P_{n,\beta}(\gamma_1, \ldots, \gamma_l) = 0, \quad n > 1.$$

When $n = 1$, we take insertion $\gamma, \gamma' \in H^4(X)$ for example (other cases follow from easier versions of the same argument). Based on Künneth decomposition:

$$H^4(X) \cong (H^0(S) \otimes H^4(T)) \oplus (H^2(S) \otimes H^2(T)) \oplus (H^4(S) \otimes H^0(T)),$$

we write

$$\begin{aligned} \gamma &= A_1 \cdot 1 \otimes \mathsf{p} + D_1 \otimes D_2 + A_2 \cdot \mathsf{p} \otimes 1, \\ \gamma' &= A_1' \cdot 1 \otimes \mathsf{p} + D_1' \otimes D_2' + A_2' \cdot \mathsf{p} \otimes 1. \end{aligned}$$

By Eqn. (2.6), the insertion becomes (see also [COT22, Proof of Thm. 5.8]):

$$(2.14) \qquad \tau(\gamma) = (D_1 \cdot \beta) \otimes D_2 + A_2 f^* \pi_{M*}(\pi_S^*\mathsf{p} \cdot \operatorname{ch}_1(\mathbb{F}_S)) \otimes 1,$$

where π_S, π_M are projections from $S \times M_n(S,\beta)$ to its factors. Hence

$$\tau(\gamma) \cdot \tau(\gamma') = (D_1 \cdot \beta) \cdot (D_1' \cdot \beta) \otimes (D_2 \cdot D_2') + A_2 A_2' f^* \left(\pi_{M*}(\pi_S^*\mathsf{p} \cdot \operatorname{ch}_1(\mathbb{F}_S))\right)^2 \otimes 1 + \text{others},$$

where "others" lie in $H^2(P_1(S, \beta)) \otimes H^2(T)$. By Theorem 2.8, we get

$$
\begin{aligned}
P_{1,\beta}(\gamma, \gamma') &= (D_1 \cdot \beta)(D_1' \cdot \beta) \int_T (D_2 \cdot D_2') \int_{P_1(S,\beta)} f^* e(T_{M_1(S,\beta)}) \\
&\quad - e(T) A_2 A_2' \int_{P_1(S,\beta)} f^* \left(c_{\beta^2}(T_{M_1(S,\beta)}) \cdot \pi_{M*} \left(\pi_S^* \mathsf{p} \cdot \mathrm{ch}_1(\mathbb{F}_S) \right)^2 \right) \\
&= (D_1 \cdot \beta)(D_1' \cdot \beta) \int_T (D_2 \cdot D_2') \int_{M_1(S,\beta)} e(T_{M_1(S,\beta)}) \\
&\quad - e(T) A_2 A_2' \int_{M_1(S,\beta)} c_{\beta^2}(T_{M_1(S,\beta)}) \cdot \pi_{M*} \left(\pi_S^* \mathsf{p} \cdot \mathrm{ch}_1(\mathbb{F}_S) \right)^2 \\
&= (D_1 \cdot \beta)(D_1' \cdot \beta) \int_T (D_2 \cdot D_2') \, e(M_1(S, \beta)),
\end{aligned}
$$

where the second equality follows from Proposition 2.9 and the last equality is proved using Fujiki formula (Theorem 2.10) and the evaluation

$$
q_M(\pi_{M*}(\pi_S^* \mathsf{p} \cdot \mathrm{ch}_1(\mathbb{F}_S))) = 0,
$$

(which follows for example from [COT22, Proof of Thm. 5.8]). Conjecture 1.10 (2) then reduces to [COT22, Thm. 5.8]. □

2.6. **Transport of integrals to the Hilbert schemes.** To compute the stable pair theory on $P_n(S, \beta)$ for $n \leqslant 0$ we will need to handle more complicated descendent integrals on $M_n(S, \beta)$. As in [COT22, §4.4] which deals with the $n = 1$ case, we use here the general framework of monodromy operators of Markman [M08] (see also [O22]) to reduce to the Hilbert schemes.

Consider the Mukai lattice, which is the lattice $\Lambda = H^*(S, \mathbb{Z})$ endowed with the Mukai pairing

$$
\langle x, y \rangle := -\int_S x^\vee y,
$$

where, if we decompose an element $x \in \Lambda$ according to degree as (r, D, n), we have written $x^\vee = (r, -D, n)$. Given a sheaf or complex E on S the Mukai vector of E is defined by

$$
v(E) = \sqrt{\mathrm{td}_S} \cdot \mathrm{ch}(E) \in \Lambda.
$$

Let $M(v)$ be a proper smooth moduli space of stable sheaves on S with Mukai vector $v \in \Lambda$ (where stability is with respect to some fixed polarization). We assume that there exists a universal family $\mathbb{F}$ on $M(v) \times S$. If it does not exists, everything below can be made to work by working with the Chern character $\mathrm{ch}(\mathbb{F})$ of a quasi-universal family, see [M08] or [O22]. Let π_M, π_S be the projections to $M(v)$ and S. One has the Mukai morphism

$$
\theta_{\mathbb{F}} : \Lambda \to H^2(M(v)),
$$

$$
\theta_{\mathbb{F}}(x) = \left[\pi_{M*}(\mathrm{ch}(\mathbb{F}) \cdot \sqrt{\mathrm{td}_S} \cdot x^\vee) \right]_{\deg=2},
$$

where $[-]_{\deg=k}$ stands for extracting the degree k component and (as we will also do below) we have suppressed the pullback maps from the projection to S. Define the universal class

$$
u_v = \exp\left(\frac{\theta_{\mathbb{F}}(v)}{\langle v, v \rangle} \right) \mathrm{ch}(\mathbb{F}) \sqrt{\mathrm{td}_S},
$$

which is independent of the choice of universal family $\mathbb{F}$. For $x \in \Lambda$, consider the normalized descendents:

$$
B(x) := \pi_{M*}(u_v \cdot x^\vee),
$$

and let $B_k(x) = [B(x)]_{\deg=2k}$ its degree $2k$ component.

Example 2.12. *For* $v = (1, 0, 1 - d)$, *the moduli space becomes the punctual Hilbert scheme:* $M(v) = S^{[d]}$. *Then we have*

$$
u_v = \exp\left(\frac{-\delta}{2d - 2} \right) \mathrm{ch}(\mathcal{I}_{\mathcal{Z}}) \sqrt{\mathrm{td}_S},
$$

where we let $\delta = \pi_* \mathrm{ch}_3(\mathcal{O}_{\mathcal{Z}})$ *(so that* -2δ *is the class of the locus of non-reduced subschemes).*

We define the standard descendents on the Hilbert scheme by

$$
\mathfrak{G}_d(\alpha) = \pi_*(\pi_S^*(\alpha) \, \mathrm{ch}_d(\mathcal{O}_{\mathcal{Z}})) \in H^*(S^{[d]}).
$$

One obtains that

$$B_1(\mathsf{p}) = -\frac{\delta}{2d-2},$$

$$B_2(\mathsf{p}) = \frac{1}{2}\frac{\delta^2}{(2d-2)^2} - \mathfrak{G}_2(\mathsf{p}).$$

For a divisor $D \in H^2(S)$, one finds

$$B_1(D) = \mathfrak{G}_2(D),$$

$$B_2(D) = \mathfrak{G}_3(D) - \frac{\delta}{2d-2}\mathfrak{G}_2(D).$$

Using the descendents $B_k(x)$, one allows to move between any two moduli spaces of stable sheaves on S just by specifying a Mukai lattice isomorphism $g : \Lambda \to \Lambda$. We give the details in the case of our interest, see [M08, O22] for the general case.

As before let $\beta \in \mathrm{Pic}(S)$ be an irreducible effective class of square $\beta \cdot \beta = 2d - 2$, and let $n \in \mathbb{Z}$. We want to connect the moduli spaces

$$M_n(S, \beta) \rightsquigarrow S^{[d]}.$$

Let $\beta = e + (d-1)f$ where $e, f \in H^2(S, \mathbb{Z})$ span a hyperbolic lattice: $\mathbb{Z}e \oplus \mathbb{Z}f \cong \left(\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}\right)$. We do not require e, f to be effective here. Define the isomorphism $g : \Lambda \to \Lambda$ by

$$1 \mapsto (0, -e, n), \quad \mathsf{p} \mapsto (0, f, 0), \quad e \mapsto (1, -nf, 0), \quad f \mapsto (0, 0, -1), \quad g|_{\{1, \mathsf{p}, e, f\}^\perp} = \mathrm{id}.$$

One sees that g is an isometry of the Mukai lattice and that

$$g_*(0, \beta, n) = (1, 0, 1 - d).$$

Then one has:

Theorem 2.13. (*Markman* [M08], *reformulation as in* [O22, Thm. 4]) *For any $k_i \geqslant 0$ and $\alpha_i \in H^*(S)$ and any polynomial P,*

$$\int_{M_n(S,\beta)} P(B_{k_i}(\alpha_i), c_j(T_{M_n(S,\beta)})) = \int_{S^{[d]}} P(B_{k_i}(g\alpha_i), c_j(T_{S^{[d]}})).$$

2.7. **Genus 1 in irreducible classes.** Recall the genus 1 Gopakumar-Vafa invariants (Proposition 2.1). On the stable pair side, we have the following:

Theorem 2.14. *Let $\beta \in H_2(S, \mathbb{Z}) \subseteq H_2(X, \mathbb{Z})$ be an irreducible curve class. Then for certain choice of orientation, we have*

(2.15) $$P_{0,\beta}(\gamma) = e(T) N_1\left(\frac{\beta^2}{2}\right) \int_{S \times \mathsf{p}} \gamma.$$

In particular, Conjecture 1.10 (3) holds in this case.

Proof. The strategy is as follows: First we write our stable pair invariants as integrals on the moduli spaces $M_0(S, \beta)$, then express the integrand in terms of the classes $B_k(x)$ and then use Markman's Theorem 2.13 to reduce to an integral over the Hilbert scheme, which is known by the results of [COT22].

By Eqn. (2.14) and Theorem 2.8 (choose the inverse orientation there), we have

$$P_{0,\beta}(\gamma) = e(T) \int_{S \times \mathsf{p}} \gamma \cdot \int_{P_0(S,\beta)} f^*\left(c_{\beta^2}(T_{M_0(S,\beta)}) \cdot \pi_{M*}\left(\pi_S^*(\mathsf{p}) \cdot \mathrm{ch}_1(\mathbb{F}_S)\right)\right).$$

Using Proposition 2.9, we find

$$P_{0,\beta}(\gamma) = e(T) \int_{S \times \mathsf{p}} \gamma \cdot \int_{M_0(S,\beta)} c_{\beta^2}(T_{M_0(S,\beta)}) \cdot c_1(-\mathbf{R}\pi_{M*}(\mathbb{F}_S)) \cdot \pi_{M*}\left(\mathrm{ch}_1(\mathbb{F}_S) \cdot \pi_S^*(\mathsf{p})\right).$$

A calculation shows that we have

$$B_1(\mathsf{p}) = \pi_*(\mathrm{ch}_1(\mathbb{F}_S)\,\pi_S^*(\mathsf{p})).$$

Moreover, the expressions (2.13) are invariant under replacing $\mathrm{ch}(\mathbb{F}_S)$ by $\mathrm{ch}(\mathbb{F}_S)\exp(\ell)$ for any line bundle $\ell \in H^2(M_n(S, \beta))$. Hence we can use $\mathrm{ch}(\mathbb{F}_S') := \mathrm{ch}(\mathbb{F}_S)\exp(\theta_{\mathbb{F}_S}(v)/\langle v, v \rangle)$ which shows that

$$c_1(-\mathbf{R}\pi_{M*}(\mathbb{F}_S)) = -\pi_{M*}(\mathrm{ch}_3(\mathbb{F}_S')) - 2\pi_{M*}(\mathrm{ch}_1(\mathbb{F}_S')\,\pi_S^*\mathsf{p})$$

$$= -B_1\left(\sqrt{\mathrm{td}_S}^{-1}\right) - 2B_1(\mathsf{p})$$

$$= -B_1(1 + \mathsf{p}).$$

We obtain that:

$$\int_{M_0(S,\beta)} c_{2d-2}(T_{M_0(S,\beta)}) \cdot c_1(-\mathbf{R}\pi_{M*}(\mathbb{F}_S)) \cdot \pi_{M*}(\mathrm{ch}_1(\mathbb{F}_S) \cdot \pi_S^*\mathsf{p})$$

$$= -\int_{M_0(S,\beta)} c_{2d-2}(T_{M_0(S,\beta)})B_1(1+\mathsf{p})B_1(\mathsf{p})$$

$$= -\int_{S^{[d]}} c_{2d-2}(T_{S^{[d]}})B_1(-e+f)B_1(f)$$

$$= -\int_{S^{[d]}} c_{2d-2}(T_{S^{[d]}})\mathfrak{G}_2(-e+f)\mathfrak{G}_2(f)$$

$$= -((-e+f)\cdot f)C(c_{2d-2}(T_{S^{[d]}}))$$

$$= N_1(d-1),$$

where we used the $k=1$ case of [COT22, Thm. 4.2] in the last step. $\qquad\square$

2.8. Genus 2 in irreducible classes. Let $\beta_d \in H_2(S,\mathbb{Z}) \subseteq H_2(X,\mathbb{Z})$ be an irreducible curve class of square $\beta_d^2 = 2d-2$. Below, we use similar method to compute stable pair invariants P_{-1,β_d} on X for all d.

Theorem 2.15. *For certain choice of orientation, we have*

$$\sum_{d\in\mathbb{Z}} P_{-1,\beta_d}\, q^d = \left(\prod_{n\geqslant 1}(1-q^n)^{-24}\right)\left(24q\frac{d}{dq}G_2(q) - 24G_2(q) - 1\right)$$

$$= 72q^2 + 1920q^3 + 28440q^4 + 305280q^5 + 2639760q^6 + 19450368q^7 + \cdots.$$

In particular, Conjecture 1.10 (4) holds in this case.

Proof. As in the genus 1 case, by Theorem 2.8 and Proposition 2.9 we have:

$$P_{-1,\beta} = -e(T)\int_{M_{-1}(S,\beta)} c_{2d-2}(T_{M_{-1}(S,\beta)}) \cdot c_2(-\mathbf{R}\pi_{M*}(\mathbb{F}_S)).$$

With the same discussion as before one gets:

$$c_2(-\mathbf{R}\pi_{M*}(\mathbb{F}_S)) = \frac{1}{2}B_1(1+\mathsf{p})^2 + B_2(1+\mathsf{p}).$$

Hence applying Markman's Theorem 2.13, we conclude

$$\int_{M_{-1}(S,\beta)} c_{2d-2}(T_{M_{-1}(S,\beta)}) \cdot c_2(-\mathbf{R}\pi_{M*}(\mathbb{F}_S))$$

$$= \int_{M_{-1}(S,\beta)} c_{2d-2}(T_{M_{-1}(S,\beta)}) \cdot \left(\frac{1}{2}B_1(1+\mathsf{p})^2 + B_2(1+\mathsf{p})\right)$$

$$= \int_{S^{[d]}} c_{2d-2}(T_{S^{[d]}})\left(\frac{1}{2}B_1(-e+f-\mathsf{p})^2 + B_2(-e+f-\mathsf{p})\right)$$

$$= \int_{S^{[d]}} c_{2d-2}(T_{S^{[d]}})\frac{1}{2}\left[\mathfrak{G}_2(-e+f) + \frac{\delta}{2d-2}\right]^2$$

$$\quad + \int_{S^{[d]}} c_{2d-2}(T_{S^{[d]}})\left(\mathfrak{G}_3(-e+f) - \frac{\delta}{2d-2}\mathfrak{G}_2(-e+f) - \frac{1}{2}\frac{\delta^2}{(2d-2)^2} + \mathfrak{G}_2(\mathsf{p})\right)$$

$$= \frac{1}{2}\left((-e+f)^2 + \frac{\delta\cdot\delta}{(2d-2)^2}\right)N_1(d-1) - \frac{1}{2}\frac{\delta\cdot\delta}{(2d-2)^2}N_1(d-1) + \int_{S^{[d]}} c_{2d-2}(T_{S^{[d]}})\mathfrak{G}_2(\mathsf{p})$$

$$= -N_1(d-1) + \int_{S^{[d]}} c_{2d-2}(T_{S^{[d]}})\mathfrak{G}_2(\mathsf{p}).$$

Thus we conclude that

$$P_{-1,\beta} = e(T)\left(N_1(d-1) - \int_{S^{[d]}} c_{2d-2}(T_{S^{[d]}})\mathfrak{G}_2(\mathsf{p})\right).$$

The desired formula now follows by the evaluation given in [COT22, Prop. 4.6]:

$$\sum_{d\geqslant 0} q^d \int_{S^{[d]}} c_{2d-2}(T_{S^{[d]}})\mathfrak{G}_2(\mathsf{p}) = \prod_{n=1}(1-q^n)^{-24}\left(G_2(q) + \frac{1}{24}\right).$$

Finally, comparing with Proposition 2.1, we are done. $\qquad\square$

Remark 2.16. *By the global Torelli theorem, primitive curve classes on K3 surfaces can be deformed to irreducible curve classes. Combining Theorem 2.11, Theorem 2.14, Theorem 2.15, we know Conjecture 1.10 also holds for primitive curve classes $\beta \in H_2(S) \subseteq H_2(X)$.*

2.9. **Genus 1: multiple fiber classes of elliptic fibrations.** Let $X = E \times E \times T$ be the product two copies of an elliptic curve E and a K3 surface T. It gives the trivial elliptic fibration

$$(2.16) \qquad \pi : X \to Y := E \times T.$$

For multiple fiber classes of π (2.16), we have the following closed evaluation:

Theorem 2.17. *Let $t > 0$ and $\gamma \in H^4(X)$. For certain choice of orientation, we have*

$$(2.17) \qquad \sum_{r \geqslant 0} P_{0,r[E]}^t(\gamma)\, q^r = 24 \left(\int_{E \times E \times \mathsf{p}} \gamma \right) \cdot \sum_{m \geqslant 1} \sum_{n \mid m} n^2 q^m.$$

Proof. By [CT19, Prop. 5.3], we know $P_0^t(X, n[E])$ is independent of the choice of $t > 0$, so we may set $t \to \infty$ and work with PT stability. As in [CMT19, Lem. 3.5], there is an isomorphism

$$\pi^* : \mathrm{Hilb}^n(Y) \cong P_0(X, n[E]), \quad I_Z \mapsto \pi^* I_Z.$$

For $I = \pi^* I_Z \in P_0(X, n[E])$, by projection formula and

$$\pi_* \mathcal{O}_X \cong \mathcal{O}_Y \oplus K_Y[-1],$$

we obtain

$$\mathbf{R}\mathrm{Hom}_X(I, I) \cong \mathbf{R}\mathrm{Hom}_Y(I_Z, I_Z) \oplus \mathbf{R}\mathrm{Hom}_Y(I_Z, I_Z \otimes K_Y)[-1].$$

By taking the traceless part, we get

$$(2.18) \qquad \mathrm{Ext}_X^2(I, I)_0 \cong \mathrm{Ext}_Y^2(I_Z, I_Z)_0 \oplus \mathrm{Ext}_Y^1(I_Z, I_Z)_0$$
$$\cong \mathrm{Ext}_Y^2(I_Z, I_Z)_0 \oplus \mathrm{Ext}_Y^2(I_Z, I_Z)_0^\vee,$$

where we use Serre duality in the second isomorphism.

Next we compare cosections on these obstruction spaces. By [KiP, Lem. 9.4], we have a surjective isotropic cosection

$$\phi_X : \mathrm{Ext}_X^2(I, I)_0 \xrightarrow{\mathrm{At}(I)} \mathrm{Ext}_X^3(I, I \otimes T^*X) \xrightarrow{\mathrm{tr}} H^3(X, T^*X) \xrightarrow{H\sigma_X} H^4(X, \wedge^4 T^*X) \xrightarrow{\int} \mathbb{C},$$

where $\mathrm{At}(I) \in \mathrm{Ext}_X^1(I, I \otimes T^*X)$ denotes the Atiyah class of I, $H \in H^1(X, T^*X)$ is an ample divisor and $\sigma_X \in H^0(X, \wedge^2 T^*X)$ is a holomorphic symplectic form of X.

By the compatibility of Atiyah classes with map $\pi : X \to Y$ (ref. [BuFl, Prop. 3.14]), we have a commutative diagram

(2.19)

$$
\begin{array}{ccccccc}
\mathrm{Ext}_X^2(I, I)_0 & \xrightarrow{\mathrm{At}(I)} & \mathrm{Ext}_X^3(I, I \otimes T^*X) & \xrightarrow{\mathrm{tr}} & H^3(X, T^*X) & \xrightarrow{\mathrm{pr}} & H^{1,1}(S) \otimes H^{0,2}(T) \\
\Big\uparrow{\scriptstyle i} & & \Big\uparrow & & \Big\uparrow & & \Big\uparrow{\scriptstyle \pi^*} \\
\mathrm{Ext}_Y^2(I_Z, I_Z)_0 & \xrightarrow{\mathrm{At}(I_Z)} & \mathrm{Ext}_Y^3(I_Z, I_Z \otimes T^*Y) & \xrightarrow{\mathrm{tr}} & H^3(Y, T^*Y) & \xrightarrow{\cong} & H^{1,1}(E) \otimes H^{0,2}(T),
\end{array}
$$

where i is the embedding in (2.18), tr denotes the trace map and pr is the projection with respect to Künneth decomposition. We define a cosection

$$\phi_Y : \mathrm{Ext}_Y^2(I_Z, I_Z)_0 \xrightarrow{\mathrm{At}(I_Z)} \mathrm{Ext}_Y^3(I_Z, I_Z \otimes T^*Y) \xrightarrow{\mathrm{tr}} H^3(Y, T^*Y) \cong H^{1,1}(E) \otimes H^{0,2}(T) \xrightarrow{\epsilon} \mathbb{C},$$

$$\text{where} \quad \epsilon(\alpha) = \int_X H\sigma_X \cdot \pi^* \alpha, \quad \alpha \in H^{1,1}(E) \otimes H^{0,2}(T).$$

It is easy to see ϕ_Y is a positive multiple of the standard cosection of $\mathrm{Hilb}^n(Y)$ (see e.g. [O18b, Eqn. (6)]), hence its reduced virtual class keeps the same.

By diagram (2.19), we have a commutative diagram:

$$
\begin{array}{ccc}
\mathrm{Ext}_X^2(I, I)_0 & \xrightarrow{\phi_X} & \mathbb{C} \\
\Big\uparrow{\scriptstyle i} & \nearrow{\scriptstyle \phi_Y} & \\
\mathrm{Ext}_Y^2(I_Z, I_Z)_0. & &
\end{array}
$$

We claim that $\mathrm{Ker}(\phi_Y)$ is a maximal isotropic subspace of $\mathrm{Ker}(\phi_X)/\mathrm{Im}(\phi_X^\vee)$. In fact, by taking dual, we have a commutative diagram

$$
\begin{array}{ccccc}
\mathbb{C} & \xrightarrow{\phi_X^\vee} & \mathrm{Ext}^2_X(I,I)_0^\vee & \xrightarrow[\cong]{Q_{\mathrm{Serre}}} & \mathrm{Ext}^2_X(I,I)_0 \cong \mathrm{Ext}^2_Y(I_Z,I_Z)_0 \oplus \mathrm{Ext}^2_Y(I_Z,I_Z)_0^\vee \\
\Big\downarrow{\scriptstyle =} & & \Big\downarrow{\scriptstyle i^\vee} & & \\
\mathbb{C} & \xrightarrow{\phi_Y^\vee} & \mathrm{Ext}^2_Y(I_Z,I_Z)_0^\vee. & &
\end{array}
$$

Since ϕ_Y is surjective, so $\phi_Y^\vee$ is injective, therefore

$$
\mathrm{Im}(\phi_X^\vee) \cap \mathrm{Ker}(\phi_Y) \subseteq \mathrm{Im}(\phi_X^\vee) \cap \mathrm{Ext}^2_Y(I_Z,I_Z)_0 = 0,
$$

and $\mathrm{Ker}(\phi_Y)$ defines a subspace of $\mathrm{Ker}(\phi_X)/\mathrm{Im}(\phi_X^\vee)$. This is a maximal isotropic subspace as $i : \mathrm{Ext}^2_Y(I_Z,I_Z)_0 \to \mathrm{Ext}^2_X(I,I)_0$ is so.

The above construction works in family and therefore we have

$$
[P_0(X,n[E])]^{\mathrm{vir}} = [\mathrm{Hilb}^n(Y)]^{\mathrm{vir}} \in A_1(P_0(X,n[E])),
$$

for certain choice of orientation in the LHS. Consider a commutative diagram:

$$
\begin{array}{ccccc}
X & \xleftarrow{\pi_X} & X \times P_0(X,n[E]) & \xrightarrow{\pi_P} & P_0(X,n[E]) \\
{\scriptstyle \pi}\Big\downarrow & & {\scriptstyle \bar\pi=(\pi,(\pi^*)^{-1})}\Big\downarrow & & {\scriptstyle (\pi^*)^{-1}}\Big\downarrow{\scriptstyle \cong} \\
Y & \xleftarrow{\pi_Y} & Y \times \mathrm{Hilb}^n(Y) & \xrightarrow{\pi_M} & \mathrm{Hilb}^n(Y),
\end{array}
$$

and denote $\mathcal{Z} \hookrightarrow Y \times \mathrm{Hilb}^n(Y)$ to be the universal 0-dimensional subscheme. Then

$$
\begin{aligned}
P_{0,n[E]}(\gamma) &= \int_{[P_0(X,n[E])]^{\mathrm{vir}}} \pi_{P*}(\pi_X^*\gamma \cdot \bar\pi^*\,\mathrm{ch}_3(\mathcal{O}_{\mathcal{Z}})) \\
&= \int_{[\mathrm{Hilb}^n(Y)]^{\mathrm{vir}}} \pi_{M*}\bar\pi_*(\pi_X^*\gamma \cdot \bar\pi^*\,\mathrm{ch}_3(\mathcal{O}_{\mathcal{Z}})) \\
&= \int_{[\mathrm{Hilb}^n(Y)]^{\mathrm{vir}}} \pi_{M*}(\mathrm{ch}_3(\mathcal{O}_{\mathcal{Z}}) \cdot \bar\pi_*(\pi_X^*\gamma)) \\
&= \int_{[\mathrm{Hilb}^n(Y)]^{\mathrm{vir}}} \pi_{M*}(\mathrm{ch}_3(\mathcal{O}_{\mathcal{Z}}) \cdot \pi_Y^*\pi_*\gamma).
\end{aligned}
$$

The statement now follows from Proposition 2.18 below. $\qquad\square$

Proposition 2.18. *Let $\omega \in H^2(E,\mathbb{Z})$ be the class of point and $D \in H^2(T,\mathbb{Q})$ any class. Then for any $n \geqslant 1$ we have:*

$$
\int_{[\mathrm{Hilb}^n(T \times E)]^{\mathrm{vir}}} \pi_{M*}\big(\mathrm{ch}_3(\mathcal{O}_{\mathcal{Z}})\pi_Y^*(\omega \otimes 1)\big) = (-1)^{n+1}e(T)\sum_{d|n} d^2,
$$

$$
\int_{[\mathrm{Hilb}^n(T \times E)]^{\mathrm{vir}}} \pi_{M*}\big(\mathrm{ch}_3(\mathcal{O}_{\mathcal{Z}})\pi_Y^*(1 \otimes D)\big) = 0.
$$

Proof. Write $\mathrm{Hilb} = \mathrm{Hilb}^n(T \times E)$ and consider the diagram

$$
\begin{array}{ccccc}
T \times E & \xleftarrow{\pi_{T \times E}} & \mathrm{Hilb} \times T \times E & \xrightarrow{\pi_M} & \mathrm{Hilb} \\
& & {\scriptstyle \tilde p}\Big\downarrow & & {\scriptstyle p}\Big\downarrow \\
& & \frac{\mathrm{Hilb} \times T \times E}{E} & \xrightarrow{\pi_{M/E}} & \mathrm{Hilb}/E,
\end{array}
$$

where the quotient by E is taken in the stacky sense. The universal subscheme $\mathcal{Z} \subset \mathrm{Hilb} \times T \times E$ has a natural E-linearization and hence arises from the pullback of a subscheme $\mathcal{Z}/E \subset (\mathrm{Hilb} \times T \times E)/E$. Moreover, as in [O18b], there exists a natural (0-dimensional) virtual class $[\mathrm{Hilb}/E]^{\mathrm{vir}}$ such that

$$
[\mathrm{Hilb}]^{\mathrm{vir}} = p^*[\mathrm{Hilb}/E]^{\mathrm{vir}}.
$$

Since the virtual class of Hilb/E arises from a symmetric obstruction theory (on an étale cover of Hilb/E), its degree can be computed by as an Behrend weighted Euler characteristic [B]:

$$
\int_{[\mathrm{Hilb}/E]^{\mathrm{vir}}} 1 = e\left(\mathrm{Hilb}/E, \nu\right).
$$

We argue now as follows: Applying the pushpull formula and using $p \circ \pi_M = \pi_{M/E} \circ \tilde{p}$ we have

$$N_n := \int_{[\mathrm{Hilb}]^{\mathrm{vir}}} \pi_{M*}\big(\mathrm{ch}_3(\mathcal{O}_{\mathcal{Z}})\pi_{T \times E}^*(\omega \otimes 1)\big)$$

$$= \int_{[\mathrm{Hilb}\,/E]^{\mathrm{vir}}} \pi_{M/E*}\tilde{p}_*\big(\mathrm{ch}_3(\mathcal{O}_{\mathcal{Z}})\pi_{T \times E}^*(\omega \otimes 1)\big)$$

$$= \int_{[\mathrm{Hilb}\,/E]^{\mathrm{vir}}} \pi_{M/E*}\big(\mathrm{ch}_3(\mathcal{O}_{\mathcal{Z}/E})\tilde{p}_*(\pi_{T \times E}^*(\omega \otimes 1))\big).$$

Then by checking on fibers we have $\tilde{p}_*(\pi_{T \times E}^*(\omega \otimes 1)) = 1$ as well as $\pi_{M/E*}\,\mathrm{ch}_3(\mathcal{O}_{\mathcal{Z}/E}) = n$. This implies that

$$N_n = n \int_{[\mathrm{Hilb}\,/E]^{\mathrm{vir}}} 1 = n \cdot e\,(\mathrm{Hilb}\,/E, \nu) = 24(-1)^{n-1} \sum_{d|n} d^2,$$

where for the last equality we have used [OS, Cor. 1].

For the second integral we argue identically, but observe that we have

$$\mathrm{ch}_3(\mathcal{O}_{\mathcal{Z}})\pi_{T \times E}^*(1 \otimes D) = \tilde{p}^*(\mathrm{ch}_3(\mathcal{O}_{\mathcal{Z}/E})\pi_T^*(D)),$$

so when pushing forward by $\tilde{p}$ the integral vanishes. $\qquad\square$

Similarly, we can consider a nontrivial elliptic fibration:

$$\bar{p} = (p, \mathrm{id}_T) : X = S \times T \to \mathbb{P}^1 \times T,$$

where $p : S \to \mathbb{P}^1$ is an elliptic $K3$ surface with a section i. Let f be a generic fiber of $\bar{p}$.

Theorem 2.19. *Let $t > 0$ and $\gamma \in H^4(X)$. Then for certain choice of orientation, we have*

$$(2.20) \qquad \sum_{r \geqslant 0} P_{0,r[f]}^t(\gamma)\,q^r = 24 \left(\int_{S \times \mathsf{p}} \gamma\right) \cdot \sum_{m \geqslant 1} \sum_{n|m} n^2 q^m.$$

Proof. The first proof is parallel to the proof of Theorem 2.17. For the second part, we need to evaluate

$$(2.21) \qquad \int_{[\mathrm{Hilb}^n(T \times \mathbb{P}^1)]^{\mathrm{vir}}} \pi_{M*}\big(\mathrm{ch}_3(\mathcal{O}_{\mathcal{Z}})\,\pi_Y^*(\omega \otimes 1)\big),$$

where $\omega \in H^2(\mathbb{P}^1)$ is the class of a point. We consider the degeneration of $T \times \mathbb{P}^1$ given by the product of T with the degeneration of $\mathbb{P}^1$ into a chain of three $\mathbb{P}^1$'s. By specializing the insertion ω to the middle factor, we are reduced to an integral of the relative Hilbert schemes $\mathrm{Hilb}^n(T \times \mathbb{P}^1/T_0 \cup T_\infty)$ with the same integrand. But this integral is also the outcome of applying the degeneration formula to the integrals considered in Proposition 2.18 (under the degeneration of E to a nodal $\mathbb{P}^1$). Hence (2.21) is given by $(-1)^{n+1}e(T)\sum_{d|n} d^2$ as well. For the analogue of the second integral in Proposition 2.18, the localization formula applied to the scaling action of $\mathbb{C}^*$ on $\mathbb{P}^1$ shows that it vanishes. $\qquad\square$

Remark 2.20. *On the product of two $K3$ surfaces, genus 1 Gopakumar-Vafa invariants in imprimitive classes are defined in [COT22, Def. A.1]. In particular, for multiple fiber classes $\beta = r[f]$ above, by using [COT22, Eqn. (5.7)], we know $n_{1,r[f]}(\gamma) = 0$ if $r > 1$.*

3. Hilbert schemes of two points on $K3$

3.1. Rational curves on exceptional locus.
Let S be a $K3$ surface. Consider the Hilbert-Chow map

$$\pi : \mathrm{Hilb}^2(S) \to \mathrm{Sym}^2(S)$$

to the symmetric product of S. Let D be the exceptional divisor fitting into Cartesian diagram:

$$\begin{array}{ccc} D & \xrightarrow{\ i\ } & \mathrm{Hilb}^2(S) \\ \pi \downarrow & & \downarrow \pi \\ S & \xrightarrow{\ \Delta\ } & \mathrm{Sym}^2(S), \end{array}$$

where Δ is the diagonal embedding. Note that $\pi : D \to S$ is a $\mathbb{P}^1$-bundle and any fiber of it has normal bundle $\mathcal{O}_{\mathbb{P}^1}(-2, 0, 0)$.

Theorem 3.1. *When $t = \frac{n}{\omega \cdot \beta} + 0^+$ (i.e. in JS chamber), Conjecture 1.10 (1), (2) hold for multiple fiber classes $\beta = r[\mathbb{P}^1]$ $(r \geqslant 1)$ of π as above.*

Proof. By Jordan-Hölder filtration, the JS moduli space $P_n^{\mathrm{JS}}(X, d[\mathbb{P}^1])$ is nonempty only if

$$d \mid n, \ n > 0.$$

so we may assume $n = m \cdot d$ for $m \in \mathbb{Z}_{\geqslant 1}$. Consider the map

$$(3.1) \qquad f : P_{md}^{\mathrm{JS}}(X, d[\mathbb{P}^1]) \to \mathrm{Sym}^d(S), \quad (F, s) \mapsto \pi_*[F].$$

As the insertion (1.7) only involves fundamental cycle of the universal one dimensional sheaf $\mathbb{F}$, we have

$$[\mathbb{F}] = \bar{f}^*[\mathcal{Z}],$$

where $[\mathcal{Z}] \hookrightarrow \mathrm{Sym}^d(S) \times S$ is the class of incident subvariety and $\bar{f} = (f, \mathrm{id}_S)$. Therefore

$$\int_{[P_{md}^{\mathrm{JS}}(X, d[\mathbb{P}^1])]^{\mathrm{vir}}} \prod_{i=1}^l \tau(\gamma_i) = \int_{[P_{md}^{\mathrm{JS}}(X, d[\mathbb{P}^1])]^{\mathrm{vir}}} f^*\Phi,$$

for some $\Phi \in H^{2(md+1)}(\mathrm{Sym}^d(S))$. When $m > 1$, we have $md + 1 > 2d$, therefore $\Phi = 0$ and

$$P_{md, d[\mathbb{P}^1]}^{\mathrm{JS}}(\gamma_1, \ldots, \gamma_l) = 0, \quad \text{if } m > 1.$$

For $m = 1$, we have an isomorphism

$$\mathrm{Hilb}^d(S) \overset{\pi^*}{\cong} P_d^{\mathrm{JS}}(D, d[\mathbb{P}^1]) \cong P_d^{\mathrm{JS}}(X, d[\mathbb{P}^1]),$$
$$I_Z \mapsto \pi^* I_Z \mapsto (\mathcal{O}_X \to i_* \pi^* \mathcal{O}_Z).$$

For $I_X = (\mathcal{O}_X \to i_* \pi^* \mathcal{O}_Z)$, we write $I_D = (\mathcal{O}_D \to \pi^* \mathcal{O}_Z)$. As in [CMT19, Prop. 4.3], [CKM20, Prop. 4.2], we have a canonical isomorphism

$$\mathrm{Ext}_D^0(I_D, \pi^* \mathcal{O}_Z) \cong \mathrm{Ext}_X^1(I_X, I_X)_0,$$

and an inclusion of maximal isotropic subspace

$$(3.2) \qquad \mathrm{Ext}_D^1(I_D, \pi^* \mathcal{O}_Z) \hookrightarrow \mathrm{Ext}_X^2(I_X, I_X)_0.$$

From distinguished triangle

$$I_D \to \mathcal{O}_D \to \pi^* \mathcal{O}_Z,$$

we obtain a distinguished triangle

$$\mathbf{R}\mathrm{Hom}_D(\pi^* \mathcal{O}_Z, \pi^* \mathcal{O}_Z) \to \mathbf{R}\mathrm{Hom}_D(\mathcal{O}_D, \pi^* \mathcal{O}_Z) \to \mathbf{R}\mathrm{Hom}_D(I_D, \pi^* \mathcal{O}_Z).$$

By projection formula, we have

$$\mathbf{R}\mathrm{Hom}_D(\pi^* \mathcal{O}_Z, \pi^* \mathcal{O}_Z) \cong \mathbf{R}\mathrm{Hom}_S(\mathcal{O}_Z, \mathcal{O}_Z), \ \ \mathbf{R}\mathrm{Hom}_D(\mathcal{O}_D, \pi^* \mathcal{O}_Z) \cong \mathbf{R}\mathrm{Hom}_S(\mathcal{O}_S, \mathcal{O}_Z).$$

Therefore we get an exact sequence

$$(3.3) \qquad 0 = H^1(S, \mathcal{O}_Z) \cong H^1(D, \pi^* \mathcal{O}_Z) \to \mathrm{Ext}_D^1(I_D, \pi^* \mathcal{O}_Z) \to \mathrm{Ext}_S^2(\mathcal{O}_Z, \mathcal{O}_Z) \to 0.$$

By Serre duality, we have

$$(3.4) \qquad \mathrm{Ext}_S^2(\mathcal{O}_Z, \mathcal{O}_Z) \cong \mathrm{Ext}_S^0(\mathcal{O}_Z, \mathcal{O}_Z)^\vee \cong H^0(S, \mathcal{O}_Z)^\vee.$$

Combining Eqns. (3.2), (3.3), (3.4), we obtain a maximal isotropic subspace

$$H^0(S, \mathcal{O}_Z)^\vee \hookrightarrow \mathrm{Ext}_X^2(I_X, I_X)_0.$$

Working in family, we see that the dual of tautological bundle $\mathcal{O}_S^{[d]}$ on $\mathrm{Hilb}^d(S)$ is a maximal isotropic subbundle of the obstruction bundle of $P_d^{\mathrm{JS}}(X, d[\mathbb{P}^1])$. By Lemma 2.6, we obtain

$$[P_d^{\mathrm{JS}}(X, d[\mathbb{P}^1])]^{\mathrm{vir}} = [\mathrm{Hilb}^d(S)] \cap c_{d-1}\left(\mathcal{O}_S^{[d]}\right),$$

for certain choice of orientation. As for insertions, consider the following diagram

let $\mathcal{Z} \hookrightarrow \operatorname{Hilb}^d(S) \times S$ denote the universal zero dimensional subscheme, then

$$
\begin{aligned}
\tau(\gamma) &= \pi_{M*}\left(\pi_X^* \gamma \cdot \operatorname{ch}_3(\bar{i}_* \bar{\pi}^* \mathcal{O}_{\mathcal{Z}})\right) \\
&= \pi_{M*}\left(\pi_X^* \gamma \cdot \bar{i}_* \bar{\pi}^*[\mathcal{Z}]\right) \\
&= \pi_{M*}\bar{i}_*\left(\bar{i}^* \pi_X^* \gamma \cdot \bar{\pi}^*[\mathcal{Z}]\right) \\
&= \pi_{M*}\bar{\pi}_*\left(\pi_D^* i^* \gamma \cdot \bar{\pi}^*[\mathcal{Z}]\right) \\
&= \pi_{M*}\left(\bar{\pi}_* \pi_D^* i^* \gamma \cdot [\mathcal{Z}]\right) \\
&= \pi_{M*}\left(\pi_S^* \pi_* i^* \gamma \cdot [\mathcal{Z}]\right) \in H^2(\operatorname{Hilb}^d(S)),
\end{aligned}
$$

which depends only on $[\mathcal{Z}]$ and hence it is a pullback from $\operatorname{Sym}^d(S)$ by the Hilbert-Chow map

$$
\mathrm{HC}\colon \operatorname{Hilb}^d(S) \to \operatorname{Sym}^d(S).
$$

To sum up, we have

(3.5) $$ P_{d,d[\mathbb{P}^1]}^{\mathrm{JS}}(\gamma_1, \ldots, \gamma_l) = \int_{\operatorname{Hilb}^d(S)} c_{d-1}\left(\mathcal{O}_S^{[d]}\right) \cdot \prod_{i=1}^l \pi_{M*}\left(\pi_S^* \pi_* i^* \gamma_i \cdot [\mathcal{Z}]\right). $$

When $d = 1$, this reduces to [COT22, Lem. 3.7]. When $d > 1$, we claim the above integral is zero. In fact, by [Lehn, Thm. 4.6], we have the formula

$$
\sum_{m \geqslant 0} c\left(\mathcal{O}_S^{[m]}\right) z^m = \exp\left(\sum_{m \geqslant 1} \frac{(-1)^{m-1}}{m} q_m(1) z^m\right) \cdot 1
$$

where $q_m(1)$ are linear maps (called Nakajima operators)

$$
q_m(1) \in \operatorname{End}(\mathbb{H}), \quad \mathbb{H} = \bigoplus_{m \geqslant 0} H^*(\operatorname{Hilb}^m(S), \mathbb{Q}),
$$

which is of bidegree $(m, 2m - 2)$. By looking at the bidegree $(d, 2d - 2)$-part, we have

$$
c_{d-1}\left(\mathcal{O}_S^{[d]}\right) = q_d(1)(1), \quad \text{where } 1 \in H^0(\operatorname{Hilb}^0(S)).
$$

By the definition of $q_d(1)$ in [Lehn, Def. 2.3], we have $q_d(1)(1) = p_{1*}[\mathcal{Q}]$ where $\mathcal{Q}$ is the cycle on $\operatorname{Hilb}^d(S) \times S$ supported on (x, ξ) with $\operatorname{Supp}(\xi) = x$. Therefore we know $c_{d-1}\left(\mathcal{O}_S^{[d]}\right)$ is supported on $\mathrm{HC}^{-1}(\Delta)$, where

$$
\Delta = \left\{(x, \cdots, x) \in \operatorname{Sym}^d(S)\right\} \subseteq \operatorname{Sym}^d(S)
$$

is the small diagonal. Our insertion is a pullback from $\operatorname{Sym}^d(S)$ and gives $(d + 1)$-dimensional constrain on $\operatorname{Sym}^d(S)$. If $d > 1$, $d + 1 > 2 = \dim_{\mathbb{C}} \Delta$, therefore the integral (3.5) is zero. $\qquad \square$

3.2. **Small degree curve classes on $X = T^*\mathbb{P}^2$.** When the $K3$ surface S has a (-2)-curve $C \subset S$, the Hilbert scheme $\operatorname{Hilb}^2(S)$ contains $\operatorname{Sym}^2(C) \subset \operatorname{Hilb}^2(S)$ as a Lagrangian subvariety. For curve classes coming from $\operatorname{Sym}^2(C) \cong \mathbb{P}^2$, our invariants can be studied on the local model $X = T^*\mathbb{P}^2$.

We have an identification of curve classes:

$$
H_2(X, \mathbb{Z}) = H_2(\mathbb{P}^2, \mathbb{Z}) = \mathbb{Z}[\ell],
$$

where $\ell \subset \mathbb{P}^2$ is a line. Let $H \in H^2(T^*\mathbb{P}^2)$ be the pullback of hyperplane class and identify $H_2(T^*\mathbb{P}^2, \mathbb{Z}) \equiv \mathbb{Z}$ by its degree against H. Gopakumar-Vafa invariants are given as follows:

Proposition 3.2. ([COT22, Cor. 6.2])

$$
n_{0,d}(H^2, H^2) = \begin{cases} 1 & \text{if } d = 1, \\ -1 & \text{if } d = 2, \\ 0 & \text{otherwise.} \end{cases}
$$

$$
n_{1,1}(H^2) = 0, \quad n_{2,1} = 0.
$$

In the stable pair side, we compute invariants for small degree curve classes.

Proposition 3.3. *For certain choice of orientation, we have*

$$
P_{1,1}(H^2, H^2) = 1, \quad P_{1,2}(H^2, H^2) = -1, \quad P_{1,3}(H^2, H^2) = 0,
$$

$$
P_{0,1}(H^2) = P_{0,2}(H^2) = 0, \quad P_{0,3}(H^2) = 1, \quad P_{-1,1} = P_{-1,2} = P_{-1,3} = 0.
$$

Moreover, $P_n^t(X, d)$ is independent of the choice of $t > n/d$ in the listed cases above.

In particular, for $X = T^*\mathbb{P}^2$, we have

- *Conjecture 1.10 (2) holds when $d \leqslant 3$.*

- *Conjecture 1.10 (3), (4) hold.*

Proof. As noted in [COT22, Proof of Lem. 6.3], we have a diagram

$$X = T^*\mathbb{P}^2 \overset{i}{\hookrightarrow} \mathcal{O}_{\mathbb{P}^2}(-1)^{\oplus 3}$$
$$\downarrow \pi$$
$$T,$$

where i is a closed imbedding and π contracts $\mathbb{P}^2$ to a point in an affine scheme T. It is easy to see that any one dimensional closed subscheme $C \subset X$ with $[C] = d$ ($d = 1, 2$) satisfies $\chi(\mathcal{O}_C) \geqslant 1$. Therefore by [CT19, Prop. 1,12], we know for $n = -1, 0, 1$ and $d \leqslant 3$, the moduli space $P_n^t(X, d)$ is independent of the choice of $t > n/d$. So we may take $t \to \infty$ and work with PT stability. Using similar analysis as [CKM20, Prop. 3.9], we know all stable pairs $(\mathcal{O}_X \overset{s}{\to} F)$ in the above cases are scheme theoretically supported on the zero section $\mathbb{P}^2 \subset X$ and F are stable. Then obviously $P_{-1}(X, d) = \emptyset$ if $d \leqslant 3$ and corresponding invariants vanish.

When $n = 1$, $d \leqslant 3$, the isomorphism

$$P_1(X, d) \cong M_1(X, d), \quad (\mathcal{O}_X \to F) \mapsto F,$$

to the moduli space of one dimensional stable sheaves F with $[F] = d[\ell]$ and $\chi(F) = 1$ will reduce the computation to the corresponding one on $M_1(X, d)$ [COT22, Prop. 6.5].

When $d = 1, 2$, we have $P_0(X, d) = \emptyset$, so invariants are zero. For $d = 3$, the support map

$$P_0(X, 3) \cong P_0(\mathbb{P}^2, 3) \overset{\cong}{\to} |\mathcal{O}_{\mathbb{P}^2}(3)| \cong \mathbb{P}^9, \quad F \mapsto \mathrm{supp}(F)$$

is an isomorphism. The universal one dimensional sheaf satisfies $\mathbb{F} = \mathcal{O}_{\mathcal{C}}$ for the universal $(1, 3)$-divisor $\mathcal{C} \hookrightarrow \mathbb{P}^9 \times \mathbb{P}^2$. Let $\pi_M \colon P_0(X, 3) \times \mathbb{P}^2 \to P_0(X, 3)$ be the projection. Bott's formula implies that

$$\mathbf{R}\mathcal{H}om_{\pi_M}(\mathcal{O}, \mathcal{O}(-\mathcal{C}) \boxtimes T^*\mathbb{P}^2) \cong \mathcal{O}_{\mathbb{P}^9}(-1)[-2]^{\oplus 8},$$
$$\mathbf{R}\mathcal{H}om_{\pi_M}(\mathcal{O}, \mathcal{O}(\mathcal{C}) \boxtimes T^*\mathbb{P}^2) \cong \mathcal{O}_{\mathbb{P}^9}(-1)^{\oplus 8},$$
$$\mathbf{R}\mathcal{H}om_{\pi_M}(\mathcal{O}, \mathcal{O} \boxtimes T^*\mathbb{P}^2) \cong \mathcal{O}_{\mathbb{P}^9}[-1].$$

Therefore, we have

$$\mathbf{R}\mathcal{H}om_{\pi_M}(\mathcal{O}_{\mathcal{C}}, \mathcal{O}_{\mathcal{C}} \boxtimes T^*\mathbb{P}^2)[1]$$
$$\cong \mathbf{R}\mathcal{H}om_{\pi_M}(\mathcal{O}(-\mathcal{C}) \to \mathcal{O}, (\mathcal{O}(-\mathcal{C}) \to \mathcal{O}) \boxtimes T^*\mathbb{P}^2)[1]$$
$$\cong \mathcal{O}_{\mathbb{P}^9}(-1)^{\oplus 8} \oplus \mathcal{O}_{\mathbb{P}^9}(1)^{\oplus 8} \oplus \mathcal{O}_{\mathbb{P}^9} \oplus \mathcal{O}_{\mathbb{P}^9}.$$

By Grothendieck-Verdier duality, it is easy to see that

$$\mathcal{O}_{\mathbb{P}^9}(-1)^{\oplus 8} \oplus \mathcal{O}_{\mathbb{P}^9}$$

is a maximal isotropic subbundle of $\mathbf{R}\mathcal{H}om_{\pi_M}(\mathcal{O}_{\mathcal{C}}, \mathcal{O}_{\mathcal{C}} \boxtimes T^*\mathbb{P}^2)[1]$. The reduced virtual class satisfies

$$[P_0(X, 3)]^{\mathrm{vir}} = \pm e\left(\mathcal{O}_{\mathbb{P}^9}(-1)^{\oplus 8}\right) \cap [\mathbb{P}^9] \in H_2(\mathbb{P}^9).$$

Let $h \in H^2(\mathbb{P}^9)$ denote the hyperplane class. It is straightforward to check

$$\tau_0(H^2) = [h].$$

By integration again the virtual class, we have the desired result. $\square$

APPENDIX A. A CONJECTURAL VIRTUAL PUSHFORWARD FORMULA

Let $\beta \in H_2(X, \mathbb{Z})$ be an irreducible curve class on a holomorphic symplectic 4-fold X. There is a well-defined forgetful map

$$f \colon P_n(X, \beta) \to M_n(X, \beta), \quad (\mathcal{O}_X \to F) \mapsto [F],$$

to the coarse moduli scheme of one dimensional stable sheaves F with $[F] = \beta$, $\chi(F) = n$.

Motivated by the Thom-Porteus formula (Proposition 2.9), we conjecture the following:

Conjecture A.1. *In the above setting, there exists a choice of orientation such that*

$$f_*[P_n(X, \beta)]^{\mathrm{vir}} = c_{1-n}(-\mathbf{R}\pi_{M*}(\mathbb{F})) \cap [M_n(X, \beta)]^{\mathrm{vir}},$$

where $\pi_M \colon M_n(X, \beta) \times X \to M_n(X, \beta)$ is the projection and $\mathbb{F}$ is a universal sheaf (if exists).

Remark A.2. *This should be proved by adapting Park's beautiful work on virtual pullback [Park] to the cosection localized version.*

We can rewrite the degree of $[P_{-1}(X, \beta)]^{\mathrm{vir}}$ as a descendent integral on $[M_1(X, \beta)]^{\mathrm{vir}}$. Let $\mathbb{F}_{\mathrm{norm}}$ be the normalized universal sheaf on $M_1(X, \beta) \times X$, i.e. $\det(\mathbf{R}\pi_{M*}\mathbb{F}_{\mathrm{norm}}) \cong \mathcal{O}_{M_1(X,\beta)}$.

Proposition A.3. *Assume Conjecture A.1. For certain choice of orientation, we have*

$$P_{-1,\beta} = -\int_{[M_1(X,\beta)]^{\mathrm{vir}}} \pi_{M*}\left(\mathrm{ch}_6(\mathbb{F}_{\mathrm{norm}})\right) - \frac{1}{12}\int_{[M_1(X,\beta)]^{\mathrm{vir}}} \pi_{M*}\left(\mathrm{ch}_4(\mathbb{F}_{\mathrm{norm}})\, \pi_X^*(c_2(X))\right).$$

Using notations from [COT22, §2.1], *this is written as*

$$(A.1) \qquad\qquad P_{-1,\beta} = -\langle \tau_3(1) \rangle_\beta^{\mathrm{DT}_4} - \frac{1}{12}\langle \tau_1(c_2(X)) \rangle_\beta^{\mathrm{DT}_4}.$$

Proof. The derived dual gives an isomorphism

$$M_{-1}(X, \beta) \cong M_1(X, \beta), \quad F \mapsto F^\vee.$$

Then the computation is finished by applying the Grothendieck-Riemann-Roch formula. $\qquad\square$

Remark A.4. *Based on Conjecture 1.10 (4), this reproduces genus 2 Gopakumar-Vafa invariants of X and therefore providing a sheaf theoretic approach to them using descendent integrals on moduli spaces of one dimensional stable sheaves as* [CT20a].

Proposition A.5. *Conjecture A.1 holds on the product $X = S \times T$ of two K3 surfaces. In particular, Eqn. (A.1) holds in this case.*

Proof. Say $\beta \in H_2(S, \mathbb{Z}) \subseteq H_2(X, \mathbb{Z})$, we have

$$f = f_S \times \mathrm{id}_T : P_n(X, \beta) \cong P_n(S, \beta) \times T \to M_n(X, \beta) \cong M_n(S, \beta) \times T,$$

for forgetful map $f_S : P_n(S, \beta) \to M_n(S, \beta)$.

By Theorem 2.8 and [COT22, Thm. 5.7], for certain choice of orientation, we have

$$(A.2)$$
$$[P_n(X, \beta)]^{\mathrm{vir}} = \left([P_n(S, \beta)] \cap f_S^* e(T_{M_n(S,\beta)})\right) \times [T] - e(T)\left([P_n(S, \beta)] \cap f_S^* c_{\beta^2}(T_{M_n(S,\beta)})\right),$$
$$[M_n(X, \beta)]^{\mathrm{vir}} = \left([M_n(S, \beta)] \cap e(T_{M_n(S,\beta)})\right) \times [T] - e(T)\left([M_n(S, \beta)] \cap c_{\beta^2}(T_{M_n(S,\beta)})\right).$$

Also note that a universal sheaf $\mathbb{F}$ on $M_n(X, \beta) \times X$ (if exists) is of form

$$\mathbb{F} = \mathbb{F}_S \boxtimes \mathcal{O}_{\Delta_T},$$

where $\mathbb{F}_S$ is a universal sheaf on $M_n(S, \beta) \times S$ and Δ_T is the diagonal of $T \times T$. So

$$(A.3) \qquad\qquad \mathbf{R}\pi_{M*}\mathbb{F} = \mathbf{R}\pi_{M_S*}\mathbb{F}_S,$$

where $\pi_{M_S} : M_n(S, \beta) \times S \to M_n(S, \beta)$ is the projection. Combining Eqns. (A.2), (A.3), we are reduced to Proposition 2.9. $\qquad\square$

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RIKEN Interdisciplinary Theoretical and Mathematical Sciences Program (iTHEMS), 2-1, Hirosawa, Wako-shi, Saitama, 351-0198, Japan
Email address: yalong.cao@riken.jp

University of Bonn, Institut für Mathematik
Email address: georgo@math.uni-bonn.de

Kavli Institute for the Physics and Mathematics of the Universe (WPI), The University of Tokyo Institutes for Advanced Study, The University of Tokyo, Kashiwa, Chiba 277-8583, Japan
Email address: yukinobu.toda@ipmu.jp