GOPAKUMAR-VAFA TYPE INVARIANTS OF HOLOMORPHIC SYMPLECTIC 4-FOLDS

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Abstract. Using reduced Gromov-Witten theory, we define new invariants which capture the enumerative geometry of curves on holomorphic symplectic 4-folds. The invariants are analogous to the BPS counts of Gopakumar and Vafa for Calabi-Yau 3-folds, Klemm and Pandharipande for Calabi-Yau 4-folds, and Pandharipande and Zinger for Calabi-Yau 5-folds.

We conjecture that our invariants are integers and give a sheaf-theoretic interpretation in terms of reduced 4-dimensional Donaldson-Thomas invariants of one-dimensional stable sheaves. We check our conjectures for the product of two $K3$ surfaces and for the cotangent bundle of $\mathbb{P}^2$. Modulo the conjectural holomorphic anomaly equation, we compute our invariants also for the Hilbert scheme of two points on a $K3$ surface. This yields a conjectural formula for the number of isolated genus 2 curves of minimal degree on a very general hyperkähler 4-fold of $K3^{[2]}$-type. The formula may be viewed as a 4-dimensional analogue of the classical Yau-Zaslow formula concerning counts of rational curves on $K3$ surfaces.

In the course of our computations, we also derive a new closed formula for the Fujiki constants of the Chern classes of tangent bundles of both Hilbert schemes of points on $K3$ surfaces and generalized Kummer varieties.

Contents

0. Introduction

0.1. Gopakumar-Vafa invariants. Gromov-Witten invariants of a smooth projective variety X are defined by integration over the virtual class [BF, LT] of the moduli space $\overline{M}_{g,n}(X,\beta)$ of genus g degree $\beta \in H_2(X,\mathbb{Z})$ stable maps:

$$\left\langle \tau_{k_1}(\gamma_1)\cdots\tau_{k_n}(\gamma_n)\right\rangle_{g,\beta}^{\mathrm{GW}} = \int_{[\overline{M}_{g,n}(X,\beta)]^{\mathrm{vir}}} \prod_{i=1}^{n} \mathrm{ev}_i^*(\gamma_i)\cdot\psi_i^{k_i}. \tag{0.1}$$

Here $\mathrm{ev}_i\colon \overline{M}_{g,n}(X,\beta)\to X$ is the evaluation map at the i-th marking, ψ_i is the i-th cotangent line class, and $\gamma_i \in H^*(X,\mathbb{Q})$ are cohomology classes. Since $\overline{M}_{g,n}(X,\beta)$ is a Deligne-Mumford stack, Gromov-Witten invariants are in general rational numbers, even if all γ_i are integral. Moreover the enumerative meaning of Gromov-Witten invariants is often not clear.

For Calabi-Yau 3-folds, Gopakumar and Vafa [GV] found explicit linear transformations which transform the Gromov-Witten invariants to a set of invariants (called *Gopakumar-Vafa invariants*) which they conjectured to be integers. In an ideal geometry, where all curves are isolated, disjoint and smooth, Gopakumar-Vafa invariants should be the actual count of curves of given genus and degree. The integrality of Gopakumar-Vafa invariants was proven recently in [IP]. A similar transformation of Gromov-Witten invariants into (conjectural) $\mathbb{Z}$-valued invariants has been proposed for Calabi-Yau 4-folds by Klemm and Pandharipande [KP], and for Calabi-Yau 5-folds by Pandharipande and Zinger [PZ]. Universal transformations are expected in every dimension [KP].

Let X be a holomorphic symplectic 4-fold, by which we mean a smooth complex projective 4-fold which is equipped with a non-degenerate holomorphic 2-form $\sigma \in H^0(X, \Omega_X^2)$. Since the obstruction sheaf has a trivial quotient, the ordinary Gromov-Witten invariants of X vanish for all non-zero curve classes. As a result, also all Klemm-Pandharipande invariants of X vanish. Instead a reduced Gromov-Witten theory is obtained by Kiem-Li's cosection localization [KiL]. It is defined as in (0.1) but by integration over the *reduced* virtual fundamental class:[1]

$$[\overline{M}_{g,n}(X,\beta)]^{\mathrm{vir}} \in A_{2-g+n}(\overline{M}_{g,n}(X,\beta)). \tag{0.2}$$

We are interested here in integer-valued invariants, which underlie the (reduced) Gromov-Witten invariants (0.1) of the holomorphic symplectic 4-fold X. In genus 0, all Gromov-Witten invariants can be reconstructed from the *primary invariants*, i.e. the integrals (0.1) where all $k_i = 0$. Our proposal for the genus 0 primary invariants is as follows:

Definition 0.1. (Definition 1.5) *For any $\gamma_1, \ldots, \gamma_n \in H^*(X,\mathbb{Z})$, we define the genus 0 Gopakumar-Vafa invariant $n_{0,\beta}(\gamma_1, \ldots, \gamma_n) \in \mathbb{Q}$ recursively by:*

$$\left\langle \tau_0(\gamma_1)\cdots\tau_0(\gamma_n)\right\rangle_{0,\beta}^{\mathrm{GW}} = \sum_{k\geqslant 1, k|\beta} k^{n-3}\, n_{0,\beta/k}(\gamma_1,\ldots,\gamma_n).$$

In fact, through a twistor space construction, this definition follows immediately from a similar definition on Calabi-Yau 5-folds given in [PZ] (see §1.3 for more explanations).

In genus 1, the situation is more complicated and does not follow from 5-fold geometry. Since the virtual dimension of (0.2) is $1+n$, we require one marked point and an insertion $\gamma \in H^4(X,\mathbb{Z})$. Because curves in imprimitive curve classes are very difficult to control in an ideal geometry (see Section 1.4) we will restrict us to a primitive curve class (i.e. where β is not a multiple of a class in $H_2(X,\mathbb{Z})$).

Definition 0.2. (Definition 1.6) *Assume that $\beta \in H_2(X,\mathbb{Z})$ is primitive. For any $\gamma \in H^4(X,\mathbb{Z})$, we define the genus 1 Gopakumar-Vafa invariant $n_{1,\beta}(\gamma) \in \mathbb{Q}$ by*

$$\left\langle \tau_0(\gamma)\right\rangle_{1,\beta}^{\mathrm{GW}} = n_{1,\beta}(\gamma) - \frac{1}{24}\left\langle \tau_0(\gamma)\tau_0(c_2(T_X))\right\rangle_{0,\beta}^{\mathrm{GW}}.$$

In genus 2, the situation is even more complicated and attracting. In fact, the appearance of genus 2 invariants is a new phenomenon that is not available on ordinary Calabi-Yau 4-folds and Calabi-Yau 5-folds. By the virtual dimension of (0.2), one expects a finite number of isolated genus 2 curves. The genus 2 Gopakumar-Vafa invariant should be a count of these curves.

Definition 0.3. (Definition 1.7) *Assume that $\beta \in H_2(X,\mathbb{Z})$ is primitive. We define the genus 2 Gopakumar-Vafa invariant $n_{2,\beta} \in \mathbb{Q}$ by*

$$\left\langle \varnothing\right\rangle_{2,\beta}^{\mathrm{GW}} = n_{2,\beta} - \frac{1}{24} n_{1,\beta}(c_2(X)) + \frac{1}{2\cdot 24^2}\left\langle \tau_0(c_2(X))\tau_0(c_2(X))\right\rangle_{0,\beta}^{\mathrm{GW}} + \frac{1}{24} N_{\mathrm{nodal},\beta},$$

where $N_{\mathrm{nodal},\beta} \in \mathbb{Q}$ is the virtual count of rational nodal curves as defined in Eqn. (1.4).

Our first main conjecture is about the integrality of these definitions:

Conjecture 0.4. (Conjecture 1.9) *With the notations as above, we have*

$$n_{0,\beta}(\gamma_1,\ldots,\gamma_n),\ \ n_{1,\beta}(\gamma),\ \ n_{2,\beta} \in \mathbb{Z}.$$

The definitions above are found via computations in an 'ideal' geometry where we assume that algebraic curves behave in the expected way, see §1.4, §1.5.[2] We justify Conjecture 0.4 in such an ideal case, which takes the whole §1.6, §1.7.

0.2. **GV/DT_4 correspondence.** The second main theme of this paper is to give a sheaf theoretic interpretation of Gopakumar-Vafa invariants. This is motivated by the parallel work of [CMT18, CT20a] on ordinary Calabi-Yau 4-folds.

Let M_β be the moduli scheme of one dimensional stable sheaves F on X with $\mathrm{ch}_3(F) = \beta$, $\chi(F) = 1$. By [KiP, Sav], the ordinary DT_4 virtual class [BJ, OT] (see also [CL14]) of M_β vanishes. By Kiem-Park's cosection localization [KiP], we instead have a (reduced) virtual class

$$[M_\beta]^{\mathrm{vir}} \in A_2(M_\beta, \mathbb{Q}). \tag{0.3}$$

As usual, the virtual class depends on a choice of orientation [CGJ, CL17]. More precisely, for each connected component of M_β, there are two choices of orientation which affect the

[1] We will only work with the reduced virtual class in this paper, hence we will denote it simply by $[-]^{\mathrm{vir}}$.

[2] Similar considerations in ideal geometries were taken by Klemm-Pandharipande on Calabi-Yau 4-folds [KP] and Pandharipande-Zinger on Calabi-Yau 5-folds [PZ], though our case looks more complicated (see §1.4 for details).

virtual class by a sign (component-wise). To define descendent invariants, consider the insertion operators:

$$\tau_i : H^*(X,\mathbb{Z}) \to H^{*+2i-2}(M_\beta,\mathbb{Q}),$$

$$\tau_i(\bullet) := (\pi_M)_* \left(\pi_X^*(\bullet) \cup \mathrm{ch}_{3+i}(\mathbb{F}_{\mathrm{norm}})\right),$$

where $\mathbb{F}_{\mathrm{norm}}$ is the normalized universal sheaf, i.e. $\det(\pi_{M*}\mathbb{F}_{\mathrm{norm}}) \cong \mathcal{O}_{M_\beta}$. As in Gromov-Witten theory, for any $\gamma_1,\ldots,\gamma_n \in H^*(X,\mathbb{Z})$ and $k_1,\ldots,k_n \in \mathbb{Z}_{\geqslant 0}$, we define DT_4 invariants:

$$\left\langle \tau_{k_1}(\gamma_1),\ldots,\tau_{k_n}(\gamma_n)\right\rangle_\beta^{\mathrm{DT}_4} := \int_{[M_\beta]^{\mathrm{vir}}} \prod_{i=1}^n \tau_{k_i}(\gamma_i) \in \mathbb{Q}. \tag{0.4}$$

Here is the second main conjecture of this paper, which gives a sheaf theoretic interpretation of our Gopakumar-Vafa invariants.

Conjecture 0.5. (Conjecture 2.2) *For certain choice of orientation, the following holds. When β is an effective curve class,*

(i) $$\left\langle \tau_0(\gamma_1),\ldots,\tau_0(\gamma_n)\right\rangle_\beta^{\mathrm{DT}_4} = n_{0,\beta}(\gamma_1,\ldots,\gamma_n).$$

When β is a primitive curve class,

(ii) $$\left\langle \tau_1(\gamma)\right\rangle_\beta^{\mathrm{DT}_4} = -\frac{1}{2}\left\langle \tau_1(\gamma)\right\rangle_{0,\beta}^{\mathrm{GW}} - n_{1,\beta}(\gamma).$$

When β is a primitive curve class,

(iii) $$-\left\langle \tau_3(1)\right\rangle_\beta^{\mathrm{DT}_4} - \frac{1}{12}\left\langle \tau_1(c_2(X))\right\rangle_\beta^{\mathrm{DT}_4} = n_{2,\beta}.$$

As in Conjecture 0.4, we verify these equalities in the ideal geometry (see §2.3, §2.4 and also §3 for details). An exception is the last equality involving genus 2 invariants, which we obtain indirectly through stable pair theory [COT22] (see Remark 2.3).

Besides computations in the ideal geometry mentioned above, we study several examples and prove our conjectures in those cases.

0.3. **Verification of conjectures I:** $K3 \times K3$**.** Let $X = S \times T$ be the product of two $K3$ surfaces. When the curve class $\beta \in H_2(S\times T,\mathbb{Z})$ is of non-trivial degree over both S and T, then the obstruction sheaf of the moduli space of stable maps has two linearly independent cosections, which implies that the (reduced) Gromov-Witten invariants of X in this class vanish. Therefore we always restrict ourselves to curve classes of form

$$\beta \in H_2(S) \subseteq H_2(X). \tag{0.5}$$

By Behrend's product formula [B99] (see Eqn. (5.1)), we can easily compute all Gromov-Witten invariants and determine the Gopakumar-Vafa invariants as follows.

Theorem 0.6. (Proposition 5.1) *Let $\gamma,\gamma' \in H^4(X)$, $\alpha \in H^6(X)$ and let*

$$\gamma = A_1\cdot 1\otimes \mathsf{p} + D_1\otimes D_2 + A_2\cdot \mathsf{p}\otimes 1, \quad \gamma' = A_1'\cdot 1\otimes \mathsf{p} + D_1'\otimes D_2' + A_2'\cdot \mathsf{p}\otimes 1,$$

$$\alpha = \theta_1 \otimes \mathsf{p} + \mathsf{p}\otimes\theta_2$$

be their Künneth decompositions. Then we have

$$n_{0,\beta}(\gamma,\gamma') = (D_1\cdot\beta)\cdot(D_1'\cdot\beta)\cdot\int_T (D_2\cdot D_2')\cdot N_0\left(\frac{\beta^2}{2}\right),$$

$$n_{0,\beta}(\alpha) = (\theta_1\cdot\beta)\, N_0\left(\frac{\beta^2}{2}\right).$$

If β is primitive, we have

$$n_{1,\beta}(\gamma) = 24A_2\, N_1(\beta^2/2), \quad n_{2,\beta} = N_2\left(\frac{\beta^2}{2}\right),$$

where

$$\sum_{l\in\mathbb{Z}} N_0(l)\, q^l = \frac{1}{q}\prod_{n\geqslant 1}\frac{1}{(1-q^n)^{24}},$$

$$\sum_{l\in\mathbb{Z}} N_1(l)\, q^l = \left(\frac{1}{q}\prod_{n\geqslant 1}\frac{1}{(1-q^n)^{24}}\right)\left(q\frac{d}{dq}G_2(q)\right),$$

$$\sum_{l\in\mathbb{Z}} N_2(l)\, q^l = \left(\frac{1}{q}\prod_{n\geqslant 1}\frac{1}{(1-q^n)^{24}}\right)\left(24q\frac{d}{dq}G_2 - 24G_2 - 1\right),$$

with Eisenstein series:

$$G_2(q) = -\frac{1}{24} + \sum_{n\geqslant 1}\sum_{d|n} d q^n.$$

In particular, Conjecture 0.4 holds for $X = S\times T$.

On the Donaldson-Thomas side, a main result of this paper is the explicit computation of all DT_4 invariants of $X = S\times T$ for the classes (0.5), see Theorem 5.8 for the formulae. We obtain a perfect match with our prediction:

Theorem 0.7 (Corollary 5.9)**.** *Conjecture 0.5 holds for $X = S\times T$ and all effective curve classes $\beta\in H_2(S,\mathbb{Z})\subseteq H_2(X,\mathbb{Z})$.*

Here, since the moduli space M_β is connected, there are precisely two choices of orientation. We pick the one specified in Eqn. (5.10) (invariants for the other differ only by an overall sign).

Contrary to the case of Gromov-Witten invariants, the computation of DT_4 invariants on $S\times T$ is highly non-trivial. In Theorem 5.7, we first identify the virtual class explicitly. This expresses the DT_4 invariants as tautological integrals on a (smooth) moduli space of one dimensional stable sheaves on the $K3$ surface S. By Markman's framework of monodromy operators [M08], we then relate such integrals to tautological integrals on the Hilbert schemes of points on S (see §4.3 and §4.4 for details). Finally, we determine these integrals explicitly in §4.1 and §4.2 by a combination of the universality result of Ellingsrud-Göttsche-Lehn [EGL], constraints from Looijenga-Lunts-Verbitsky Lie algebra [LL, Ver13] and known computations of Euler characteristics.

In particular, we found a remarkable closed formula for Fujiki constants of Chern classes of Hilbert schemes $S^{[n]}$ of points on S, which takes the following beautiful form (see also Proposition 4.3 for the formula on generalized Kummer varieties):

Theorem 0.8. (Theorem 4.2) *Let S be a $K3$ surface. For any $k\geqslant 0$,*

$$\sum_{n\geqslant k} C(c_{2n-2k}(T_{S^{[n]}}))\, q^n = \frac{(2k)!}{k!2^k}\left(q\frac{d}{dq}G_2(q)\right)^k\prod_{n\geqslant 1}\frac{1}{(1-q^n)^{24}}.$$

The right hand side, up to the combinatorical prefactor $(2k)!/(k!2^k)$, is precisely the generating series of counts of genus k curves on a $K3$ surface passing through k generic points as considered by Bryan and Leung [BL]. This suggests a relationship to the work of Göttsche on curve counting on surfaces [G98], which will be taken up in a follow-up work.

0.4. **Verification of conjectures II: $T^*\mathbb{P}^2$.** Let $T^*\mathbb{P}^2$ be the total space of the cotangent bundle on $\mathbb{P}^2$, which is holomorphic symplectic. Let $H\in H^2(T^*\mathbb{P}^2)$ be the pullback of the hyperplane class and use the identification $H_2(T^*\mathbb{P}^2,\mathbb{Z})\equiv\mathbb{Z}$ given by taking the degree against H. By Graber-Pandharipande's virtual localization formula [GP], we can compute all genus Gromov-Witten invariants (Proposition 6.1) and determine the Gopakumar-Vafa invariants.

Proposition 0.9. (Corollary 6.2)

$$n_{0,d}(H^2,H^2) = \begin{cases} 1 & \text{if } d=1,\\ -1 & \text{if } d=2,\\ 0 & \text{otherwise.}\end{cases}$$

$$n_{1,1}(H^2) = 0, \quad n_{2,1} = 0.$$

In particular, Conjecture 0.4 holds for $T^*\mathbb{P}^2$.

On the sheaf side, we can compute DT_4 invariants for small degree curve classes.

Proposition 0.10. (Proposition 6.5) *For certain choice of orientation, we have*

$$\big\langle\tau_0(H^2),\tau_0(H^2)\big\rangle_1^{\mathrm{DT}_4}=1,\quad \big\langle\tau_0(H^2),\tau_0(H^2)\big\rangle_2^{\mathrm{DT}_4}=-1,\quad \big\langle\tau_0(H^2),\tau_0(H^2)\big\rangle_3^{\mathrm{DT}_4}=0,$$

$$\big\langle\tau_1(H^2)\big\rangle_1^{\mathrm{DT}_4}=-\frac{1}{2},\quad \big\langle\tau_1(H^2)\big\rangle_2^{\mathrm{DT}_4}=\frac{1}{2},\quad \big\langle\tau_1(H^2)\big\rangle_3^{\mathrm{DT}_4}=0,$$

$$\big\langle\tau_2(H)\big\rangle_1^{\mathrm{DT}_4}=-\frac{1}{4},\quad \big\langle\tau_2(H)\big\rangle_2^{\mathrm{DT}_4}=-\frac{1}{4},\quad \big\langle\tau_2(H)\big\rangle_3^{\mathrm{DT}_4}=0,$$

$$\big\langle\tau_3(1)\big\rangle_1^{\mathrm{DT}_4}=-\frac{1}{8},\quad \big\langle\tau_3(1)\big\rangle_2^{\mathrm{DT}_4}=\frac{1}{8},\quad \big\langle\tau_3(1)\big\rangle_3^{\mathrm{DT}_4}=0.$$

In particular, Conjecture 0.5 (i) *holds for all $d\leqslant 3$, and Conjecture 0.5* (ii), (iii) *hold.*

0.5. **Verification of conjectures III: $K3^{[2]}$.** Consider the Hilbert scheme $S^{[2]}$ of two points on a $K3$ surface S. By a result of Beauville [Bea], $S^{[2]}$ is irreducible hyperkähler, i.e. it is simply connected and the space of its holomorphic 2-forms is spanned by a (unique) symplectic form. Because the genus 0 Gromov-Witten theory of $S^{[2]}$ is completely known by [O18, O21a, O21c] (see Theorem 7.3 for the primitive case), all genus 0 Gopakumar-Vafa invariants are easily computed. For simplicity, we check the integrality conjecture in the following basic case (ref. §7.7):

Theorem 0.11. *Conjecture 0.4 holds for all effective curve classes on $S^{[2]}$ in genus* 0 *and with one marked point.*

Higher genus Gromov-Witten invariants are more difficult to compute even for primitive curve classes. Nevertheless there are several conjectures on the structure of these invariants, including (i) a *quasi-Jacobi form property*, and (ii) a *holomorphic anomaly equation* (see [O22b, Conj. A & C], see also [O21b] for a progress report). Assuming these conjectures and using several explicit evaluations of Gromov-Witten invariants, we obtain a complete computation of all genus 1 and 2 Gromov-Witten invariants of $S^{[2]}$ in primitive classes, see Theorem 7.4. From this, all Gopakumar-Vafa invariants are computed in Theorems 7.6 and 7.10.

With the help of a computer program, we obtain the following check of integrality:

Theorem 0.12. (Corollaries 7.8 and 7.11) *Assume Conjectures A and C of* [O22b]. *Then the genus 1 and 2 part of Conjecture 0.4 hold for all primitive curve classes $\beta\in H_2(S^{[2]},\mathbb{Z})$ satisfying $(\beta,\beta)\leqslant 100$, where $(-,-)$ is the Beauville-Bogomolov-Fujiki pairing as in §7.2.*

0.6. **A Yau-Zaslow type formula on $K3^{[2]}$.** A hyperkähler variety is of $K3^{[2]}$-type if it is deformation-equivalent to the Hilbert scheme of 2 points of a $K3$ surface S. Given a primitive curve class $\beta\in H_2(S^{[2]},\mathbb{Z})$, consider the very general deformation (X,β') of a pair $(S^{[2]},\beta)$, where β stays of Hodge type on all fibers. By the deformation theory of hyperkähler varieties, the variety X then has Picard rank 1 and the algebraic classes in $H_2(X,\mathbb{Z})$ are generated by β'. In particular β' is irreducible. In this case, it is natural to expect that curves in (X,β') forms an ideal geometry in the sense of §1.4, §1.5. In other words, after a generic deformation, our Gopakumar-Vafa invariants should give enumerative information about curves in these hyperkähler varieties of $K3^{[2]}$-type.

In genus 2, this yields the following conjectural formula for the number of isolated (rigid) genus 2 curves on a very general hyperkähler variety of $K3^{[2]}$-type of minimal degree. This may be viewed as a 4-dimensional analogue of the classical Yau-Zaslow formula concerning counts of rational curves on $K3$ surfaces:

Theorem 0.13. (Theorem 7.10) *Assume Conjectures A and C of* [O22b]. *For any hyperkähler variety X of $K3^{[2]}$-type and primitive curve class $\beta\in H_2(X,\mathbb{Z})$, the genus* 2 *Gopakumar-Vafa invariant $n_{2,\beta}$ is the coefficient determined by β (see Definition 7.1) of the quasi-Jacobi form*

$$\begin{aligned}\widetilde{I}(y,q)=\frac{\Theta^2}{\Delta}\Bigg[&\frac{5}{384}\wp E_2^3+\frac{25}{6144}E_2^4+\frac{35}{384}\wp E_2^2-\frac{5}{512}E_2^3+\frac{5}{384}\wp E_2E_4+\frac{7}{3072}E_2^2E_4\\&-\frac{71}{64}\wp E_2+\frac{27}{512}E_2^2-\frac{47}{384}\wp E_4+\frac{5}{4608}E_2E_4-\frac{13}{18432}E_4^2-\frac{1}{96}\wp E_6\\&+\frac{1}{1152}E_2E_6+\frac{9}{8}\wp-\frac{5}{32}E_2-\frac{23}{1536}E_4-\frac{5}{1152}E_6+\frac{1}{8}\Bigg],\end{aligned}$$

where the functions $\Theta,\Delta,\wp,E_i$ are defined in §7.1.

In genus 1, it is convenient to encode the invariants in the *genus 1 Gopakumar-Vafa class*

$$n_{1,\beta} \in H^4(X, \mathbb{Q})$$

which is defined by

$$\forall\, \gamma \in H^4(X, \mathbb{Q}) : \quad \int_X n_{1,\beta} \cup \gamma = n_{1,\beta}(\gamma),$$

where $n_{1,\beta}(\gamma)$ is given in Definition 0.2. In an ideal geometry, $n_{1,\beta}$ is the class of the surface swept out by elliptic curves in class β. Theorem 7.6 then yields a conjectural formula for this class. We list the first values of the genus 1 and 2 Gopakumar-Vafa invariants of hyperkähler varieties of $K3^{[2]}$-type in Table 1 and Table 2 below. Since the deformation class of a pair (X, β) where β is a primitive curve class, only depends on the square (β, β) (see [O21a]), the Gopakumar-Vafa invariants only depend on (β, β).

It is interesting to compare the enumerative significance of the listed invariants with the known geometry of curves on very general hyperkähler 4-folds of $K3^{[2]}$-type with curve class β. In the case $(\beta, \beta) = -5/2$, any curve in class β is a line in a Lagrangian $\mathbb{P}^2 \subset X$, see [HT]. In particular, there are no higher genus curves, and indeed we observe the vanishing of the $g = 1, 2$ Gopakumar-Vafa invariants in this case. Similarly, the case $(\beta, \beta) = -1/2$ corresponds to the exeptional curve class on $K3^{[2]}$ (the class of the exceptional curve of the Hilbert-Chow morphism $K3^{[2]} \to \mathrm{Sym}^2(K3)$), and again there are no higher genus curves. The case $(\beta, \beta) = -2$ is similar, see [HT]. The first time we see elliptic curves is in case $(\beta, \beta) = 0$, which corresponds to the fiber class of a Lagrangian fibration $X \to \mathbb{P}^2$. Elliptic curves appear here in fibers over the discriminant. The case $(\beta, \beta) = 3/2$ corresponds to a very general Fano variety of lines on a cubic 4-fold, with β the minimal curve class (of degree 3 against the Plücker polarization). Since there are no cubic genus 2 curves in a projective space (see also Example 1.10), there are no genus 2 curves in this class; again, this matches the vanishing observed in the table. The case $(\beta, \beta) = 2$ are the double covers of EPW sextics [O06]. The first time we should see isolated smooth genus 2 curves is the case $(\beta, \beta) = 11/2$, which are precisely the Debarre-Voisin 4-folds [DV]. Here, the explicit geometry of curves has not been studied yet. It would be very interesting to construct the expected 3465 isolated smooth genus 2 curves explicitly. In fact, to the best of the authors' knowledge, there exists so far no known example of a smooth isolated (rigid) genus 2 curves on a hyperkähler 4-fold, and this may be perhaps the simplest case.

(β,β)	a_β	b_β
$-5/2$	0	0
-2	0	0
$-1/2$	0	0
0	6	1
$3/2$	105	$35/8$
2	360	30
$7/2$	3840	40
4	9360	300
$11/2$	74970	$-6405/4$
6	157080	-1540
$15/2$	1034496	-55224
8	1982820	-94570
$19/2$	11288760	-965720
10	20371680	-1702680

(β,β)	a_β	b_β
$23/2$	103461120	-12187560
12	178607520	-21135240
$27/2$	826591920	-124077800
14	1378589520	-210090760
$31/2$	5903493120	-1077138720
16	9574935480	-1781067420
$35/2$	38376042111	$-65957272227/8$
18	60812926920	-13338391770
$39/2$	230147470080	-56902511160
20	357559991712	-90266652168
$43/2$	1286717384040	-359854419320
22	1965075202440	-560881363980
$47/2$	6762292992000	-2110582343520
24	10172904142800	-3237985250920

TABLE 1. The first coefficients of the genus 1 Gopakumar-Vafa class[3]

$$n_{1,\beta} = \frac{1}{2} a_\beta h_\beta^2 + b_\beta c_2(T_X)$$

for a hyperkähler 4-fold of $K3^{[2]}$-type with primitive curve class β (see §7.2 for the definition of the dual divisor h_β). In an ideal geometry (ref. §1.5), $n_{1,\beta}$ is the class of the surface swept out by the elliptic curves in class β.

0.7. **Appendix.** In the appendix §A, we discuss several cases where we can extend the above GW/GV/DT_4 correspondence to imprimitive curve classes.

(β,β)	$n_{2,\beta}$
$-5/2$	0
-2	0
$-1/2$	0
0	0
$3/2$	0
2	0
$7/2$	0
4	0
$11/2$	3465
6	7920
$15/2$	153720
8	321300
$19/2$	3527370
10	6902280

(β,β)	$n_{2,\beta}$
$23/2$	55981800
12	104091120
$27/2$	691537770
14	1234210950
$31/2$	7087424400
16	12229093800
$35/2$	62706694050
18	105164743320
$39/2$	492018813720
20	805306494960
$43/2$	3490512517800
22	5593478602320
$47/2$	22715949849120
24	35731375344000

(β,β)	$n_{2,\beta}$
$51/2$	137145316350735
26	212193639864360
$55/2$	775018459086480
28	1181532282033600
$59/2$	4129199523398880
30	6211686830906340
$63/2$	20865837137909400
32	31011424430679000
$67/2$	100506478032240210
34	147733008377317200
$71/2$	463428612330788160
36	674306145117002160
$75/2$	2052965259390710250
38	2959299345635755920
$79/2$	8765107896801841200

TABLE 2. The first genus 2 Gopakumar-Vafa invariants of a hyperkähler 4-fold of $K3^{[2]}$-type in a primitive curve class β.

Notation and convention. All varieties and schemes are defined over $\mathbb{C}$. For a morphism $\pi\colon X\to Y$ of schemes and objects $\mathcal{F},\mathcal{G}\in \mathrm{D^b}(\mathrm{Coh}(X))$ we will use

$$\mathbf{R}\mathcal{H}om_\pi(\mathcal{F},\mathcal{G}) := \mathbf{R}\pi_*\mathbf{R}\mathcal{H}om_X(\mathcal{F},\mathcal{G}).$$

A class $\beta\in H_2(X,\mathbb{Z})$ is called *effective* if there exists a non-empty curve $C\subset X$ with class $[C]=\beta$. An effective class β is called *irreducible* if it is not the sum of two effective classes, and it is called *primitive* if it is not a positive integer multiple of an effective class.

A holomorphic symplectic variety is a smooth projective variety together with a non-degenerate holomorphic two form $\sigma\in H^0(X,\Omega_X^2)$. A holomorphic symplectic variety is *irreducible hyperkähler* if X is simply connected and $H^0(X,\Omega_X^2)$ is generated by a symplectic form. A $K3$ surface is an irreducible hyperkähler variety of dimension 2.

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1. Gopakumar-Vafa invariants

Let X be a holomorphic symplectic 4-fold with symplectic form $\sigma\in H^0(X,\Omega_X^2)$.

In this section we first recall the definition of (reduced) Gromov-Witten invariants, and then give our definition of Gopakumar-Vafa invariants. In Section 1.4, we justify the definition by working in an ideal geometry of curves.

1.1. **Gromov-Witten invariants.** Let $\overline{M}_{g,n}(X,\beta)$ be the moduli space of n-pointed genus g stable maps to X representing the non-zero curve class $\beta\in H_2(X,\mathbb{Z})$. The moduli space $\overline{M}_{g,n}(X,\beta)$ admits a perfect obstruction theory [BF, LT]. By the construction of [MP13, §2.2] the symplectic form σ induces an everywhere surjective cosection of the obstruction sheaf. By Kiem-Li's theory of cosection localization [KiL] it follows that the standard virtual class as defined in [BF, LT] vanishes and instead there exists a reduced virtual fundamental class:

$$[\overline{M}_{g,n}(X,\beta)]^{\mathrm{vir}}\in A_{2-g+n}(\overline{M}_{g,n}(X,\beta)).$$

[3]Although there are fractional numbers in Table 1, the corresponding classes $n_{1,\beta}$ are integral, see Lemma 7.7.

In this paper we will always work with the reduced virtual fundamental class which we will hence simply denote by $[-]^{\mathrm{vir}}$.

Given cohomology classes $\gamma_i \in H^*(X)$ and integers $k_i \geqslant 0$ the (reduced) Gromov-Witten invariants of X in class β are defined by

$$\big\langle \tau_{k_1}(\gamma_1) \cdots \tau_{k_n}(\gamma_n) \big\rangle_{g,\beta}^{\mathrm{GW}} = \int_{[\overline{M}_{g,n}(X,\beta)]^{\mathrm{vir}}} \prod_{i=1}^{n} \mathrm{ev}_i^*(\gamma_i) \cdot \psi_i^{k_i}, \tag{1.1}$$

where $\mathrm{ev}_i \colon \overline{M}_{g,n}(X,\beta) \to X$ is the evaluation map at the i-th marking and ψ_i is the i-th cotangent line class. By the properties of the reduced virtual class, the integral (1.1) is invariant under deformations of the pair (X,β) with preserve the Hodge type of the class β. We call the invariant (1.1) a *primary Gromov-Witten invariant* if all the k_i are zero.

1.2. **Relations.** We record several basic relations among genus 0 Gromov-Witten invariants which will be used later on in the text. For the first reading, this section map be skipped.

Lemma 1.1. *Let D be a divisor on X such that $d := D \cdot \beta \neq 0$. Then*

$$\big\langle \tau_1(\gamma) \big\rangle_{0,\beta}^{\mathrm{GW}} = \frac{1}{d^2} \big\langle \tau_0(\gamma)\tau_0(D^2) \big\rangle_{0,\beta}^{\mathrm{GW}} - \frac{2}{d} \big\langle \tau_0(\gamma \cdot D) \big\rangle_{0,\beta}^{\mathrm{GW}}.$$

Proof. By the divisor equation (e.g. [CK, pp. 305])

$$\big\langle \tau_1(\gamma)\tau_0(D)^2 \big\rangle_{0,\beta}^{\mathrm{GW}} = d^2 \big\langle \tau_1(\gamma) \big\rangle_{0,\beta}^{\mathrm{GW}} + 2d \big\langle \tau_0(\gamma \cdot D) \big\rangle_{0,\beta}^{\mathrm{GW}}.$$

On the other hand, by rewriting ψ_1 in terms of boundary divisors and using the splitting formula for reduced virtual classes as in [MPT, §7.3] one gets

$$\big\langle \tau_1(\gamma)\tau_0(D)^2 \big\rangle_{0,\beta}^{\mathrm{GW}} = \big\langle \tau_0(\gamma)\tau_0(D^2) \big\rangle_{0,\beta}^{\mathrm{GW}}.$$

□

Lemma 1.2. *For any $\gamma \in H^4(X)$, we have: $\big\langle \tau_1(\gamma) \big\rangle_{0,\beta}^{\mathrm{GW}} = \big\langle \tau_2(1)\tau_0(\gamma) \big\rangle_{0,\beta}^{\mathrm{GW}}$.*

Proof. Arguing as in Lemma 1.1 we can express both sides in terms of primary Gromov-Witten invariants, which yields the result. □

Lemma 1.3. $\big\langle \tau_3(1) \big\rangle_{0,\beta}^{\mathrm{GW}} = \big\langle \tau_2(1)\tau_2(1) \big\rangle_{0,\beta}^{\mathrm{GW}}$.

Proof. Let $D \in H^2(X)$ such that $d := D \cdot \beta \neq 0$. Consider the following invariants:

$$\begin{aligned}
A_0 &= \big\langle \tau_3(1) \big\rangle_{0,\beta}^{\mathrm{GW}} & B_0 &= \big\langle \tau_2(1)\tau_2(1) \big\rangle_{0,\beta}^{\mathrm{GW}} \\
A_1 &= \big\langle \tau_2(D) \big\rangle_{0,\beta}^{\mathrm{GW}} & B_1 &= \big\langle \tau_2(1)\tau_1(D) \big\rangle_{0,\beta}^{\mathrm{GW}} \\
A_2 &= \big\langle \tau_1(D^2) \big\rangle_{0,\beta}^{\mathrm{GW}} & B_0 &= \big\langle \tau_2(1)\tau_0(D^2) \big\rangle_{0,\beta}^{\mathrm{GW}} \\
A_3 &= \big\langle \tau_0(D^3) \big\rangle_{0,\beta}^{\mathrm{GW}} & & \\
C_2 &= \big\langle \tau_1(D)\tau_1(D) \big\rangle_{0,\beta}^{\mathrm{GW}} & C_3 &= \big\langle \tau_1(D)\tau_0(D^2) \big\rangle_{0,\beta}^{\mathrm{GW}} \\
F &= \big\langle \tau_0(D^2)\tau_0(D^2) \big\rangle_{0,\beta}^{\mathrm{GW}}. & &
\end{aligned}$$

Applying topological recursions to the invariants on the left then yields the following relations on the right:

$$\begin{aligned}
&\big\langle \tau_3(1)\tau_0(D)\tau_0(D) \big\rangle_{0,\beta}^{\mathrm{GW}} : & B_2 &= d^2 A_0 + 2dA_1 + A_2 \\
&\big\langle \tau_2(D)\tau_0(D)\tau_0(D) \big\rangle_{0,\beta}^{\mathrm{GW}} : & C_3 &= d^2 A_1 + 2dA_2 + A_3 \\
&\big\langle \tau_1(D^2)\tau_0(D)\tau_0(D) \big\rangle_{0,\beta}^{\mathrm{GW}} : & F &= d^2 A_2 + 2dA_3 \\
&\big\langle \tau_2(1)\tau_0(D^3) \big\rangle_{0,\beta}^{\mathrm{GW}} : & -A_3 &= dB_2 + C_3 \\
&\big\langle \tau_1(D)\tau_0(D^2)\tau_0(D) \big\rangle_{0,\beta}^{\mathrm{GW}} : & dA_3 &= dC_3 + F \\
&\big\langle \tau_2(1)\tau_2(1)\tau_0(D) \big\rangle_{0,\beta}^{\mathrm{GW}} : & 0 &= dB_0 + 2B_1 \\
&\big\langle \tau_2(1)\tau_1(D)\tau_0(D) \big\rangle_{0,\beta}^{\mathrm{GW}} : & 0 &= dB_1 + C_2 + B_2 \\
&\big\langle \tau_1(D)\tau_1(D)\tau_0(D) \big\rangle_{0,\beta}^{\mathrm{GW}} : & 0 &= dC_2 + 2C_3.
\end{aligned}$$

Putting all together (using the assistance of a computer) one finds:

$$\big\langle \tau_3(1) \big\rangle_{0,\beta}^{\mathrm{GW}} = A_0 = -\frac{8}{d^3} A_3 + \frac{6}{d^4} F = B_0 = \big\langle \tau_2(1)\tau_2(1) \big\rangle_{0,\beta}^{\mathrm{GW}}. \tag{1.2}$$

□

Lemma 1.4. *Assume that all fibers of the universal curve $p : \mathcal{C} \to \overline{M}_{0,0}(X,\beta)$ are isomorphic to $\mathbb{P}^1$. Let $\pi : \overline{M}_{0,1}(X,\beta) \to \overline{M}_{0,0}(X,\beta)$ be the forgetful morphism. Then*

$$c_1(\omega_\pi) = \psi_1.$$

In particular, with $f : \mathcal{C} \to X$ the universal map, we have

$$\big\langle \tau_1(\gamma) \big\rangle^{\mathrm{GW}}_{0,\beta} = \int_{p^*[\overline{M}_{0,0}(X,\beta)]^{\mathrm{vir}}} f^*(\gamma)\, c_1(\omega_p). \tag{1.3}$$

Proof. Let $\tilde{p} : \mathcal{C}_1 \to \overline{M}_{0,1}(X,\beta)$ be the universal curve and let $s : \overline{M}_{0,1}(X,\beta) \to \mathcal{C}_1$ be the universal section. By definition, we have

$$\psi_1 = s^*(c_1(\omega_{\tilde{p}})) = s^* c_1(\Omega_{\tilde{p}}).$$

Recall that we have $\mathcal{C} \cong \overline{M}_{0,1}(X,\beta)$. Moreover, since $\mathcal{C} \to \overline{M}_{0,0}(X,\beta)$ parametrizes only smooth curves, we have

$$\mathcal{C}_1 \cong \mathcal{C} \times_{\overline{M}_{0,0}(X,\beta)} \mathcal{C}.$$

Under this isomorphism, the section s is identified with the diagonal morphism. We have the fiber diagram

$$\begin{array}{ccc} \mathcal{C} \times_{\overline{M}_{0,0}(X,\beta)} \mathcal{C} & \xrightarrow{\ \tilde{\pi}\ } & \mathcal{C} \\ s \uparrow \downarrow \tilde{p} & & \downarrow p \\ \overline{M}_{0,1}(X,\beta) & \xrightarrow{\ \pi\ } & \overline{M}_{0,0}(X,\beta). \end{array}$$

Hence since $\tilde{\pi} \circ s = \mathrm{id}$, we have

$$\psi_1 = s^* c_1(\Omega_{\tilde{p}}) = s^* \tilde{\pi}^* c_1(\Omega_p) = c_1(\Omega_p).$$

The second part follows since

$$[\overline{M}_{0,1}(X,\beta)]^{\mathrm{vir}} = \pi^* [\overline{M}_{0,0}(X,\beta)]^{\mathrm{vir}}.$$ ◻

1.3. **Definition of GV invariants.** We consider the definition of Gopakumar-Vafa invariants.

In genus 0, by [BL, MP13], reduced Gromov-Witten invariants of X are equal to the (ordinary) Gromov-Witten invariants in fiber classes of the twistor space $\mathcal{X} \to \mathbb{P}^1$ associated to the symplectic form σ (alternatively, we can view X embedded in a suitable 1-parameter family of holomorphic symplectic 4-folds such that the corresponding classifying map is transverse to the Noether-Lefschetz divisor defined by β). The definition of genus 0 Gopakumar-Vafa invariants for Calabi-Yau 5-folds proposed by Pandharipande and Zinger in [PZ, Eqn. (0.2)] hence can be viewed as a definition for genus 0 Gopakumar-Vafa invariants of X as follows:

Definition 1.5. *For any $\gamma_1, \ldots, \gamma_n \in H^*(X,\mathbb{Z})$, we define the genus 0 Gopakumar-Vafa invariant $n_{0,\beta}(\gamma_1, \ldots, \gamma_n) \in \mathbb{Q}$ by*

$$\big\langle \tau_0(\gamma_1) \cdots \tau_0(\gamma_n) \big\rangle^{\mathrm{GW}}_{0,\beta} = \sum_{k \geqslant 1, k | \beta} k^{n-3}\, n_{0,\beta/k}(\gamma_1, \ldots, \gamma_n).$$

The case of genus 1 does *not* follow from the 5-fold geometry, since the virtual class of the moduli spaces differ by a factor of $(-1)^g \lambda_g$, see [MP13, O21a]. Instead we propose a definition of genus 1 Gopakumar-Vafa invariants based on computations in an ideal geometry of curves in class β. Because curves in imprimitive curve classes are very difficult to control, we restrict hereby to the primitive case (i.e. to those β which are not a multiple in $H_2(X,\mathbb{Z})$). Consider the Chern classes of the tangent bundle of X:

$$c_k(X) := c_k(T_X) \in H^{2k}(X,\mathbb{Z}).$$

Definition 1.6. *Assume that $\beta \in H_2(X,\mathbb{Z})$ is primitive. For any $\gamma \in H^4(X,\mathbb{Z})$, we define the genus 1 Gopakumar-Vafa invariant $n_{1,\beta}(\gamma) \in \mathbb{Q}$ by*

$$\big\langle \tau_0(\gamma) \big\rangle^{\mathrm{GW}}_{1,\beta} = n_{1,\beta}(\gamma) - \frac{1}{24} \big\langle \tau_0(\gamma) \tau_0(c_2(T_X)) \big\rangle^{\mathrm{GW}}_{0,\beta}.$$

Next we come to the genus 2 Gopakumar-Vafa invariants. Since the virtual dimension of the moduli space $\overline{M}_{2,0}(X,\beta)$ is zero, GV invariants are defined without cohomological constraints. In other words, we expect that $n_{2,\beta}$ should be given by the enumerative count of genus 2 curves in class β. For the definition we require the following invariant introduced in [NO]:

$$N_{\mathrm{nodal},\beta} := \frac{1}{2} \left[\int_{[\overline{M}_{0,2}(X,\beta)]^{\mathrm{vir}}} (\mathrm{ev}_1 \times \mathrm{ev}_2)^*(\Delta_X) - \int_{[\overline{M}_{0,1}(X,\beta)]^{\mathrm{vir}}} \frac{\mathrm{ev}_1^*(c(T_X))}{1 - \psi_1} \right], \tag{1.4}$$

where

- $\Delta_X \in H^8(X \times X)$ is the class of the diagonal, and
- $c(T_X) = 1 + c_2(T_X) + c_4(T_X)$ is the total Chern class of T_X.

The invariant $N_{\mathrm{nodal},\beta}$ is the expected number of rational nodal curves in class β [NO, Prop. 1.2][4].

Definition 1.7. *Assume that $\beta \in H_2(X, \mathbb{Z})$ is primitive. We define the genus 2 Gopakumar-Vafa invariant $n_{2,\beta} \in \mathbb{Q}$ by*

$$\langle \varnothing \rangle^{\mathrm{GW}}_{2,\beta} = n_{2,\beta} - \frac{1}{24} n_{1,\beta}(c_2(X)) + \frac{1}{2 \cdot 24^2} \langle \tau_0(c_2(X)) \tau_0(c_2(X)) \rangle^{\mathrm{GW}}_{0,\beta} + \frac{1}{24} N_{\mathrm{nodal},\beta}.$$

Remark 1.8. *For primitive $\beta \in H_2(X, \mathbb{Z})$, we obtain the following:*

$$n_{0,\beta}(\gamma_1, \ldots, \gamma_n) = \langle \tau_0(\gamma_1) \cdots \tau_0(\gamma_n) \rangle^{\mathrm{GW}}_{0,\beta},$$

$$n_{1,\beta}(\gamma) = \langle \tau_0(\gamma) \rangle^{\mathrm{GW}}_{1,\beta} + \frac{1}{24} \langle \tau_0(\gamma) \tau_0(c_2(X)) \rangle^{\mathrm{GW}}_{0,\beta},$$

$$n_{2,\beta} = \langle \varnothing \rangle^{\mathrm{GW}}_{2,\beta} + \frac{1}{24} \langle \tau_0(c_2(X)) \rangle^{\mathrm{GW}}_{1,\beta} + \frac{1}{2 \cdot 24^2} \langle \tau_0(c_2(X)) \tau_0(c_2(X)) \rangle^{\mathrm{GW}}_{0,\beta} - \frac{1}{24} N_{\mathrm{nodal},\beta}.$$

It would be interesting to obtain a conceptual understanding for the form of these formulae.

As in the cases of Calabi-Yau 4-folds and 5-folds [KP, PZ], our first main conjecture concerns the integrality of the Gopakumar-Vafa invariants on holomorphic symplectic 4-folds.

Conjecture 1.9 (Integrality)**.** *We have*

$$n_{0,\beta}(\gamma_1, \ldots, \gamma_n), \;\; n_{1,\beta}(\gamma), \;\; n_{2,\beta} \in \mathbb{Z}.$$

1.4. **Ideal geometry.** We will justify our definition of Gopakumar-Vafa invariants by working in an 'ideal' geometry where we assume curves on X deform in families of expected dimensions and have expected genericity properties. This discussion is inspired by the 'ideal' geometry of curves on Calabi-Yau 4-folds by [KP] and on Calabi-Yau 5-folds by [PZ]. Concretely, since the virtual dimension of $\overline{M}_{g,0}(X, \beta)$ is $2 - g$, we expect that:

Any genus g curve moves in a smooth compact $(2 - g)$-dimensional family.

In particular, there are no curves of genus $g \geqslant 3$.

We discuss now the expected behaviour of the curves in these families. We start with genus zero. Let $p : \mathcal{C}^0_\beta \to S^0_\beta$ be a family of rational curves in class β over a smooth 2-dimensional surface S^0_β, fiberwise embedded in X. Then we can have the following behaviour:

(i) All the curves parametrized by S^0_β can be reducible.
Reason: Let $\beta = \beta_1 + \beta_2$ and let $\mathcal{C}^0_{\beta_i} \to S^0_{\beta_i}$ be a 2-dimensional family of rational curves in class β_i. Let $S^0_{\beta_1,\beta_2}$ be the preimage of the diagonal under the evaluation maps
$$j_1 \times j_2 : \mathcal{C}^0_{\beta_1} \times \mathcal{C}^0_{\beta_2} \to X \times X.$$
Then $S^0_{\beta_1,\beta_2}$ is of expected dimension $3 + 3 - 4 = 2$, so by gluing the curves we can obtain a 2-dimensional family of reducible rational curves in class β.

(ii) Given a generic curve $\mathcal{C}^0_s := p^{-1}(s) \subset X$ in the family, there exists another curve $\mathcal{C}^0_{s'} \subset X$ in the family which meets it.
Reason: This follows by the same reasoning as in (i).

(iii) For finitely many $s \in S$, we expect the curve $\mathcal{C}^0_s \subset X$ to be nodal[5].
Reason: The moduli space $\overline{M}_{0,2}(X, \beta)$ is of expected dimension 4, and hence the preimage of the diagonal under $\mathrm{ev}_1 \times \mathrm{ev}_2$ is of expected dimension 0.[6]

(iv) Even if all fibers of $\mathcal{C}^0_\beta \to S_\beta$ are smooth $\mathbb{P}^1$'s, the natural morphism $j : \mathcal{C}^0_\beta \to X$ is not necessarily an immersion.
(The differential $dj : T_{\mathcal{C}^0_\beta} \to j^*(T_X)$ is expected to have a kernel in codimension $\geqslant 2$.)

Similarly given a family $p : \mathcal{C}^1_\beta \to S^1_\beta$ of elliptic curves in class β over a smooth 1-dimensional curve S^1_β, fiberwise embedded in X, all the curves parametrized by S^1_β can be reducible. The argument is similar to (i) above, by considering the preimage of the diagonal under the evaluation maps

$$j_1 \times j_2 : \mathcal{C}^0_{\beta_1} \times \mathcal{C}^1_{\beta_2} \to X \times X, \quad \text{where } \beta = \beta_1 + \beta_2,$$

where $\mathcal{C}^0_{\beta_1}$ (resp. $\mathcal{C}^1_{\beta_2}$) is a family of rational curves in class β_1 (resp. elliptic curves in class β_2).

[4]Here we use genus reduction to rewrite the term N_X in [NO, §1.3] as the first term in Eqn. (1.4).

[5]A naive model for (ii,iii) would be for the image of $\mathcal{C}^0_\beta$ in X to be $(S \times \mathbb{P}^1)/((s, 0) \sim (gs, \infty))$, where $g \in \mathrm{Aut}(S)$ is an automorphism with a finite number of fixed points (which lead to nodal curves).

[6]This is related to what is called the double point number.

The genus 2 curves we expect to be smooth and finite. By dimension reasons they should be disjoint from elliptic curves, but can have finite intersection points with the family of rational curves. In the moduli space $\overline{M}_{2,0}(X,\beta)$ we will hence also see genus 2 curves with rational tails.

In summary, the geometry of curves is more complicated then for both CY 4-folds and CY 5-folds. Especially for imprimitive curve classes β, it becomes increasingly difficult to control.

1.5. **Ideal geometry: Primitive case.** We make the following additional assumptions:

- X is irreducible hyperkähler,
- the effective curve class $\beta \in H_2(X,\mathbb{Z})$ is primitive.

By the global Torelli for (irreducible) hyperkähler varieties [Ver13, Huy] (in fact, the local surjectivity of the period map is sufficient) the pair (X,β) is deformation equivalent (through a deformation with keeps β of Hodge type) to a pair (X',β') where $\beta' \in H_2(X,\mathbb{Z})$ is irreducible. Hence we may without loss of generality make the following stronger assumption:[7]

- the effective curve class $\beta \in H_2(X,\mathbb{Z})$ is irreducible.

Under these assumptions our ideal geometry of curves simplifies to the following form:

(1) The rational curves in X of class β move in a proper 2-dimensional smooth family of embedded irreducible rational curves. Except for a finite number of rational nodal curves, the rational curves are smooth, with normal bundle $\mathcal{O}_{\mathbb{P}^1} \oplus \mathcal{O}_{\mathbb{P}^1} \oplus \mathcal{O}_{\mathbb{P}^1}(-2)$.
(2) The arithmetic genus 1 curves in X of class β move in a proper 1-dimensional smooth family of embedded irreducible genus 1 curves. Except for a finite number of rational nodal curves, the genus one curves are smooth elliptic curves. For the convenience of computations, we also assume the normal bundle of elliptic curves is $L \oplus L^{-1} \oplus \mathcal{O}$, where L is a generic degree zero line bundle.
(3) All genus two curves are smooth and rigid.
(4) There are no curves of genus $g \geqslant 3$.

Example 1.10. *Let $Y \subset \mathbb{P}^5$ be a very general smooth cubic 4-fold and let $F(Y) \subset \mathrm{Gr}(2,6)$ be the Fano variety of lines on Y. By a result of Beauville and Donagi* [BD]*, $F(Y)$ is an irreducible hyperkähler 4-fold, and together with its Plücker polarization it is the generic member of a locally complete family of polarized hyperkähler varieties deformation equivalent to the second punctual Hilbert scheme of a $K3$ surface. The algebraic classes in $H_2(F(Y),\mathbb{Z})$ are of rank 1. Let β be the generator which pairs positively with the polarization (it is of degree 3 with respect to the Plücker polarization). The geometry of curves in class β has been studied in* [OSY, NO, GK]*. The Chow variety of curves in class β is given by*

$$\mathrm{Chow}_\beta(F(Y)) = S \cup \Sigma,$$

where $S \subset F(Y)$ is the smooth irreducible surface of lines of second type, and Σ is a smooth curve parametrizing genus 1 curves. There are precisely 3780 rational nodal curves corresponding to the intersection points $S \cap \Sigma$, and all other rational curves are isomorphic to $\mathbb{P}^1$. Moreover, there are no curves of genus $\geqslant 2$ in class β. We see that the curves in $F(Y)$ of class β satisfy the requirements of the ideal geometry.

1.6. **Justification: GV in genus 1.** For a given class $\gamma \in H^4(X,\mathbb{Z})$ let $\Gamma \subset X$ be a generic topological cycle whose class is Poincaré dual to γ. In an ideal geometry, the Gopakumar-Vafa invariant $n_{1,\beta}(\gamma)$ should be the (enumerative) number $n(\Gamma)$ of arithmetic genus 1 curves in X which are incident to Γ. To derive an expression for it using Gromov-Witten invariants, we start with the genus 1 Gromov-Witten invariant:

$$\big\langle \tau_0(\gamma) \big\rangle^{\mathrm{GW}}_{1,\beta}, \tag{1.5}$$

where $\beta \in H_2(X,\mathbb{Z})$ is primitive. Assuming the ideal geometry of Section 1.5 we will analyze the contributions from genus 0 and genus 1 curves to it (there are no contributions from genus 2 curves since they never meet the cycle Γ). We show that the contribution from genus 1 curves is precisely $n(\Gamma)$. This will yield the expression for $n_{1,\beta}(\gamma)$.

[7] The statement is false if we do not assume that X is irreducible hyperkähler, for example on X the product of two $K3$ surfaces, a class $\beta = (\beta_1, \beta_2)$ with both β_i non-zero effective, any deformation that keeps β Hodge, will keep both β_i Hodge. In particular β stays reducible under deformations.

1.6.1. *Contribution from genus one curves.* Let $p : \mathcal{C}^1_\beta \to S^1_\beta$ be a 1-dimensional family of elliptic curves of class β as in Section 1.5, and let $j : \mathcal{C}^1_\beta \to X$ be the evaluation map.

Since Γ (which represents $\gamma \in H^4(X,\mathbb{Z})$) is chosen generic, it intersects $\mathcal{C}^1_\beta$ in precisely $(\mathcal{C}^1_\beta \cdot \gamma)$ many points. Following Section 1.5, we assume that the incident curves are smooth elliptic curves E with normal bundle $N_{E/X} = L \oplus L^{-1} \oplus \mathcal{O}$. We find the contribution of this family to the invariant (1.5) is

$$(\mathcal{C}^1_\beta \cdot \gamma) \int_{[\overline{M}_{1,1}(E,1)]^{\mathrm{vir}}} \mathrm{ev}_1^*(\omega) = (\mathcal{C}^1_\beta \cdot \gamma) = n(\Gamma),$$

where $\omega \in H^2(E,\mathbb{Z})$ is the class of a point and the trivial factor $H^1(E, N_{E/X}) = H^1(E, \mathcal{O}_E) = \mathbb{C}$ in the obstruction sheaf does not appear because we used the reduced virtual fundamental class.

1.6.2. *Contribution from genus zero curves.* Let $p : \mathcal{C}^0_\beta \to S^0_\beta$ be a 2-dimensional family of embedded rational curves of class β in X parametrized by a smooth surface S^0_β. The generic fiber of p is isomorphic to $\mathbb{P}^1$ but over finitely many points we can have a rational nodal curve. The insertion Γ intersects the divisor $\mathcal{C}^0_\beta$ in a curve that we can assume maps to a curve in S^0_β. In particular, it avoids the singular fibers. For simplicity we may hence assume that there are no nodal fibers, and that this is the only family of rational curves in class β. We will compute the contribution of this family to the genus 1 GW invariant (1.5).

Under these assumptions, for any genus 1 degree β stable map $f : C \to X$, the source curve splits canonically as $C \cong E \cup \mathbb{P}^1$, where E is an elliptic curve glued to $\mathbb{P}^1$ at one point p. The map f is of degree 0 on E, and of degree β on $\mathbb{P}^1$. Hence

$$\overline{M}_{1,0}(X,\beta) = \overline{M}_{0,1}(X,\beta) \times \overline{M}_{1,1}.$$

By comparing the obstruction theories on the level of virtual classes, we get

$$\begin{aligned}[\overline{M}_{1,0}(X,\beta)]^{\mathrm{vir}} &= \left[\frac{c(\mathbb{E}^\vee \otimes \mathrm{ev}_1^*(T_X))}{1 - \lambda_1 - \psi_1} \right]_3 \cap \left([\overline{M}_{1,1}] \times [\overline{M}_{0,1}(X,\beta)]^{\mathrm{vir}} \right) \\ &= \left(\psi_1^3 - \psi_1^2 \lambda_1 + \mathrm{ev}_1^*(c_2(X))(\psi_1 - \lambda_1) \right) \left([\overline{M}_{1,1}] \times [\overline{M}_{0,1}(X,\beta)] \right),\end{aligned}$$

where $[-]_d$ denotes taking the degree d part, $\mathbb{E} \to \overline{M}_g$ is the Hodge bundle over the moduli space of curves (having fiber $H^0(C, \omega_C)$ over a point $[C]$) with first Chern class $\lambda_1 = c_1(\mathbb{E}) \in A^1(\overline{M}_{1,1})$, and ψ_1 is the usual psi class on the moduli space $\overline{M}_{0,1}(X,\beta)$. In the last line we have used that the dimension of $\overline{M}_{0,k}(X,\beta)$ is equal to the expected dimension, so

$$[\overline{M}_{0,k}(X,\beta)]^{\mathrm{vir}} = [\overline{M}_{0,k}(X,\beta)]. \tag{1.6}$$

Finally, as we will do often, we have suppressed pullback maps along the projection to the factors.

Consider the forgetful morphism $\pi \colon \overline{M}_{1,1}(X,\beta) \to \overline{M}_{1,0}(X,\beta)$ which at the same time is the universal curve over the moduli space. In particular, we have a decomposition

$$\overline{M}_{1,1}(X,\beta) = \mathcal{E} \cup \mathcal{P},$$

where $\mathcal{E} \to \overline{M}_{1,1}$ and $\mathcal{P} \to \overline{M}_{0,1}(X,\beta)$ are the universal curves. Since π is flat of relative dimension 1, we have

$$[\overline{M}_{1,1}(X,\beta)] = \pi^*[\overline{M}_{1,0}(X,\beta)] = a_* \left([\overline{M}_{1,2}] \times [\overline{M}_{0,1}(X,\beta)] \right) + b_* \left([\overline{M}_{1,1}] \times [\overline{M}_{0,2}(X,\beta)] \right),$$

where $a : \mathcal{E} \to \overline{M}_{1,1}(X,\beta)$ and $b : \mathcal{P} \to \overline{M}_{1,1}(X,\beta)$ are the natural inclusions. We find

$$\begin{aligned}\langle \tau_0(\gamma) \rangle^{\mathrm{GW}}_{1,\beta} &= \int_{[\overline{M}_{1,1}] \times [\overline{M}_{0,2}(X,\beta)]^{\mathrm{vir}}} \mathrm{ev}_2^*(\gamma) \pi^*(\psi_1^3 - \psi_1^2 \lambda_1 + \mathrm{ev}_1^*(c_2(X))(\psi_1 - \lambda_1)) \\ &+ \int_{[\overline{M}_{1,2}] \times [\overline{M}_{0,1}(X,\beta)]^{\mathrm{vir}}} \mathrm{ev}_1^*(\gamma) \pi^*(\psi_1^3 - \psi_1^2 \lambda_1 + \mathrm{ev}_1^*(c_2(X))(\psi_1 - \lambda_1)),\end{aligned} \tag{1.7}$$

where for the second summand the γ is pulled back along the evaluation map $\mathrm{ev}_1 : \overline{M}_{0,1}(X,\beta) \to X$ (since the map is constant on the elliptic curve).

In the second term in Eqn. (1.7), the integrand over the factor $\overline{M}_{1,2}$ is pulled back from $\overline{M}_{1,1}$; hence this term vanishes. We conclude that Eqn. (1.7) is equal to:

$$\begin{aligned}&\int_{\overline{M}_{1,1}} (-\lambda_1) \int_{[\overline{M}_{0,2}(X,\beta)]^{\mathrm{vir}}} \mathrm{ev}_2^*(\gamma) \pi^*(\psi_1^2 + \mathrm{ev}_1^*(c_2(X))) \\ &= -\frac{1}{24} \left(\langle \tau_2(1) \tau_0(\gamma) \rangle^{\mathrm{GW}}_{0,\beta} + \langle \tau_0(c_2(X)) \tau_0(\gamma) \rangle^{\mathrm{GW}}_{0,\beta} - \langle \tau_1(\gamma) \rangle^{\mathrm{GW}}_{0,\beta} \right) \\ &= -\frac{1}{24} \langle \tau_0(c_2(X)) \tau_0(\gamma) \rangle^{\mathrm{GW}}_{0,\beta}.\end{aligned}$$

Here in the second step, we used that $\pi^*(\psi_1) = \psi_1 - s_*(1)$, where $s : \overline{M}_{0,1}(X,\beta) \to \overline{M}_{0,2}(X,\beta)$ is the section, so that $\pi^*(\psi_1^2) = \psi_1^2 - s_*(\psi_1)$, and in the last step we used Lemma 1.2.

1.6.3. *Conclusion.* By the discussion above we have obtained that in the ideal geometry we have

$$\big\langle \tau_0(\gamma) \big\rangle^{\mathrm{GW}}_{1,\beta} = n(\Gamma) - \frac{1}{24} \big\langle \tau_0(c_2(X)) \tau_0(\gamma) \big\rangle^{\mathrm{GW}}_{0,\beta}.$$

Since $n(\Gamma)$ is the Gopakuma-Vafa invariant $n_{1,\beta}(\gamma)$ in the ideal geometry, this ends the justification for both Definition 1.6 and integrality of genus 1 invariants in Conjecture 1.9.

1.7. **Justification: GV in genus** 2. In the ideal geometry of Section 1.5, the genus two Gopakumar-Vafa invariant $n_{2,\beta}$ should be the (enumerative) number of genus 2 curves in the irreducible curve class β. We hence make the ansatz

$$\big\langle \varnothing \big\rangle^{\mathrm{GW}}_{2,\beta} = \int_{[\overline{M}_{2,0}(X,\beta)]^{\mathrm{vir}}} 1 = n_{2,\beta} + \cdots, \tag{1.8}$$

where the dots stand for the contributions from curves of genus $\leqslant 1$. In this section we derive an expression for these lower genus contributions.

1.7.1. *Contribution from genus one curves.* We consider first the contributions from a 1-dimensional family of elliptic curves $\mathcal{C}^1_\beta \to S^1_\beta$ parametrized by a smooth curve S^1_β, but with the *additional assumption* that there are no nodal rational curves in the family.

For simplicity of notation we also assume that the family $\mathcal{C}^1_\beta$ parametrizes all curves in class β (so there are no rational or genus 2 curves). We compute the invariant $\big\langle \varnothing \big\rangle^{\mathrm{GW}}_{2,\beta}$ in this geometry.

Under the above assumption we have the isomorphism

$$\overline{M}_{2,0}(X,\beta) = \overline{M}_{1,1}(X,\beta) \times \overline{M}_{1,1},$$

and with an argument parallel to Section 1.6.2, the virtual class is:

$$[\overline{M}_{2,0}(X,\beta)]^{\mathrm{vir}} = \left[\frac{c(\mathbb{E}^\vee \otimes \mathrm{ev}_1^*(T_X))}{1 - \lambda_1 - \psi_1} \right]_3 \cap [\overline{M}_{1,1}(X,\beta)]^{\mathrm{vir}} \times [\overline{M}_{1,1}],$$

where ψ_1 is the cotangent line class on $\overline{M}_{1,1}(X,\beta)$ and $\lambda_1 \in H^2(\overline{M}_{1,1})$, both pulled back to the product via the projection to the factors. One obtains that:

$$\begin{aligned} \big\langle \varnothing \big\rangle^{\mathrm{GW}}_{2,\beta} &= \int_{[\overline{M}_{1,1}(X,\beta)]^{\mathrm{vir}} \times [\overline{M}_{1,1}]} -\lambda_1 \psi_1^2 - \lambda_1 \,\mathrm{ev}_1^*(c_2(T_X)) \\ &= -\frac{1}{24} \int_{[\overline{M}_{1,1}(X,\beta)]^{\mathrm{vir}}} \psi_1^2 + \mathrm{ev}_1^*(c_2(X)). \end{aligned}$$

By our assumption there are no family of rational curves in class β, so that we have $\psi_1 = \tau^*(\psi_1)$, where $\tau : \overline{M}_{1,1}(X,\beta) \to \overline{M}_{1,1}$ is the forgetful morphism to the moduli space of stable curves, and therefore $\psi_1^2 = 0$. We conclude that

$$\big\langle \varnothing \big\rangle^{\mathrm{GW}}_{2,\beta} = -\frac{1}{24} \big\langle \tau_0(c_2(X)) \big\rangle^{\mathrm{GW}}_{1,\beta} = -\frac{1}{24} n_{1,\beta}(c_2(X)).$$

In total hence we see that the family $\mathcal{C}^1_\beta \to S^1_\beta$ contributes $-\frac{1}{24} n_{1,\beta}(c_2(X))$ to the integral (1.8).

Assume more generally that there are both rational and elliptic curves in class β, but still no nodal rational curves. Then by the discussion in Section 1.6 and the above computation we have that $-\frac{1}{24} n_{1,\beta}(c_2(X))$ is precisely the contribution from the elliptic curves to (1.8). Hence this contribution remains valid also in the presence of rational curves.

1.7.2. *Contribution from genus zero curves.* Let $p : \mathcal{C}^0_\beta \to S^0_\beta$ be a family of degree β embedded rational curves in X parametrized by a smooth surface S^0_β. We assume that there are no curves of genus 1 or 2, and that all rational curves parametrized by S^0_β are smooth. Since β is irreducible, this means that all of them are isomorphic to $\mathbb{P}^1$.

By our assumption, we have an isomorphism of moduli spaces:

$$M := \overline{M}_{2,0}(X,\beta) \cong \overline{M}_{2,0}(\mathcal{C}^0_\beta / S^0_\beta, 1),$$

where the right hand side is the moduli space of genus 2 degree 1 stable maps to the fibers of $\mathcal{C}_\beta^0 \to S_\beta^0$. In particular, we have a diagram:

$$\begin{array}{ccccccc} & & & \tilde{f} & & & \\ C & \xrightarrow{\rho} & q^*\mathcal{C}_\beta^0 & \xrightarrow{\tilde{q}} & \mathcal{C}_\beta^0 & \xrightarrow{f_\beta} & X \\ & \searrow^{\pi} & \downarrow{\scriptstyle \tilde{p}} & & \downarrow{\scriptstyle p} & & \\ & & M & \xrightarrow{q} & S_\beta^0, & & \end{array}$$

where $C \to M$ is the universal curve over the moduli space (for both $\overline{M}_{2,0}(X,\beta)$ and $\overline{M}_{2,0}(\mathcal{C}_\beta^0/S_\beta^0, 1)$), $\tilde{f} : C \to \mathcal{C}_\beta^0$ is the universal map of $\overline{M}_{2,0}(\mathcal{C}_\beta^0/S_\beta^0, 1)$, and q is the structure morphism to the base. By definition the middle square is fibered. The moduli space $\overline{M}_{2,0}(\mathcal{C}_\beta^0/S_\beta^0, 1)$ carries naturally a virtual fundamental class which we denote by

$$[M]^{\mathrm{rel}} := [\overline{M}_{2,0}(\mathcal{C}_\beta^0/S_\beta^0, 1)]^{\mathrm{vir}} \in A_6(M).$$

We also denote the reduced virtual fundamental class of $\overline{M}_{2,0}(X,\beta)$ by

$$[M]^{\mathrm{vir}} := [\overline{M}_{2,0}(X,\beta)]^{\mathrm{vir}} \in A_0(M).$$

Since f_β is fiberwise an embedding we have the subbundle $T_p \subset f_\beta^*(T_X)$. Let

$$N = f_\beta^*(T_X)/T_p$$

be the quotient, which is locally free of rank 3. The key to our discussion is the following comparision of virtual fundamental classes.

Proposition 1.11. *We have*

$$[M]^{\mathrm{vir}} = e\Big(\tilde{p}_*\big((R^1\rho_*\mathcal{O}_C) \otimes \tilde{q}^*N\big)\Big) \cap [M]^{\mathrm{rel}}.$$

For the proof we start with the two basic lemmata:

Lemma 1.12. *We have*

$$\pi_*(\tilde{f}^*N) \cong q^*(T_{S_\beta^0}).$$

Proof. By the Cohomology and Base Change Theorem we have

$$\rho_*(\mathcal{O}_C) = \mathcal{O}_{q^*\mathcal{C}_\beta^0}.$$

Hence we find that

$$\begin{aligned} \pi_*\tilde{f}^*N &= \tilde{p}_*\rho_*\rho^*\tilde{q}^*N \\ &= \tilde{p}_*(\rho_*(\mathcal{O}_C) \otimes \tilde{q}^*N) \\ &= \tilde{p}_*\tilde{q}^*N \\ &= q^*p_*N \\ &= q^*T_{S_\beta^0}, \end{aligned}$$

where in the second equality we used that N is locally free, and in the forth equality we used flat base change. For the last step we used that $S_\beta^0 = \overline{M}_{0,0}(X,\beta)$ is smooth with tangent bundle given by p_*N (which at each point $s \in S_\beta^0$ has fiber $H^0(\mathcal{C}_{\beta,s}^0, N_{\mathcal{C}_{\beta,s}^0/X})$). ☐

Lemma 1.13. *We have the exact sequence:*

$$0 \to \tilde{p}_*\big((R^1\rho_*\mathcal{O}_C) \otimes \tilde{q}^*N\big) \to R^1\pi_*(\tilde{f}^*N) \to R^1\tilde{p}_*(\tilde{q}^*N) \to 0$$

and $R^1\tilde{p}_*(\tilde{q}^*N) = \mathcal{O}_M$.

Proof. The first statement is just an application of the Leray-Serre spectral sequence for the composition $\pi = \tilde{p} \circ \rho$. For the second statement, we have by flat base change that:

$$R^1\tilde{p}_*(\tilde{q}^*N) \cong q^*(R^1p_*N).$$

By the existence of a global cosection, we have a surjection $R^1p_*N \to \mathcal{O}_{S_\beta^0}$. Since p_*N is locally free of rank 2, R^1p_*N is locally free of rank 1, so using the cosection it is isomorphic to $\mathcal{O}_{S_\beta^0}$. ☐

Proof of Proposition 1.11. The 'standard' virtual tangent bundle[8] of $\overline{M}_{2,0}(X,\beta)$ relative to the Artin stack of prestable curves $\mathfrak{M}_2$ is by definition given by

$$T^{\mathrm{std}}_{\overline{M}_{2,0}(X,\beta)/\mathfrak{M}_2} = R\pi_* f^*(T_X),$$

where $f = f_\beta \circ \tilde{f} : C \to X$ is the universal map. The reduced virtual tangent bundle is defined to be the cone:

$$T^{\mathrm{vir}}_{\overline{M}_{2,0}(X,\beta)/\mathfrak{M}_2} = (R\pi_* f^* T_X)^{\mathrm{red}} := \mathrm{Cone}(R\pi_* f^*(T_X) \xrightarrow{\mathsf{sr}_\sigma} \mathcal{O}[-1])[-1],$$

where sr_σ is the semi-regularity map associated to the symplectic form σ, see [MP13, MPT]. The inclusion $T_p \subset f_\beta^*(T_X)$ induces a natural distinguished triangle:

$$R\pi_* \tilde{f}^* T_p \to (R\pi_* f^* T_X)^{\mathrm{red}} \to \left(R\pi_* \tilde{f}^* N\right)^{\mathrm{red}}. \tag{1.9}$$

where the third term is defined as the cone of the first map. By Lemma 1.13 and since the restriction of sr_σ to $\tilde{p}_*\big((R^1\rho_*\mathcal{O}_C) \otimes \tilde{q}^* N\big)$ vanishes, we have

$$h^1\left((R\pi_* \tilde{f}^* N)^{\mathrm{red}}\right) = \tilde{p}_*\big((R^1\rho_*\mathcal{O}_C) \otimes \tilde{q}^* N\big). \tag{1.10}$$

Similarly, the virtual tangent bundle of the perfect obstruction theory of $\overline{M}_{2,0}(\mathcal{C}^0_\beta/S^0_\beta)$ fits into the distinguished triangle

$$R\pi_*(\tilde{f}^* T_p) \to T^{\mathrm{vir}}_{\overline{M}_{2,0}(\mathcal{C}^0_\beta/S^0_\beta,1)/\mathfrak{M}_2} \to q^*(T_{S^0_\beta}). \tag{1.11}$$

By Lemma 1.12 there exists a natural morphism

$$q^*(T_{S^0_\beta}) \to \left(R\pi_* \tilde{f}^* N\right)^{\mathrm{red}},$$

which induces an isomorphism in degree 0 cohomology. This morphism induces a morphism from the complex (1.11) to the complex (1.9), and combining with Eqn. (1.10), we obtain the distinguished triangle:

$$T^{\mathrm{vir}}_{\overline{M}_{2,0}(\mathcal{C}^0_\beta/S^0_\beta,1)/\mathfrak{M}_2} \to T^{\mathrm{vir}}_{\overline{M}_{2,0}(X,\beta)/\mathfrak{M}_2} \to \tilde{p}_*\big((R^1\rho_*\mathcal{O}_C) \otimes \tilde{q}^* N\big)[-1].$$

The claim now follows from the excess intersection formula. □

The moduli space M decomposes naturally as the union

$$M = M_1 \cup (M_2/\mathbb{Z}_2),$$

where

$$M_1 = \overline{M}_{2,1} \times \overline{M}_{0,1}(\mathcal{C}^0_\beta/S^0_\beta, 1),$$
$$M_2 = \overline{M}_{1,1} \times \overline{M}_{1,1} \times \overline{M}_{0,2}(\mathcal{C}^0_\beta/S^0_\beta, 1),$$

and $\mathbb{Z}_2$ acts by interchanging the two factors of $\overline{M}_{1,1}$ and switching the markings on $\overline{M}_{0,2}(\mathcal{C}^0_\beta/S^0_\beta, 1)$. The class $[M]^{\mathrm{rel}}$ is of dimension 6, but the dimensions of M_1 and M_2 are 7 and 6 respectively. In particular, there exists some class $\alpha \in A_6(M_1)$ such that

$$[M]^{\mathrm{rel}} = \xi_{1*}(\alpha) + \frac{1}{2}\xi_{2*}[M_2],$$

where $\xi_i : M_i \to M$ are the natural (gluing) morphisms.[9] By Proposition 1.11, we find that:

$$\begin{aligned} \langle \varnothing \rangle^{\mathrm{GW}}_{2,\beta} &= \int_{[M]^{\mathrm{rel}}} e\Big(\tilde{p}_*\big((R^1\rho_*\mathcal{O}_C) \otimes \tilde{q}^* N\big)\Big) \\ &= \int_{\alpha} \xi_1^* e\Big(\tilde{p}_*\big((R^1\rho_*\mathcal{O}_C) \otimes \tilde{q}^* N\big)\Big) \\ &\quad + \frac{1}{2}\int_{\overline{M}_{1,1}\times\overline{M}_{1,1}\times\overline{M}_{0,2}(\mathcal{C}^0_\beta/S^0_\beta,1)} \xi_2^* e\Big(\tilde{p}_*\big((R^1\rho_*\mathcal{O}_C) \otimes \tilde{q}^* N\big)\Big). \end{aligned} \tag{1.12}$$

These two terms are analyzed as follows:

[8]If $E^\bullet \to \mathbb{L}_M$ is a perfect obstruction theory, then the associated virtual tangent bundle is $T^{\mathrm{vir}}_M := (E^\bullet)^\vee$.

[9]The naïve splitting of the virtual class $[\overline{M}_{2,0}(X,\beta)]^{\mathrm{vir}}$ as the sum

$$[\overline{M}_{2,1} \times \overline{M}_{0,1}(X,\beta)]\left[\frac{c(\mathbb{E}^\vee \otimes \mathrm{ev}_1^*(T_X))}{1-\psi_1-\psi_1'}\right]_7 + \frac{1}{2}[\overline{M}^2_{1,1} \times \overline{M}_{0,2}(X,\beta)]^{\mathrm{vir}}\left[\frac{c(\mathbb{E}_a^\vee \otimes \mathrm{ev}_1^*(T_X))}{1-\lambda_{a,1}-\psi_1}\frac{c(\mathbb{E}_b^\vee \otimes \mathrm{ev}_2^*(T_X))}{1-\lambda_{b,1}-\psi_2}\right]_6$$

does *not* hold. The long detour to the relative virtual class $[M]^{\mathrm{rel}}$ is necessary to decompose the virtual class!

Lemma 1.14. *We have the vanishing*

$$e\Big(\xi_1^* \tilde{p}_*\big((R^1\rho_*\mathcal{O}_C)\otimes \tilde{q}^*N\big)\Big) = 0.$$

Proof. Let $C \to M$ be the universal curve as before, and let $C' \to M_1$ be its pull back along $\xi_1 : M_1 \to M$. There exists a natural decomposition $C' = R \cup_q Z$ where R is the pullback of the universal curve over $\overline{M}_{0,1}(X,\beta)$ and Z is the pullback of the universal curve from $\overline{M}_{2,1}$. The curves R and Z are glued along the marked points $v : M_1 \to C$. In particular, we have the diagram

$$\begin{array}{ccc} C' & \longrightarrow & C \\ \big\downarrow{\scriptstyle \rho'} & & \big\downarrow{\scriptstyle \rho} \\ \xi_1^* q^*\mathcal{C}_\beta^0 & \xrightarrow{\tilde{\xi}_1} & q^*\mathcal{C}_\beta^0 \\ {\scriptstyle x}\big\uparrow\big\downarrow{\scriptstyle \tilde{p}'} & & \big\downarrow{\scriptstyle \tilde{p}} \\ M_1 & \xrightarrow{\xi_1} & M, \end{array} \qquad (v : M_1 \to C')$$

where $x = \rho' \circ v$ is the image of the gluing point. Applying ρ'_* to the normalization exact sequence

$$0 \to \mathcal{O}_{C'} \to \mathcal{O}_R \oplus \mathcal{O}_Z \to \mathcal{O}_v \to 0$$

shows that

$$\xi_1^* R^1\rho_*(\mathcal{O}_C) = R^1\rho'_*\mathcal{O}_C = R^1\rho'_*(\mathcal{O}_Z) = x_*(\mathrm{pr}_1^*\mathbb{E}^\vee),$$

where $\mathrm{pr}_1 : M_1 \to \overline{M}_{2,1}$ is the projection and $\mathbb{E} \to \overline{M}_{2,1}$ is the Hodge bundle (pulled back to the product). We obtain that:

$$\xi_1^*\tilde{p}_*\big((R^1\rho_*\mathcal{O}_C)\otimes \tilde{q}^*N\big) \cong \tilde{\mathrm{ev}}_1^*(N)\otimes \mathrm{pr}_1^*(\mathbb{E}^\vee),$$

where $\tilde{\mathrm{ev}}_1 = \tilde{q}\circ\tilde{\xi}_1\circ x : M_1 \to \mathcal{C}_\beta^0$ is the evaluation map.

Using the defining exact sequence $0 \to T_p \to f_\beta^*(T_X) \to N \to 0$ and that $\tilde{\mathrm{ev}}_1^*(T_p)$ is isomorpic to the cotangent line bundle of $\overline{M}_{0,1}(\mathcal{C}_\beta^0/S_\beta^0, 1)$ at the marking, i.e. $\tilde{\mathrm{ev}}_1^*(T_p) \cong \mathbb{L}_{p_1}^\vee$, we obtain the exact sequence

$$0 \to \mathbb{E}^\vee\otimes\mathbb{L}_{p_1}^\vee \to \mathbb{E}^\vee\otimes \mathrm{ev}_1^*(T_X) \to \tilde{\mathrm{ev}}_1^*(N)\otimes\mathrm{pr}_1^*(\mathbb{E}^\vee) \to 0,$$

where $\mathrm{ev}_1 : \overline{M}_{0,1}(\mathcal{C}_\beta^0/S_\beta^0, 1) \cong \overline{M}_{0,1}(X,\beta) \to X$ is the evaluation map to X and we surpressed the pullbacks by the projection to the factors. We conclude that

$$\begin{aligned} e\Big(\xi_1^*\tilde{p}_*\big((R^1\rho_*\mathcal{O}_C)\otimes\tilde{q}^*N\big)\Big) &= \left[\frac{c(\mathbb{E}^\vee\otimes\mathrm{ev}_1^*(T_X))}{c(\mathbb{E}^\vee\otimes\mathbb{L}_{p_1}^\vee)}\right]_6 \\ &= \left[\frac{c(\mathbb{E}^\vee)^4 + 2\,\mathrm{ev}_1^*(c_2(T_X))c(\mathbb{E}^\vee)(1-\lambda_1)}{(1-\psi_1)^2 - \lambda_1(1-\psi_1)+\lambda_2}\right]_6, \end{aligned}$$

where in the second equality we used the splitting principle and the Mumford relation

$$c(\mathbb{E})\,c(\mathbb{E}^\vee) = (1+\lambda_1+\lambda_2)(1-\lambda_1+\lambda_2) = 1 + 2\lambda_2 - \lambda_1^2 + \lambda_2^2 = 1. \tag{1.13}$$

Now a straightforward computation (using that $\overline{M}_{0,1}(X,\beta)$ is of dimension 3 and the Mumford relation, and which may be performed by a computer program) shows that this degree 6 component vanishes. □

Lemma 1.15.

$$\int_{\overline{M}_{1,1}\times\overline{M}_{1,1}\times\overline{M}_{0,2}(\mathcal{C}_\beta^0/S_\beta^0,1)} \xi_2^* e\Big(\tilde{p}_*\big((R^1\rho_*\mathcal{O}_C)\otimes\tilde{q}^*N\big)\Big) = \frac{1}{24^2}\big\langle \tau_0(c_2(X))\tau_0(c_2(X))\big\rangle_{0,\beta}^{\mathrm{GW}}.$$

Proof. Let $C' \to M_2$ be the pullback of the universal curve $C \to M$ to M_2. We have a decomposition $C' = R \cup E_1 \cup E_2$, where R is the universal 2-pointed genus 0 curve, and the E_i are the universal genus 1 curves. Let

$$x_1, x_2 : M_2 \to \xi_2^* q^*\mathcal{C}_\beta^0$$

be the image of the marked points under the evaluation map $\rho' : C' \to \xi_2^*q^*\mathcal{C}_\beta^0$. We have

$$\xi_2^* R^1\rho_*\mathcal{O}_C = R^1\rho'_*(\mathcal{O}_{C'}) = x_{1*}(\mathbb{E}_1^\vee)\oplus x_{2*}(\mathbb{E}_2^\vee),$$

where $\mathbb{E}_i = \mathrm{pr}_i^*(\mathbb{E})$ are the Hodge bundles pulled-back from the first or second copy of M_2. We argue as in Lemma 1.14, that is first we have

$$\tilde{\mathrm{ev}}_i = \tilde{q}\circ\tilde{\xi}_2\circ x_i.$$

Then with $\pi_i : \overline{M}_{0,2}(\mathcal{C}^0_\beta/S^0_\beta, 1) \to \overline{M}_{0,1}(\mathcal{C}^0_\beta/S^0_\beta, 1)$ the morphism that forgets all but the i-th marking we have that

$$\tilde{\mathrm{ev}}_i^*(T_p) = \pi_i^* \tilde{\mathrm{ev}}^*(T_p) = \pi_i^*(\mathbb{L}_{p_i}^\vee).$$

(Here, we need the precompose with the forgetful morphism because the two markings can lie on a bubble in which case the tangent space to a marking maps with zero to the tangent space of the image point; by precomposing with the forgetful map, we contract the bubbles). As in Lemma 1.14 we then obtain that

$$\begin{aligned}(1.14)\qquad &\int_{\overline{M}_{1,1}\times\overline{M}_{1,1}\times\overline{M}_{0,2}(\mathcal{C}^0_\beta/S^0_\beta,1)} \xi_2^* e\Big(\tilde{p}_*\big((R^1\rho_*\mathcal{O}_C)\otimes \tilde{q}^*N\big)\Big)\\ &= \int_{\overline{M}_{1,1}\times\overline{M}_{1,1}\times\overline{M}_{0,2}(\mathcal{C}^0_\beta/S^0_\beta,1)} \frac{c(\mathbb{E}_1^\vee\otimes \mathrm{ev}_1^*(T_X))c(\mathbb{E}_2^\vee\otimes \mathrm{ev}_1^*(T_X))}{(1-c_1(\mathbb{E}_1)-\pi_1^*(\psi_1))(1-c_1(\mathbb{E}_1)-\pi_2^*(\psi_2))}.\end{aligned}$$

For $i = 1, 2$ and $(\lambda, \psi, \mathrm{ev}) := (c_1(\mathbb{E}_i), \pi_i^*(\psi_i), \mathrm{ev}_i)$, we have

$$\begin{aligned}&\frac{c(\mathbb{E}^\vee\otimes \mathrm{ev}^*(T_X))}{1-\lambda-\psi}\\ &= ((1-\lambda)^4 + \mathrm{ev}^*(c_2(X))(1-\lambda)^2)(1+\lambda+\psi+2\lambda\psi+\psi^2+3\lambda\psi^2+\psi_1^3+4\lambda\psi^3)\\ &= \lambda\Big((1+\mathrm{ev}^* c_2(X))(1+2\psi+3\psi^2+4\psi^3) + (-4-2\,\mathrm{ev}^* c_2(X))(1+\psi+\psi^2+\psi^3)\Big) + (\cdots)\\ &= \lambda\Big(-3-2\psi-\psi^2-\mathrm{ev}^*(c_2(X))\Big) + (\cdots),\end{aligned}$$

where $(\cdots)$ are terms that are not multiples of λ.

Using this and Eqn. (1.6), the term (1.14) becomes:

$$(1.15)\qquad \frac{1}{24^2}\int_{[\overline{M}_{0,2}(X,\beta)]^{\mathrm{vir}}} (\mathrm{ev}_1^*(c_2(X)) + \pi_1^*(\psi_1)^2)(\mathrm{ev}_2^*(c_2(X)) + \pi_2^*(\psi_2)^2).$$

On $\overline{M}_{0,2}(X,\beta)$ we have

$$\psi_1 = \pi_1^*(\psi_1) + s_*(1),$$

where $s : \overline{M}_{0,1}(X,\beta) \to \overline{M}_{0,2}(X,\beta)$ is the canonical section, and therefore

$$\pi_1^*(\psi_1)^2 = \psi_1^2 - s_*(\psi_1).$$

Applying Lemma 1.2, we find that:

$$\int_{[\overline{M}_{0,2}(X,\beta)]^{\mathrm{vir}}} \pi_1^*(\psi_1)^2\,\mathrm{ev}_2^*(c_2(X)) = \big\langle \tau_2(1)\tau_0(c_2(X))\big\rangle^{\mathrm{GW}}_{0,\beta} - \big\langle \tau_1(c_2(X))\big\rangle^{\mathrm{GW}}_{0,\beta} = 0.$$

With a similar reasoning, using Lemma 1.3, we also get that:

$$\int_{[\overline{M}_{0,2}(X,\beta)]^{\mathrm{vir}}} \pi_1^*(\psi_1)^2\pi_2^*(\psi_2)^2 = \big\langle \tau_2(1)\tau_2(1)\big\rangle^{\mathrm{GW}}_{0,\beta} - \big\langle \tau_3(1)\big\rangle^{\mathrm{GW}}_{0,\beta} = 0.$$

Inserting both these vanishings into Eqn. (1.15) concludes the claim. ☐

Inserting the two lemmata above into Eqn. (1.12), the whole computation collpases into the following simple evaluation:

$$\big\langle \varnothing\big\rangle^{\mathrm{GW}}_{2,\beta} = \frac{1}{2\cdot 24^2}\big\langle \tau_0(c_2(X))\tau_0(c_2(X))\big\rangle^{\mathrm{GW}}_{0,\beta}.$$

We hence conclude that the family $\mathcal{C}^0_\beta \to S^0_\beta$ of rational curves with only smooth fibers contributes $\big\langle \tau_0(c_2(X))\tau_0(c_2(X))\big\rangle^{\mathrm{GW}}_{0,\beta}/(2\cdot 24^2)$ to the Gopakumar-Vafa invariant $n_{2,\beta}$.

1.7.3. *Conclusion and contribution from nodal rational curves.* Consider an ideal geometry of curves as in Section 1.5 without any additional assumptions. We expect the contributions from genus 0 and genus 1 curves to the invariant $\big\langle \varnothing\big\rangle^{\mathrm{GW}}_{2,\beta}$ to be as discussed above, plus a correction term coming from the nodal rational curves. This correction term should be local, and hence a multiple of the expected number of nodal rational curves $N_{\mathrm{nodal},\beta}$. We hence make the ansatz:

$$(1.16)\qquad \big\langle \varnothing\big\rangle^{\mathrm{GW}}_{2,\beta} = n_{2,\beta} - \frac{1}{24}n_1(c_2(X)) + \frac{1}{2\cdot 24^2}\big\langle \tau_0(c_2(X))\tau_0(c_2(X))\big\rangle^{\mathrm{GW}}_{0,\beta} + aN_{\mathrm{nodal},\beta}$$

for a constant $a \in \mathbb{Q}$ independent of (X,β).

We determine now a with a test calculation. Let X be the Fano variety of lines on a very general cubic 4-fold, and let $\beta \in H_2(X, \mathbb{Z})$ be the minimal effective curve class. As we will see in Section 7 we have the evaluations (assuming the conjectural holomorphic anomaly equation):

$$\big\langle \varnothing \big\rangle^{\mathrm{GW}}_{2,\beta} = -11445/128,$$
$$n_{1,\beta}(c_2(X)) = 5985,$$
$$\big\langle \tau_0(c_2(X))\tau_0(c_2(X)) \big\rangle^{\mathrm{GW}}_{0,\beta} = 2835.$$

Moreover, by [NO, Thm. 1.3], we have

$$N_{\mathrm{nodal},\beta} = 3780.$$

Since there are no genus 2 curves on X in class β (see [NO]) we set

$$n_{2,\beta} = 0.$$

Inserting this into Eqn. (1.16) yields:

$$a = \frac{1}{24}.$$

This conclude the justification of Definition 1.7. While the last step (i.e. §1.7.3) requires two assumption (locality of the contribution of nodal rational curves, and the holomorphic anomaly equation), the remainder of the paper yields plenty of numerical support for this definition.

2. Donaldson-Thomas invariants

For a holomorphic symplectic 4-fold, we define (reduced) Donaldson-Thomas invariants (DT_4 invariants for short) of one dimensional stable sheaves. We then use them to give a sheaf theoretic approach to Gopakumar-Vafa invariants defined in the previous section. In the last section we justify the definition by computations in the ideal geometry of curves.

2.1. **Definitions.** Let M_β be the moduli scheme of one dimensional stable sheaves F on X with $[F] = \beta$, $\chi(F) = 1$. Such moduli spaces are independent of the choice of polarization (e.g. [CMT18, Rmk. 1.2]) and are used in [CMT18, CT20a] to give sheaf theoretic interpretation of Gopakumar-Vafa type invariants of ordinary Calabi-Yau 4-folds [KP]. We also refer to [CMT19, CT19, CT20b, CT20c] for related conjectures and computations, which build on the works of virtual class constructions [BJ, OT] (see also [CL14]).

Parallel to Gromov-Witten theory, the ordinary virtual class of M_β vanishes [KiP, Sav]. For a choice of ample divisor H, one can define a reduced virtual class due to Kiem-Park [KiP, Def. 8.7, Lem. 9.4]:

$$[M_\beta]^{\mathrm{vir}} \in A_2(M_\beta, \mathbb{Q}), \tag{2.1}$$

depending on the choice of orientation [CGJ, CL17]. To define descendent invariants, we need insertions:

$$\tau_i : H^*(X, \mathbb{Z}) \to H^{*+2i-2}(M_\beta, \mathbb{Q}),$$
$$\tau_i(\bullet) := (\pi_M)_* \left(\pi_X^*(\bullet) \cup \mathrm{ch}_{3+i}(\mathbb{F}_{\mathrm{norm}})\right),$$

where $\mathbb{F}_{\mathrm{norm}}$ is the normalized universal sheaf, i.e. $\det(\pi_{M*}\mathbb{F}_{\mathrm{norm}}) \cong \mathcal{O}_{M_\beta}$ (ref. [CT20a, §1.4]).

Definition 2.1. *For any $\gamma_1, \ldots, \gamma_n \in H^*(X)$ and $k_i \in \mathbb{Z}_{\geqslant 0}$ the DT_4 invariants are defined by*

$$\big\langle \tau_{k_1}(\gamma_1), \ldots, \tau_{k_n}(\gamma_n) \big\rangle^{\mathrm{DT}_4}_\beta := \int_{[M_\beta]^{\mathrm{vir}}} \prod_{i=1}^n \tau_{k_i}(\gamma_i) \in \mathbb{Q}. \tag{2.2}$$

2.2. **Conjectures.** As in [CMT18, CT20a], we propose the following sheaf theoretic interpretation of all genus Gopakumar-Vafa invariants:

Conjecture 2.2. *For certain choice of orientation, the following equalities hold.*
When β is an effective curve class,

(i) $$\big\langle \tau_0(\gamma_1), \ldots, \tau_0(\gamma_n) \big\rangle^{\mathrm{DT}_4}_\beta = n_{0,\beta}(\gamma_1, \ldots, \gamma_n).$$

When β is a primitive curve class,

(ii) $$\big\langle \tau_1(\gamma) \big\rangle^{\mathrm{DT}_4}_\beta = -\frac{1}{2}\big\langle \tau_1(\gamma) \big\rangle^{\mathrm{GW}}_{0,\beta} - n_{1,\beta}(\gamma).$$

When β is a primitive curve class,

(iii) $$-\big\langle \tau_3(1) \big\rangle^{\mathrm{DT}_4}_\beta - \frac{1}{12}\big\langle \tau_1(c_2(X)) \big\rangle^{\mathrm{DT}_4}_\beta = n_{2,\beta}.$$

By Proposition 1.1, $\langle \tau_1(\gamma) \rangle_{0,\beta}^{\mathrm{GW}}$ can be deduced by $g = 0$ primary Gromov-Witten invariants. Therefore these formulae determine all genus Gopakumar-Vafa invariants from primary and descendent DT_4 invariants, which give a sheaf theoretic interpretation for them.

Remark 2.3. *The way we write down Conjecture 2.2* (iii) *is indirect. By* [COT22, App. A]*, the LHS of* (iii) *is equal to stable pair invariant $P_{-1,\beta}$ which is conjecturally the same as genus 2 Gopakumar-Vafa invariants* [COT22, Conj. 1.10]*. We believe there is also a formula relating $\langle \tau_2(\theta) \rangle_{\beta}^{\mathrm{DT}_4}$ to genus 2 Gopakumar-Vafa invariants, which we haven't found so far.*

Remark 2.4. *Our conjecture implicitly includes the independence of* DT_4 *invariants on the choice of ample divisor in defining reduced virtual classes* (2.1)*.*

2.3. **Justification: Primary** DT_4 **invariants.** For Conjecture 2.2 (i), we consider the case $\gamma_1, \gamma_2 \in H^4(X, \mathbb{Z})$ for simplicity. These two 4-cycles (generically) cut out finite number of rational curves and miss high genus curves.

As in [CMT18, §1.4], any one dimensional stable sheaf F with $[F] = \beta$ is $\mathcal{O}_C$ for some rational curve C. Their moduli space M_β is identified with the moduli space S_β^0 of rational curves and

$$[M_\beta]^{\mathrm{vir}} = [S_\beta^0], \tag{2.3}$$

for some choice of orientation. After imposing the primary insertion, we have

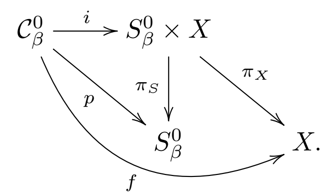
$$\int_{[M_\beta]^{\mathrm{vir}}} \tau_0(\gamma_1)\, \tau_0(\gamma_2) = \int_{S_\beta^0} p_*(f^*\gamma_1) \cdot p_*(f^*\gamma_2),$$

where $p : \mathcal{C}_\beta^0 \to S_\beta^0$ is the total space of rational curve family (RCF) of class β and $f : \mathcal{C}_\beta^0 \to X$ is the evaluation map. Therefore Conjecture 2.2 (i) is confirmed in this ideal setting as both sides of the equation are (virtually) enumerating rational curves of class β incident to cycles dual to γ_1 and γ_2.

2.4. **Justification: Descendent** DT_4 **invariants.** For Conjecture 2.2 (ii), as we put the incident condition with one 4-cycle γ in $\langle \tau_1(\gamma) \rangle_{\beta}^{\mathrm{DT}_4}$ which generically does not intersect genus 2 curves, so we only need to consider the contributions from RCF and ECF (elliptic curve family).

(1) For any RCF of class β, we have an embedding $i : \mathcal{C}_\beta^0 \hookrightarrow S_\beta^0 \times X$ fitting into the diagram:

(2.4)



By Grothendieck-Riemann-Roch (GRR) formula, we have

$$\mathrm{ch}(i_*\mathcal{O}_{\mathcal{C}_\beta^0}) = i_*(\mathrm{td}^{-1}(N_{\mathcal{C}_\beta^0/S_\beta^0 \times X})). \tag{2.5}$$

Obviously $\mathbb{F}_{\mathrm{norm}} = \mathcal{O}_{\mathcal{C}_\beta^0}$, and therefore

$$\begin{aligned} \tau_1(\gamma) &= \pi_{S*}(\mathrm{ch}_4(\mathcal{O}_{\mathcal{C}_\beta^0}) \cdot \pi_X^*\gamma) \\ &= -\frac{1}{2}\pi_{S*}(i_*c_1(\omega_p) \cdot \pi_X^*\gamma) \\ &= -\frac{1}{2}\pi_{S*}(i_*(c_1(\omega_p) \cdot f^*\gamma)) \\ &= -\frac{1}{2}p_*(c_1(\omega_p) \cdot f^*\gamma), \end{aligned}$$

where ω_p is the relative cotangent bundle of p.

Combining with Eqn. (2.3), we see RCF in class β contributes to $\langle \tau_1(\gamma) \rangle_{\beta}^{\mathrm{DT}_4}$ by

$$\int_{[M_\beta]^{\mathrm{vir}}} \tau_1(\gamma) = -\frac{1}{2}\int_{S_\beta^0} p_*(c_1(\omega_p) \cdot f^*\gamma). \tag{2.6}$$

As β is primitive, we may deform it to the irreducible case where RCF consists of smooth rational curves (except at some finite number of fibers of nodal curves which can be ignored by insertion $\gamma \in H^4(X)$). By Lemma 1.4, the RHS of Eqn. (2.6) is equal to $-\frac{1}{2}\langle \tau_1(\gamma) \rangle_{0,\beta}^{\mathrm{GW}}$. This justifies the first term in the RHS of Conjecture 2.2 (ii).

(2) Next we consider the contribution from ECF. Let $p : \mathcal{C}^1_\beta \to S^1_\beta$ be the total space of ECF of class β and $j : \mathcal{C}^1_\beta \to X$ be the evaluation map. The insertion $\gamma \in H^4(X)$ (generically) intersects $\mathcal{C}^1_\beta$ in a finite number of points. We may assume $\mathcal{C}^1_\beta = E \times S^1_\beta$ is the product, p is the projection and j is an embedding in our computations. We further assume E is smooth with normal bundle $L \oplus L^{-1} \oplus \mathcal{O}$ for a generic degree zero line bundle L on E.

Lemma 2.5. *Let $p : \mathcal{C}^1_\beta \to S^1_\beta$ be a one dimensional family of smooth elliptic curves E on X with normal bundle $N_{E/X} = L \oplus L^{-1} \oplus \mathcal{O}$ for a generic $L \in \mathrm{Pic}^0(E)$. Then any one dimensional stable sheaf F supported on this family is scheme theoretically supported on a fiber of p.*

Proof. By [CMT18, Lem. 2.2], we know F is scheme theoretically supported on $\mathrm{Tot}_E(L \oplus L^{-1})$ for a fiber E of p. By [HST, Prop. 4.4], F is scheme theoretically supported on the its zero section, so we are done. □

By the above lemma, there exists a morphism

$$M_\beta \to S^1_\beta, \tag{2.7}$$

whose fiber over $\{E\}$ is the moduli space $M_{1,1}(E)$ of stable bundles on E with rank 1 and $\chi = 1$. Note that $M_{1,1}(E) \cong E$. A family version of such isomorphism gives

$$M_\beta \cong \mathcal{C}^1_\beta, \tag{2.8}$$

such that the virtual class satisfies

$$[M_\beta]^{\mathrm{vir}} = [\mathcal{C}^1_\beta], \tag{2.9}$$

for certain choice of orientation.

Next we compute the descendent insertion. In the following diagram:

$$\begin{array}{ccccc} \mathbb{F} & & \mathbb{E} & & \\ \downarrow & & \downarrow & & \\ \mathcal{C}^1_\beta \times X & \xleftarrow{\bar{j}=(\mathrm{id},j)} & \mathcal{C}^1_\beta \times \mathcal{C}^1_\beta & \xrightarrow{\bar{p}=(p,p)} & S^1_\beta \times S^1_\beta, \end{array}$$

a universal one dimensional sheaf $\mathbb{F}$ can be chosen as

$$\mathbb{F} = \bar{j}_*\mathbb{E}, \quad \mathbb{E} := \mathcal{O}_{\bar{p}^*(\Delta_{S^1_\beta})}(\Delta_{\mathcal{C}^1_\beta}),$$

where we treat $\Delta_{\mathcal{C}^1_\beta}$ as a divisor of $\bar{p}^*(\Delta_{S^1_\beta})$ via

$$\Delta_{\mathcal{C}^1_\beta} = \left\{(x,x) \,|\, x \in \mathcal{C}^1_\beta\right\} \hookrightarrow \left\{(x, p^{-1}p(x)) \,|\, x \in \mathcal{C}^1_\beta\right\} = \bar{p}^*(\Delta_{S^1_\beta}).$$

It is straightforward to check that $\mathbb{F}$ is normalized.

Below, we use notations from the following diagram

$$\begin{array}{ccccccc} & & \mathcal{C}^1_\beta \times X & \xleftarrow{\bar{j}=(\mathrm{id},j)} & \mathcal{C}^1_\beta \times \mathcal{C}^1_\beta & & \\ & \swarrow^{\pi_C} & \downarrow^{\pi_X} & & \downarrow^{\pi_2} & \searrow^{\pi_1} & \\ \mathcal{C}^1_\beta & & X & \xleftarrow{j} & \mathcal{C}^1_\beta & & \mathcal{C}^1_\beta. \end{array}$$

The GRR formula gives

$$\mathrm{ch}_4(\bar{j}_*\mathbb{E}) = \bar{j}_*\left(\frac{1}{2}\,\mathrm{ch}_1(\mathbb{E}) \cdot \pi_2^* c_1(\mathcal{C}^1_\beta) + \mathrm{ch}_2(\mathbb{E})\right). \tag{2.10}$$

Therefore, we have

$$\begin{aligned} \tau_1(\gamma) &= \pi_{C*}(\mathrm{ch}_4(\bar{j}_*\mathbb{E}) \cdot \pi_X^*\gamma) \\ &= \pi_{C*}\bar{j}_*\left(\left(\frac{1}{2}\,\mathrm{ch}_1(\mathbb{E}) \cdot \pi_2^* c_1(\mathcal{C}^1_\beta) + \mathrm{ch}_2(\mathbb{E})\right) \cdot \bar{j}^*\pi_X^*\gamma\right) \\ &= \frac{1}{2}\pi_{1*}\left(\mathrm{ch}_1(\mathbb{E}) \cdot \pi_2^* c_1(\mathcal{C}^1_\beta) \cdot \pi_2^* j^*\gamma\right) + \pi_{1*}\left(\mathrm{ch}_2(\mathbb{E}) \cdot \pi_2^* j^*\gamma\right) \\ &= \pi_{1*}\left(\mathrm{ch}_2(\mathbb{E}) \cdot \pi_2^* j^*\gamma\right), \end{aligned} \tag{2.11}$$

where the last equality is because $\dim_{\mathbb{C}} \mathcal{C}^1_\beta = 2$ and $c_1(\mathcal{C}^1_\beta) \cdot j^*\gamma \in H^6(\mathcal{C}^1_\beta) = 0$.

From the exact sequence in $\mathrm{Coh}(\mathcal{C}^1_\beta \times \mathcal{C}^1_\beta)$:

$$0 \to \mathcal{O}_{\bar{p}^*(\Delta_{S^1_\beta})} \to \mathcal{O}_{\bar{p}^*(\Delta_{S^1_\beta})}(\Delta_{\mathcal{C}^1_\beta}) \to \mathcal{O}_{\Delta_{\mathcal{C}^1_\beta}}(\Delta_{\mathcal{C}^1_\beta}) \to 0,$$

we obtain

$$\begin{aligned}\mathrm{ch}_2(\mathbb{E}) &= \mathrm{ch}_2\left(\mathcal{O}_{\bar{p}^*(\Delta_{S^1_\beta})}\right) + \mathrm{ch}_2\left(\mathcal{O}_{\Delta_{\mathcal{C}^1_\beta}}(\Delta_{\mathcal{C}^1_\beta})\right) \\ &= \bar{p}^*\,\mathrm{ch}_2(\mathcal{O}_{\Delta_{S^1_\beta}}) + [\Delta_{\mathcal{C}^1_\beta}] \\ &= -\frac{1}{2}\bar{p}^*(\Delta_{S^1_\beta})_*(c_1(S^1_\beta)) + [\Delta_{\mathcal{C}^1_\beta}],\end{aligned} \tag{2.12}$$

where $\Delta_{S^1_\beta} : S^1_\beta \to S^1_\beta \times S^1_\beta$ denotes the diagonal embedding and we use GRR formula for the map $\Delta_{S^1_\beta}$ in the last equation.

Combining Eqns. (2.11), (2.12), we obtain

$$\tau_1(\gamma) = -\frac{1}{2}\pi_{1*}\left(\bar{p}^*(\Delta_{S^1_\beta})_*(c_1(S^1_\beta)) \cdot \pi_2^* j^*\gamma\right) + \pi_{1*}\left([\Delta_{\mathcal{C}^1_\beta}] \cdot \pi_2^* j^*\gamma\right) = j^*\gamma, \tag{2.13}$$

where we note that $\bar{p}^*(\Delta_{S^1_\beta})_*(c_1(S^1_\beta))$ is some multiple of the fiber class of $\bar{p}$, so the first term in above vanishes. Therefore, ECF of class β contributes to $\langle \tau_1(\gamma)\rangle^{\mathrm{DT}_4}_\beta$ by

$$\int_{[M_\beta]^{\mathrm{vir}}} \tau_1(\gamma) = \int_{\mathcal{C}^1_\beta} j^*\gamma,$$

which gives exactly the genus 1 GV invariant $n_{1,\beta}(\gamma)$ for primitive β as they are (virtually) enumerating elliptic curves of class β incident to the cycle dual to γ.

Remark 2.6. *For a general curve class β and any $k \geqslant 1$ such that $k|\beta$, one can similarly show that any elliptic curve family $\mathcal{C}^1_{\beta/k}$ of class β/k contributes to $\langle \tau_1(\gamma)\rangle^{\mathrm{DT}_4}_\beta$ by*

$$\int_{\mathcal{C}^1_{\beta/k}} j^*\gamma = n_{1,\beta/k}(\gamma).$$

Therefore, all elliptic curve families contribute to $\langle \tau_1(\gamma)\rangle^{\mathrm{DT}_4}_\beta$ by $\sum_{k|\beta} n_{1,\beta/k}(\gamma)$.

3. The embedded rational curve family

As a first illustration of the general case, we work out here all Gromov-Witten, Gopakumar-Vafa and Donaldson-Thomas invariants for a family of smooth irreducible rational curves *globally embedding* in a holomorphic symplectic 4-fold. We will see that the global embedding assumption forces already almost all of our invariants to vanish.

3.1. **Setting.** Let X be a holomorphic symplectic 4-fold with symplectic form $\sigma \in H^0(X, \Omega^2_X)$. Consider a family $p : \mathcal{C} \to S$ of embedded rational curves in the irreducible curve class $\beta \in H_2(X, \mathbb{Z})$ parametrized by a smooth surface S.

We make the following assumptions:

(i) All fibers of p are non-singular (isomorphic to $\mathbb{P}^1$).
(ii) The evaluation map $j : \mathcal{C} \to X$ is a (global) embedding.
(iii) All curves in class $d\beta$ for all $d \geqslant 1$ are unions of curves of the family $\mathcal{C} \to S$.

Let $\sigma \in H^0(X, \Omega^2_X)$ be the holomorphic symplectic form. Since the pullback $j^*(\sigma) \in H^0(\mathcal{C}, \Omega^2_{\mathcal{C}})$ vanishes on T_p, there exists a 2-form $\alpha \in H^0(S, \Omega^2_S)$ such that

$$p^*(\alpha) = j^*(\sigma).$$

If α vanishes at a point $s \in S$, then for every point x in the fiber $\mathcal{C}_s := p^{-1}(s)$ the form σ vanishes on the image of $T_{\mathcal{C},x} \to T_{X,j(x)}$. Since $\sigma_{j(x)}$ is non-degenerate, it can only vanish on a subspace of at most half the dimension of $T_{X,j(x)}$, so this is impossible. Hence α does not vanish. We conclude that S is a *holomorphic symplectic* surface, hence either an abelian or a $K3$ surface.

Moreover, consider the sequence

$$0 \to T_{\mathcal{C}} \to j^*(T_X) \to N_{\mathcal{C}/X} \to 0.$$

The form $\sigma' = \sigma|_{\mathcal{C}} \in H^0(\mathcal{C}, j^*\Omega^2_X)$ is non-degenerate; so the vanishing $\sigma'(T_p, T_{\mathcal{C}}) = 0$ implies that we have an isomorphism

$$\sigma' : T_p \xrightarrow{\cong} N^\vee_{\mathcal{C}/X}.$$

Example 3.1. *Let $S^{[2]}$ be the Hilbert scheme of two points on a holomorphic symplectic surface S. The Hilbert-Chow map from $S^{[2]}$ to the second symmetric product of S:*

$$\pi : S^{[2]} \to \mathrm{Sym}^2(S)$$

is a resolution of singularity [F], *whose exceptional divisor D fits into the Cartesian diagram*

$$\begin{array}{ccc} D & \xrightarrow{j} & S^{[2]} \\ {\scriptstyle p}\downarrow & & \downarrow{\scriptstyle \pi} \\ S & \xrightarrow{\Delta} & \mathrm{Sym}^2(S), \end{array}$$

where Δ is the diagonal embedding and $p : D \to S$ is a $\mathbb{P}^1$-bundle. The pair $(S^{[2]}, \beta := j_*[D_s])$ satisfies the assumptions (i-iii) for the family $D \to S$.

3.2. **Gromov-Witten invariants.** In the setting (i-iii), we have the following computation of Gromov-Witten invariants. In genus 0, one has the following description:

Lemma 3.2. *For any $\gamma_1, \ldots, \gamma_n \in H^*(X)$, we have*

$$\big\langle \tau_0(\gamma_1)\cdots\tau_0(\gamma_n)\big\rangle^{\mathrm{GW}}_{0,d\beta} = d^{-3+n}\int_S \prod_{i=1}^n p_*(j^*(\gamma_i)).$$

Proof. By condition (iii) the evaluation map factors as

$$\mathrm{ev} : \overline{M}_{0,n}(X, d\beta) \xrightarrow{\rho} \underbrace{\mathcal{C}\times_S \cdots \times_S \mathcal{C}}_{n \text{ times}} \xrightarrow{j\times\cdots\times j} X^n.$$

Since $\overline{M}_{0,n}(X, d\beta)$ is of virtual dimension $2 + n = \dim(\mathcal{C}\times_S \cdots \times_S \mathcal{C})$ we have

$$\mathrm{ev}_*[\overline{M}_{0,n}(X, d\beta)]^{\mathrm{vir}} = a_d[\mathcal{C}\times_S \cdots \times_S \mathcal{C}].$$

By restriction to a fiber and using the Aspinwall-Morrison formula (see e.g. [O18, Prop. 7(i)] for our context), we have

$$a_d = d^{-3+n}.$$

Consider the fiber diagram

$$\begin{array}{ccccc} \mathcal{C} & \xleftarrow[\pi_n]{} & \mathcal{C}^{\times_S n} & \xrightarrow{j^{\times n}} & X^n \\ {\scriptstyle p}\downarrow & & \downarrow{\scriptstyle \pi} & & \\ S & \xleftarrow[p]{} & \mathcal{C}^{\times_S (n-1)} & & \end{array}$$

where π_n and π are the projections to the n-th and the first $(n-1)$-factors respectively, and p is the structure morphism. We obtain:

$$\begin{aligned} \int_{\mathcal{C}^{\times_S n}} (j^{\times n})(\gamma_1 \otimes \cdots \otimes \gamma_n) &= \int_{\mathcal{C}^{\times_S n}} \pi^*\big((j^{\times(n-1)})^*(\gamma_1\otimes\cdots\otimes\gamma_{n-1})\big)\pi_n^* j^*(\gamma_n) \\ &= \int_{\mathcal{C}^{\times_S (n-1)}} \big((j^{\times(n-1)})^*(\gamma_1\otimes\cdots\otimes\gamma_{n-1})\big)p^*(p_*(j^*(\gamma_n))) \\ &= \int_S \prod_{i=1}^n p_*(j^*(\gamma_i)), \end{aligned}$$

where we used that $\pi_*\pi_n^*(j^*\gamma_n) = p^*p_*(j^*\gamma_n)$ and then induction in the last step. The claim follows by putting these two statements together. □

In genus 1 and 2, we have:

Lemma 3.3. *For any $\gamma \in H^4(X, \mathbb{Z})$ and $d \geqslant 1$, we have*

$$\big\langle \tau_0(\gamma)\big\rangle^{\mathrm{GW}}_{1,d\beta} = \big\langle \varnothing \big\rangle^{\mathrm{GW}}_{2,d\beta} = 0.$$

Proof. Under our assumptions we have an isomorphism of moduli spaces

$$\overline{M}_{1,1}(X, \beta) \cong \overline{M}_{1,1}(\mathcal{C}, dF) \cong \overline{M}_{1,1}(\mathcal{C}/S, d),$$

where $\overline{M}_{1,1}(\mathcal{C}, dF)$ is the moduli space of stable maps to the (total space of) $\mathcal{C}$ of degree d times the fiber class F, and $\overline{M}_{1,1}(\mathcal{C}/S, d)$ is the moduli space of stable maps into fibers of $\mathcal{C} \to S$. By comparing the perfect-obstruction theories of the first two moduli spaces one finds that:

$$\big\langle \tau_0(\gamma)\big\rangle^{\mathrm{GW}}_{1,d\beta} = \int_{[\overline{M}_{1,1}(\mathcal{C},dF)]^{\mathrm{vir}}} \mathrm{ev}_1^*(j^*(\gamma))\, e(\mathcal{V}),$$

where the fiber of the bundle $\mathcal{V}$ at a point $[f : \Sigma \to \mathcal{C}, p_1] \in \overline{M}_{1,1}(\mathcal{C}, dF)$ is the kernel of the semiregularity map $H^1(\Sigma, f^*(N_{\mathcal{C}/X})) \to H^1(\Sigma, \omega_\Sigma) = \mathbb{C}$.

Similarly, the virtual classes of the latter two moduli spaces are related by

$$[\overline{M}_{1,1}(\mathcal{C}, dF)]^{\mathrm{vir}} = [\overline{M}_{1,1}(\mathcal{C}/S, d)]^{\mathrm{vir}} \cdot e(\mathbb{E}^\vee \otimes p^*(T_S)).$$

Since S is symplectic, we have: $e(\mathbb{E}^\vee \otimes p^* T_S) = c_2(T_S) - \lambda_1 c_1(T_S) = c_2(T_S)$. Hence

$$\begin{aligned} \big\langle \tau_0(\gamma) \big\rangle^{\mathrm{GW}}_{1,d\beta} &= \int_{[\overline{M}_{1,1}(\mathcal{C},dF)]^{\mathrm{vir}}} \mathrm{ev}_1^*(j^*(\gamma))\, p^*(c_2(T_S)) e(\mathcal{V}) \\ &= \int_{[\overline{M}_{1,1}(\mathcal{C},dF)]^{\mathrm{vir}}} \mathrm{ev}_1^*(j^*(\gamma)\, p^* c_2(T_S)) e(\mathcal{V}) \\ &= 0, \end{aligned}$$

where in the last step we used that $j^*(\gamma)\, p^* c_2(T_S) = 0 \in H^*(\mathcal{C})$ for dimension reasons.

The case of genus 2 is similar (using the Mumford relation (1.13)). □

We also have the following vanishing:

Lemma 3.4. *For any $\gamma \in H^4(X, \mathbb{Q})$ and $d \geqslant 1$ we have:*

$$\big\langle \tau_0(\gamma) \tau_0(c_2(X)) \big\rangle^{\mathrm{GW}}_{0,d\beta} = 0.$$

Proof. Consider the invariant $\big\langle \tau_1(\gamma) \big\rangle^{\mathrm{GW}}_{0,\beta}$. By Lemma 1.4, we have

$$\big\langle \tau_1(\gamma) \big\rangle^{\mathrm{GW}}_{0,\beta} = \int_{\mathcal{C}} j^*(\gamma)\, c_1(\omega_p). \tag{3.1}$$

Applying Lemma 1.1 to the divisor $[\mathcal{C}] \in H^2(X, \mathbb{Z})$ which satisfies $[\mathcal{C}] \cdot \beta = -2$, we also have:

$$\big\langle \tau_1(\gamma) \big\rangle^{\mathrm{GW}}_{0,\beta} = \frac{1}{4} \big\langle \tau_0(\gamma) \tau_0([\mathcal{C}]^2) \big\rangle^{\mathrm{GW}}_{0,\beta} + \big\langle \tau_0(\gamma \cdot [\mathcal{C}]) \big\rangle^{\mathrm{GW}}_{0,\beta}. \tag{3.2}$$

Since $N_{\mathcal{C}/X} \cong T_p^\vee = \omega_p$, we have

$$\big\langle \tau_0(\gamma \cdot [\mathcal{C}]) \big\rangle^{\mathrm{GW}}_{0,\beta} = \int_{\mathcal{C}} j^*(\gamma \cdot [\mathcal{C}]) = \int_{\mathcal{C}} j^*(\gamma)\, c_1(\omega_p).$$

Comparing Eqn. (3.1) and Eqn. (3.2), we conclude with the help of Lemma 3.2 that:

$$0 = \big\langle \tau_0(\gamma) \tau_0([\mathcal{C}]^2) \big\rangle^{\mathrm{GW}}_{0,\beta} = \int_S p_*(j^*(\gamma)) \cdot p_*(c_1(T_p)^2). \tag{3.3}$$

The pair of short exact sequences

$$\begin{aligned} &0 \to T_C \to j^* T_X \to T_p^\vee \to 0, \\ &0 \to T_p \to T_C \to p^*(T_S) \to 0 \end{aligned}$$

shows that

$$j^*(c(X)) = 1 + p^*(c_2(S)) - c_1(T_p)^2$$

and hence

$$j^*(c_2(X)) = p^*(c_2(S)) - c_1(T_p)^2. \tag{3.4}$$

Inserting into Eqn. (3.3), we find

$$\int_S p_*(j^*(\gamma)) \cdot p_*(j^*(c_2(X))) = 0.$$

By Lemma 3.2 this implies the claim (for all $d \geqslant 1$). □

We will also require the following evaluation.

Lemma 3.5. *For any $\gamma \in H^4(X, \mathbb{Q})$, we have*

$$\big\langle \tau_1(\gamma) \big\rangle^{\mathrm{GW}}_{0,d\beta} = \frac{1}{d^3} \int_{\mathcal{C}} j^*(\gamma)\, c_1(\omega_p). \tag{3.5}$$

Proof. By Lemma 1.1 applied to $D = [\mathcal{C}] \in H^2(X, \mathbb{Z})$ (which satisfies $D \cdot \beta = -2$) we have:

$$\big\langle \tau_1(\gamma) \big\rangle^{\mathrm{GW}}_{0,d\beta} = \frac{1}{4d^2} \big\langle \tau_0(\gamma) \tau_0([\mathcal{C}]^2) \big\rangle^{\mathrm{GW}}_{0,d\beta} + \frac{1}{d} \big\langle \tau_0(\gamma \cdot [\mathcal{C}]) \big\rangle^{\mathrm{GW}}_{0,d\beta}.$$

By Eqn. (3.3) and Lemma 3.2 the first term vanishes. And by Lemma 3.2 again we get:

$$\big\langle \tau_0(\gamma \cdot [\mathcal{C}]) \big\rangle^{\mathrm{GW}}_{0,d\beta} = \frac{1}{d^2} \int_{\mathcal{C}} j^*(\gamma \cdot [\mathcal{C}]) = \frac{1}{d^2} \int_{\mathcal{C}} j^*(\gamma) c_1(\omega_p).$$ □

3.3. **Gopakumar-Vafa invariants.** We compute all $g \geqslant 1$ Gopakumar-Vafa invariants in the setting (i-iii).

Lemma 3.6. *For any $\gamma \in H^4(X, \mathbb{Z})$, we have*

$$n_{1,\beta}(\gamma) = n_{2,\beta} = 0.$$

Proof. By Lemmata 3.3 and 3.4 and the definition of Gopakumar-Vafa invariants it suffices to show that $N_{\mathrm{nodal},\beta}$ vanishes. Since $\overline{M}_{0,2}(X,\beta) = \mathcal{C} \times_S \mathcal{C}$ we have

$$N_{\mathrm{nodal},\beta} = \frac{1}{2}\left[\int_{\mathcal{C}\times_S\mathcal{C}} (j\times j)^*(\Delta_X) - \int_{[\overline{M}_{0,1}(X,\beta)]^{\mathrm{vir}}} \psi_1^3 + \mathrm{ev}_1^*(c_2(X))\psi_1\right].$$

To evaluate the first term we use that the preimage of the diagonal under $j \times j : \mathcal{C}\times_S\mathcal{C} \to X \times X$ is equal to $\mathcal{C}$ and that the refined intersection has an excess bundle which is an extension of T_S and $T_p^\vee$. For the second term we use Eqn. (3.4) and that by Lemma 1.4 we have $\psi_1 = -c_1(T_p)$ under the isomorphism $\overline{M}_{0,1}(X,\beta) \cong \mathcal{C}$. With this the above becomes:

$$\begin{aligned} &= \frac{1}{2}\left[\int_{\mathcal{C}} e(T_S)c_1(T_p^\vee) - \int_{\mathcal{C}} (-c_1(T_p))^3 + (p^*(c_2(S)) - c_1(T_p)^2)(-c_1(T_p))\right] \\ &= \frac{1}{2}\left[-2e(S) + 2e(S)\right] = 0, \end{aligned}$$

where we used $\psi_1 = -c_1(T_p)$ under the isomorphism $\overline{M}_{0,1}(X,\beta) \cong \mathcal{C}$ by Lemma 1.4. □

3.4. DT_4 **invariants.**

Lemma 3.7. *In the setting (i-iii), for certain choice of orientation, we have*

$$\begin{aligned} \left\langle \tau_0(\gamma_1), \cdots, \tau_0(\gamma_n)\right\rangle_\beta^{\mathrm{DT}_4} &= \int_S \prod_{i=1}^n (p_* j^* \gamma_i), \\ \left\langle \tau_1(\gamma)\right\rangle_\beta^{\mathrm{DT}_4} &= -\frac{1}{2}\int_{\mathcal{C}} j^*(\gamma)\, c_1(\omega_p), \\ \left\langle \tau_2(\theta)\right\rangle_\beta^{\mathrm{DT}_4} &= \frac{1}{12}\int_{\mathcal{C}} j^*(\theta)\, c_1(\omega_p)^2 - \frac{1}{12}\int_{\mathcal{C}} j^*(c_2(X)\cdot\theta), \\ \left\langle \tau_3(1)\right\rangle_\beta^{\mathrm{DT}_4} &= \frac{1}{24}\int_{\mathcal{C}} j^*(c_2(X))\, c_1(\omega_p). \end{aligned}$$

Moreover, all DT_4 *invariants vanish in curve class $d\beta$ for $d > 1$.*

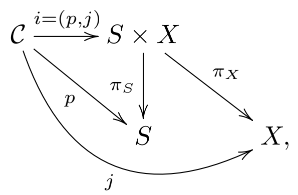
Proof. The computation is essentially done in §2.4. By [CMT18, Lem. 2.2], any one dimensional stable sheaf in class $d\beta$ is scheme theoretically supported on a fiber of $p : \mathcal{C} \to S$. Therefore

$$M_\beta \cong S, \quad M_{d\beta} = \emptyset, \text{ if } d \geqslant 2. \tag{3.6}$$

And their virtual classes satisfy

$$[M_\beta]^{\mathrm{vir}} = \pm[S], \quad [M_{d\beta}]^{\mathrm{vir}} = 0, \text{ if } d \geqslant 2. \tag{3.7}$$

Under the isomorphism (3.6) and the commutative diagram



the normalized universal family $\mathbb{F}_{\mathrm{norm}}$ is $i_*\mathcal{O}_{\mathcal{C}}$. By Grothendieck-Riemann-Roch formula,

$$\begin{aligned} \mathrm{ch}(i_*\mathcal{O}_{\mathcal{C}}) &= i_*(\mathrm{td}^{-1}(N_{\mathcal{C}/S\times X})) \\ &= i_*\left(1 - \frac{1}{2}c_1(\omega_p) + \frac{1}{12}\left(c_1(\omega_p)^2 - j^*c_2(X)\right) + \frac{1}{24}c_1(\omega_p)\cdot j^*c_2(X)\right). \end{aligned} \tag{3.8}$$

Therefore

$$\begin{aligned} \tau_k(\gamma) &= \pi_{S*}\left(\pi_X^*\gamma \cup \mathrm{ch}_{k+3}(i_*\mathcal{O}_{\mathcal{C}})\right) \\ &= \pi_{S*}\left(\pi_X^*\gamma \cup i_*\left[\mathrm{td}^{-1}(N_{\mathcal{C}/S\times X})\right]_{\deg_{\mathbb{C}} k}\right) \\ &= p_*\left(j^*\gamma \cup \left[\mathrm{td}^{-1}(N_{\mathcal{C}/S\times X})\right]_{\deg_{\mathbb{C}} k}\right). \end{aligned}$$

Combining with Eqns. (3.7), (3.8), we are done. □

To sum up, combining Lemmata 3.2–3.7, we obtain:

Theorem 3.8. *Conjecture 1.9 and Conjecture 2.2 hold in the setting specified in §3.1.*

4. Tautological integrals on moduli spaces of sheaves on $K3$ surfaces

In this section, we compute several tautological integrals on moduli spaces of one dimensional stable sheaves on $K3$ surfaces. These will be used in Section 5 to compute DT_4 invariants on the product of $K3$ surfaces, though they are interesting in their own right.

4.1. **Fujiki constants.** The second cohomology $H^2(M,\mathbb{Z})$ of an irreducible hyperkähler variety carries a integral non-degenerate quadratic form

$$\mathsf{q} : H^2(M,\mathbb{Z}) \to \mathbb{Z},$$

called the Beauville-Bogomolov-Fujiki form. By the following result of Fujiki [Fuji] (and its generalization in [GHJ]) it controls the intersection numbers of products of divisors against Hodge cycles which stay Hodge type on all deformations of M:

Theorem 4.1. ([Fuji], [GHJ, Cor. 23.17]) *Assume $\alpha \in H^{4j}(M,\mathbb{C})$ is of type $(2j,2j)$ on all small deformation of M. Then there exists a unique constant $C(\alpha)\in\mathbb{C}$ depending only on α and called the Fujiki constant of α such that for all $\beta \in H^2(M,\mathbb{C})$ we have*

$$\int_M \alpha\cdot\beta^{2n-2j} = C(\alpha)\cdot \mathsf{q}(\beta)^{n-j}. \tag{4.1}$$

In this section, we consider the Hilbert scheme $S^{[n]}$ of n-points of a $K3$ surface S, which by the work of Beauville [Bea] is irreducible hyperkähler. We will prove a closed formula for the Fujiki constants of all Chern classes of its tangent bundle.

For $k \geqslant 2$ even, we define the classical Eisenstein series

$$G_k(q) = -\frac{B_k}{2\cdot k} + \sum_{n\geqslant 1}\sum_{d|n} d^{k-1}q^n, \tag{4.2}$$

where B_k are Bernoulli numbers, i.e. $B_2=\frac{1}{6}$, $B_4=-\frac{1}{30}$, $\cdots$. For example, we have

$$G_2(q) = -\frac{1}{24} + \sum_{n\geqslant 1}\sum_{d|n} dq^n.$$

Theorem 4.2. *Let S be a $K3$ surface. For any $k\geqslant 0$,*

$$\sum_{n\geqslant k} C(c_{2n-2k}(T_{S^{[n]}}))\, q^n = \frac{(2k)!}{k!2^k}\left(q\frac{d}{dq}G_2(q)\right)^k \prod_{n\geqslant 1}(1-q^n)^{-24}.$$

The first coefficients are listed in Table 3. Remarkablely, the right hand side in Theorem 4.2 is up to the prefactor $(2k)!/(k!2^k)$ precisely the generating series of counts of genus k curves on a $K3$ surface passing through k generic points [BL]. This suggests a relationship to the work of Göttsche on curve counting on surfaces [G98]. The proof presented below uses similar ideas as in [G98], but we could not directly deduce it from there. The relationship to curve counting on $K3$ surfaces will be taken up in a follow-up work.

Proof. Let $L\in\mathrm{Pic}(S)$ be a line bundle on an arbitrary surface S. Consider the series

$$\Phi_{S,L} = \sum_{n=0}^{\infty} q^n \int_{S^{[n]}} c(T_{S^{[n]}})\, e^{c_1(L_n)},$$

where we let $L_n = \det(L^{[n]})\otimes\det(\mathcal{O}_S^{[n]})^{-1}$. Since the integrand is multiplicative, by [NW, Prop 3] (which immediately follows from [EGL]), there exists power series $\mathsf{A},\mathsf{B},\mathsf{C},\mathsf{D}$ in q such that for any surface S and line bundle L, we have

$$\Phi_{S,L} = \exp\left(c_1(L)^2\mathsf{A} + c_1(L)\cdot c_1(S)\mathsf{B} + c_1(S)^2\mathsf{C} + c_2(S)\mathsf{D}\right).$$

The Göttsche formula

$$\Phi_{S,0} = \sum_{n\geqslant 0} q^n\int_{S^{[n]}} c_{2n}(T_{S^{[n]}}) = \prod_{n\geqslant 1}\frac{1}{(1-q^n)^{e(S)}}$$

shows then that $\mathsf{C}=0$ and provides an explicit expression for D. Hence we have

$$\Phi_{S,L} = \prod_{n\geqslant 1}(1-q^n)^{-e(S)}\exp\left(c_1(L)^2\mathsf{A} + c_1(L)c_1(S)\mathsf{B}\right).$$

	c_0	c_2	c_4	c_6	c_8	c_{10}	c_{12}
$S^{[0]}$	1						
$S^{[1]}$	1	24					
$S^{[2]}$	3	30	324				
$S^{[3]}$	15	108	480	3200			
$S^{[4]}$	105	630	2016	5460	25650		
$S^{[5]}$	945	5040	13500	26184	49440	176256	
$S^{[6]}$	10395	51030	122220	198300	266490	378420	1073720

TABLE 3. The first non-trivial Fujiki constants of the Chern classes $c_k := c_k(T_{S^{[n]}})$ of Hilbert schemes of points on a $K3$ surface. The modularity of Theorem 4.2 appears in the diagonals, e.g. the cases $k = 0, 1$ are the functions:

$$\prod_{n\geqslant 1}(1-q^n)^{-24} = 1 + 24q + 324q^2 + 3200q^2 + \cdots.$$

$$\left(q\frac{d}{dq}G_2(q)\right)\prod_{n\geqslant 1}(1-q^n)^{-24} = q + 30q^2 + 480q^3 + 5460q^4 + \cdots.$$

Replacing L by $L^{\otimes t}$ for $t \in \mathbb{Z}$ shows that

$$\sum_{n=0}^{\infty} q^n \int_{S^{[n]}} c(T_{S^{[n]}})\, e^{tc_1(L_n)} = \prod_{n\geqslant 1}(1-q^n)^{-e(S)} \exp\left(c_1(L)^2 t^2 \mathsf{A} + c_1(L)c_1(S)t\mathsf{B}\right). \tag{4.3}$$

Since both sides are power series with coefficients which are polynomials in t, and the equality holds for all $t \in \mathbb{Z}$, we find that Eqn. (4.3) also holds for t, a formal variable. We write $\Phi_{S,L}(t)$ for the series (4.3). We argue now in two steps.

Step 1: Specialization to $K3$ surfaces. Let S be a $K3$ surface. Since $S^{[n]}$ is holomorphic symplectic, its odd Chern classes vanish. Together with Eqn. (4.1) and $\mathsf{q}(L_n) = c_1(L)^2$ this gives

$$\begin{aligned}\Phi_{S,L}(t) &= \sum_{n\geqslant 0} q^n \sum_{k=0}^{n} \frac{1}{(2k)!}\int_{S^{[n]}} c_{2n-2k}(T_{S^{[n]}})c_1(L_n)^{2k}t^{2k} \\ &= \sum_{k\geqslant 0}\frac{t^{2k}\left(c_1(L)^2\right)^k}{(2k)!}\sum_{n\geqslant k} C(c_{2n-2k}(T_{S^{[n]}}))q^n.\end{aligned}$$

On the other hand, we have

$$\Phi_{S,L}(t) = \prod_{n\geqslant 1}(1-q^n)^{-24}\cdot \exp(c_1(L)^2t^2\mathsf{A}).$$

By taking the t^{2k} coefficient we obtain that

$$\sum_{n\geqslant k} C(c_{2n-2k}(T_{S^{[n]}}))\, q^n = \frac{(2k)!}{k!}\mathsf{A}(q)^k \prod_{n\geqslant 1}(1-q^n)^{-24}. \tag{4.4}$$

Step 2: Specialization to abelian surfaces. Let A be an abelian surface with a line bundle $L \in \mathrm{Pic}(A)$ such that $c_1(L)^2 \neq 0$. Let $\sigma : A^{[n]} \to A$ be the sum map, and let

$$\mathrm{Kum}_{n-1}(A) = \sigma^{-1}(0_A)$$

be the generalized Kummer variety of dimension $2n-2$. We have the fiber diagram

$$\begin{array}{ccc} A\times \mathrm{Kum}_{n-1}(A) & \xrightarrow{\ \nu\ } & A^{[n]} \\ \big\downarrow{\scriptstyle \mathrm{pr}_1} & & \big\downarrow{\scriptstyle \sigma} \\ A & \xrightarrow{\ n\times\ } & A. \end{array}$$

In particular ν is étale (of degree n^4), which implies that

$$\nu^* T_{A^{[n]}} \cong \mathrm{pr}_1^*(T_A) \oplus \mathrm{pr}_2^*(T_{\mathrm{Kum}_{n-1}(A)}).$$

Since the Chern classes of an abelian surface vanish, we obtain

$$\nu^* c(T_{A^{[n]}}) = \mathrm{pr}_2^* c(T_{\mathrm{Kum}_{n-1}(A)}).$$

Moreover one has (see [NW, Eqn. (2)]) that

$$\nu^*(L_n) = \mathrm{pr}_1^*(L^{\otimes n}) \otimes \left(L_n|_{\mathrm{Kum}_{n-1}(A)}\right).$$

We obtain that

$$\begin{aligned}\int_{A^{[n]}} c_1(L_n)^2 c_{2n-2}(T_{A^{[n]}}) &= \frac{1}{n^4}\int_{A\times \mathrm{Kum}_{n-1}(A)} c_1(\nu^*(L_n))^2 c_{2n-2}(\nu^* T_{\mathrm{Kum}_{n-1}(A)}) \\ &= \frac{1}{n^2}(c_1(L)^2)\cdot \int_{\mathrm{Kum}_{n-1}(A)} c_{2n-2}(T_{\mathrm{Kum}_{n-1}(A)}) \\ &= \frac{1}{n^2}(c_1(L)^2)\, e(\mathrm{Kum}_{n-1}(A)).\end{aligned}$$

Using that $e(\mathrm{Kum}_{n-1}(A)) = n^3\sum_{d|n} d$ (ref. [GS]) and Eqn. (4.3), we conclude that

$$\begin{aligned}(c_1(L)^2)\cdot \mathsf{A}(q) = [\Phi_{A,L}(t)]_{t^2} &= \frac{1}{2}\sum_{n\geqslant 0} q^n \int_{A^{[n]}} c_1(L_n)^2 c_{2n-2}(T_{A^{[n]}}) \\ &= \frac{(c_1(L)^2)}{2}\sum_{n\geqslant 1} n \sum_{d|n} dq^n,\end{aligned}$$

where $[-]_{t^2}$ denotes the coefficient of t^2 term. Hence

$$\mathsf{A}(q) = \frac{1}{2} q\frac{d}{dq} G_2(q).$$

Combining with Eqn. (4.4), we are done. □

For completeness we also state the Fujiki constants of Chern classes of the second known infinite family of hyperkähler varieties, the generalized Kummer varieties.

Proposition 4.3. *For any $k \geqslant 0$ and abelian surface A, we have*

$$\sum_{n\geqslant k} C(c_{2n-2k}(T_{\mathrm{Kum}_n(A)}))\, q^{n+1} = \frac{(2k)!}{(k+1)!2^k}\left(q\frac{d}{dq}\right)^2 \left(q\frac{d}{dq}G_2\right)^{k+1}.$$

Proof. Using the universality (4.3) and the value of $\mathsf{A}(q)$ we computed above, one concludes that for any line bundle L on A, we have:

$$\begin{aligned}&\frac{(2k)!}{k!2^k}\left(q\frac{d}{dq}G_2\right)^k (c_1(L)^2)^k \\ =&\sum_{n\geqslant 0} q^n \int_{A^{[n]}} c_{2n-2k}(T_{A^{[n]}}) c_1(L_n)^{2k} \\ =&\sum_{n\geqslant 0} q^n \frac{1}{n^4}\int_{A\times \mathrm{Kum}_{n-1}(A)} c_{2n-2k}(\nu^*(T_{A^{[n]}})) c_1(\nu^* L_n)^{2k} \\ =&\sum_{n\geqslant 0} q^n \frac{1}{n^2}\binom{2k}{2}\left(\int_A c_1(L)^2\right) \mathsf{q}(L_n|_{\mathrm{Kum}_{n-1}(A)})^{k-1} C(c_{2n-2k}(T_{\mathrm{Kum}_{n-1}(A)})).\end{aligned}$$

Using $\mathsf{q}(L_n|_{\mathrm{Kum}_{n-1}(A)}) = c_1(L)^2$ we conclude the claim. □

Remark 4.4. *It is remarkable that all Fujiki constants of $c_k(T_X)$ for $X \in \{S^{[n]}, \mathrm{Kum}_n(A)\}$ are positive integers. By the software package 'bott' of J. Song* [Son]*, the same can be checked numerically for arbitrary products of Chern classes of the tangent bundle (up to $n \leqslant 10$). We also refer to* [CJ, J] *for some general results on positivity of Todd classes of hyperkähler varieties, and to* [OSV] *for a discussion on positivity of Chern (character) numbers. This suggests the question whether all (non-trivial) Fujiki constants of products of Chern classes on irreducible hyperkähler varieties positive. This question was raised independently and then studied in* [BS, Saw].

4.2. **Descendent integrals on the Hilbert scheme.** We now turn to integrals over descendents on Hilbert schemes, which are defined for $\alpha \in H^*(S)$ and $d \geqslant 0$ by

$$\mathfrak{G}_d(\alpha) := \pi_{\mathrm{Hilb}\,*}(\pi_S^*(\alpha)\,\mathrm{ch}_d(\mathcal{O}_{\mathcal{Z}})) \in H^*(S^{[n]}),$$

where $\pi_{\mathrm{Hilb}}, \pi_S$ are projections from $S^{[n]}\times S$ to the factors. We prove the following evaluations:

Proposition 4.5. *Let $\mathsf{p} \in H^4(S)$ be the point class. Then*

$$\sum_{n\geqslant 0} q^n \int_{S^{[n]}} c(T_{S^{[n]}})\mathfrak{G}_2(\mathsf{p}) = \left(\frac{1}{24} + G_2\right)\prod_{n\geqslant 1}(1-q^n)^{-24}.$$

Proof. This is a special case of [QS], but we can give a direct argument. For any surface S and K-theory class $x \in K(S)$ with $\mathrm{ch}_0(x) = \mathrm{ch}_1(x) = 0$ consider the series

$$\Phi_{S,x} = \sum_{n \geqslant 0} q^n \int_{S^{[n]}} c(T_{S^{[n]}}) e^{\mathrm{ch}_2(x^{[n]})}.$$

By [EGL] and since we know the answer for $x = 0$, there exists a series $\mathsf{A}(q)$ such that

$$\Phi_{S,x} = \prod_{n \geqslant 1} (1-q^n)^{-24} \exp\left(\mathrm{ch}_2(x)\mathsf{A}\right).$$

Setting $x = t\mathcal{O}_{\mathsf{p}}$, we in fact get the equality of

$$\Phi_{S,t} := \sum_{n \geqslant 0} q^n \int_{S^{[n]}} c(T_{S^{[n]}}) e^{\mathrm{ch}_2(\mathcal{O}_{\mathsf{p}}^{[n]})t} = \prod_{n \geqslant 1} (1-q^n)^{-24} \exp\left(\mathsf{A}t\right).$$

Case 1: K3 surfaces. By GRR and taking the t^1-coefficient, one finds that

$$\sum_{n \geqslant 0} q^n \int_{S^{[n]}} c(T_{S^{[n]}}) \mathfrak{G}_2(\mathsf{p}) = [\Phi_{S,t}]_{t^1} = \mathsf{A}(q) \prod_{n \geqslant 1} (1-q^n)^{-24}. \tag{4.5}$$

Case 2: Abelian surfaces. For an abelian surface A, similar as before, we have

$$\begin{aligned}\int_{A^{[n]}} c_{2n-2}(T_{A^{[n]}}) \mathrm{ch}_2(\mathcal{O}_{\mathsf{p}}^{[n]}) &= \frac{1}{n^4} \int_{A \times \mathrm{Kum}_{n-1}(A)} \nu^*(\mathrm{ch}_2(\mathcal{O}_{\mathsf{p}}^{[n]})) c_{2n-2}(T_{\mathrm{Kum}_{n-1}(A)}) \\ &= \frac{e(\mathrm{Kum}_{n-1}(A))}{n^3} = \frac{1}{n^3}\left(n^3 \sum_{d|n} d\right) = \sum_{d|n} d.\end{aligned}$$

Here we used that $\nu^* \mathrm{ch}_2(\mathcal{O}_{\mathsf{p}}^{[n]})|_{A \times pt} = n\mathsf{p}$. (To see the last statement, consider the diagram

$$\begin{array}{ccc} A \times \mathrm{Kum}_{n-1}(A) \times A & \xrightarrow{\nu \times \mathrm{id}} & A^{[n]} \times A \\ \downarrow & & \downarrow \pi \\ A \times \mathrm{Kum}_{n-1}(A) & \xrightarrow{\nu} & A^{[n]} \\ \downarrow \mathrm{pr}_1 & & \downarrow \sigma \\ A & \xrightarrow{n\times} & A. \end{array}$$

Let $\mathcal{Z} \subset A^{[n]} \times A$ be the universal subscheme, and let $\mathcal{Z}_{\mathrm{Kum}} \subset \mathrm{Kum}_{n-1}(A) \times A$ be its restriction to the Kummer. Inside $A \times \mathrm{Kum}_{n-1} \to A$, we have an equality of subschemes:

$$(\nu \times \mathrm{id})^{-1}(\mathcal{Z}) = m_{13}^{-1}(Z_{\mathrm{Kum}}),$$

where m_{13} is the addition map on the outer factors. Restricting to $A \times pt \times A$ we find that $m_{13}|^*_{A \times pt \times A}(n\mathsf{p}) = n\Delta_A$. Then the claim follows from the definition). We hence obtain that

$$\mathsf{A}(q) = [\Phi_{A,t}]_{t^1} = \sum_{n \geqslant 0} q^n \int_{A^{[n]}} c_{2n-2}(T_{A^{[n]}}) \mathrm{ch}_2(\mathcal{O}_{\mathsf{p}}^{[n]}) = \frac{1}{24} + G_2(q).$$

Combining with Eqn. (4.5), we are done. $\square$

Proposition 4.6.

(i) $$\int_{S^{[d]}} c_{2d-2}(S^{[d]}) \mathfrak{G}_3(D) = 0 \quad \textit{for all divisors } D \in H^2(S, \mathbb{Q}),$$

(ii) $$\sum_{n \geqslant 0} q^n \int_{S^{[n]}} c(T_{S^{[n]}}) \mathfrak{G}_4(1) = \left(-20G_2^2 - 2G_2 - 5/3G_4 - 1/24\right) \prod_{n \geqslant 1} (1-q^n)^{-24},$$

where G_k is given in (4.2).

Proof. Recall that for any hyperkähler variety X, the Looijenga-Lunts-Verbitsky Lie algebra $\mathfrak{g}(X)$ is isomorphic to $\mathrm{so}(H^2(X,\mathbb{Q}) \oplus U_{\mathbb{Q}})$, where $U = \left(\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}\right)$ is the hyperbolic lattice [LL, Ver95, Ver96]. The degree 0 part of the Lie algebra splits as $\mathfrak{g}_0(X) = \mathbb{Q}h \oplus \mathfrak{so}(H^2(X,\mathbb{Q}))$ where h is the degree grading operator. Looijenga and Lunts show that for the natural action of $\mathfrak{g}(X)$ on cohomology, the subliealgebra $\mathfrak{so}(H^2(X,\mathbb{Q}))$ acts by derivations. In other words, if $\mathfrak{t} \subset \mathfrak{so}(H^2(X))$ is a maximal Cartan, we have a decomposition

$$H^*(X) = \bigoplus_{\lambda : \mathfrak{t} \to \mathbb{Z}} V_\lambda,$$

which is multiplicative, i.e. $V_\lambda \cdot V_\mu \subset V_{\lambda+\mu}$. Here λ runs over all weights of the torus and V_λ is the corresponding eigenspace.

For a Hilbert scheme, let $\delta = c_1(\mathcal{O}_S^{[n]})$ and recall the natural decomposition

$$H^2(S^{[d]}) = H^2(S) \oplus \mathbb{Q}\delta.$$

We consider the subliealgebra $\mathfrak{so}(H^2(S)) \subset \mathfrak{so}(H^2(X,\mathbb{Q}))$ and for a Cartan $\mathfrak{t}' \subset \mathfrak{so}(H^2(S))$ the associated decomposition

$$H^*(S^{[d]}) = \bigoplus_{\mu:\mathfrak{t}'\to\mathbb{Z}} V_\mu.$$

Since Chern classes are monodromy invariant, they lie in V_0; see [LL] for a discussion. If $D = 0$ there is nothing to prove. Otherwise, we can choose the Cartan $\mathfrak{t}'$ such that D lies in a non-zero eigenspace of the action of $\mathfrak{so}(H^2(S))$ on $H^2(S,\mathbb{Q})$. Since the map $\gamma \mapsto \pi_{\mathrm{Hilb}\,*}(\mathrm{ch}_d(\mathcal{O}_\mathcal{Z})\pi_S^*(\gamma))$ is equivariant with respect to the action of $\mathfrak{so}(H^2(S))$ on $H^*(S^{[d]})$ and $H^*(S)$ respectively, we conclude that also

$$\mathfrak{G}_3(D) = \pi_{\mathrm{Hilb}\,*}(\mathrm{ch}_3(\mathcal{O}_\mathcal{Z})\pi_S^*(\gamma))$$

is of non-trivial weight with respect to $\mathfrak{t}'$, i.e. lies in V_μ for $\mu \neq 0$.

By multiplicativity of the decomposition, it follows that the integrand $c_{2d-2}(S^{[d]}) \cdot \mathfrak{G}_3(D)$ is of non-zero weight, hence its integral must be zero. This proves (i).

For part (ii) we start with the vanishing from part (i): For any divisor $W \in H^2(S,\mathbb{Q})$ we have

$$\int_{S^{[n]}} c_{2d-2}(S^{[n]})\mathfrak{G}_3(W) = 0.$$

In the notation of [NOY, Eqn. (36)] consider the element $h_{F\delta} = F \wedge \delta$ in $\mathfrak{so}(H^2(X,\mathbb{Q}))$ for some $F \in H^2(S,\mathbb{Q})$. Since the integrated degree 0 part of the LLV algebra acts as ring isomorphisms and preserves the Chern classes (see [LL]), we have

$$\begin{aligned}(4.6)\qquad \int_{S^{[n]}} c_{2d-2}(S^{[n]})e^{t\cdot h_{F\delta}}(\mathfrak{G}_3(W)) &= \int_{S^{[n]}} e^{t\cdot h_{F\delta}}(c_{2d-2}(S^{[n]})\mathfrak{G}_3(W)) \\ &= \int_{S^{[n]}} c_{2d-2}(S^{[n]})\mathfrak{G}_3(W) = 0.\end{aligned}$$

By [NOY, Prop. 4.4], we have that

$$h_{F\delta}(\mathfrak{G}_3(D_1)) = -\mathfrak{G}_2(F)\mathfrak{G}_2(W) - \langle F, W\rangle \mathfrak{G}_2(1)\mathfrak{G}_2(\mathsf{p}) - \langle F, W\rangle \mathfrak{G}_4(1).$$

Taking the derivative $\frac{d}{dt}|_{t=0}$ of Eqn. (4.6), we find that:

$$\langle F, W\rangle \int_{S^{[n]}} c(T_{S^{[n]}})\mathfrak{G}_4(1) = -\int_{S^{[n]}} c(T_{S^{[n]}})\mathfrak{G}_2(F)\mathfrak{G}_2(W) - n\langle F, W\rangle \int_{S^{[n]}} c(T_{S^{[n]}})\mathfrak{G}_2(\mathsf{p}).$$

Note that by Theorem 4.2, we have

$$\sum_{n\geqslant 0} q^n \int_{S^{[n]}} c(T_{S^{[n]}})\mathfrak{G}_2(F)\mathfrak{G}_2(W) = \langle F, W\rangle \prod_{n\geqslant 1}(1-q^n)^{-24} q\frac{d}{dq}G_2(q).$$

Moreover, by considering log-derivative, one has

$$(4.7)\qquad q\frac{d}{dq}\prod_{n\geqslant 1}(1-q^n)^{-24} = \prod_{n\geqslant 1}(1-q^n)^{-24}(1+24G_2).$$

Therefore we find as desired

$$\begin{aligned}(4.8)\qquad \sum_{n\geqslant 0} q^n \int_{S^{[n]}} c(T_{S^{[n]}})\mathfrak{G}_4(1) &= -\prod_{n\geqslant 1}(1-q^n)^{-24}\left(2q\frac{d}{dq}G_2(q) + (\frac{1}{24}+G_2)(1+24G_2)\right) \\ &= -\prod_{n\geqslant 1}(1-q^n)^{-24}(5/3G_4 + 20G_2^2 + 2G_2 + 1/24),\end{aligned}$$

where we used the following Ramanujan differential equation [BGHZ, pp. 49, Prop. 15]

$$q\frac{d}{dq}G_2(q) = -2G_2(q)^2 + \frac{5}{6}G_4(q). \qquad \square$$

4.3. **Descendent integrals on moduli spaces of 1-dimensional sheaves.** Let $\beta \in H_2(S,\mathbb{Z})$ be an effective curve class and let $M_{S,\beta}$ be the moduli space of one dimensional stable sheaves F on S with $[F]=\beta$ and $\chi(F)=1$. By a result of Mukai [M], $M_{S,\beta}$ is a smooth projective holomorphic symplectic variety of dimension β^2+2. Let $\mathbb{F}$ be the normalized universal family, i.e. which satisfies $\det \mathbf{R}\pi_{M*}\mathbb{F} = \mathcal{O}_{M_{S,\beta}}$. For $\alpha \in H^*(S)$, we define the descendents

$$\sigma_d(\alpha) = \pi_{M*}(\pi_S^*(\alpha)\,\mathrm{ch}_d(\mathbb{F})).$$

We have the following evaluations:

Proposition 4.7. *Let $\beta \in H_2(S,\mathbb{Z})$ be an effective curve class. For the point class $\mathsf{p} \in H^4(S)$ and $D \in H^2(S)$, we have*

(i) $$\int_{M_{S,\beta}} c(T_{M_{S,\beta}})\sigma_2(\mathsf{p}) = N_1\left(\frac{\beta^2}{2}\right),$$

(ii) $$\int_{M_{S,\beta}} c(T_{M_{S,\beta}})\sigma_3(D) = -(D\cdot\beta)N'\left(\frac{\beta^2}{2}\right),$$

(iii) $$\int_{M_{S,\beta}} c(T_{M_{S,\beta}})\sigma_4(1) = -N'\left(\frac{\beta^2}{2}\right),$$

where $N_1(l)$, $N'(l)$ for all $l \in \mathbb{Z}$ are defined by the generating series

$$\sum_{l\in\mathbb{Z}} N_1(l)\,q^l = \left(\frac{1}{q}\prod_{n\geqslant 1}\frac{1}{(1-q^n)^{24}}\right)\left(q\frac{d}{dq}G_2(q)\right) \tag{4.9}$$
$$= 1 + 30q + 480q^2 + 5460q^3 + 49440q^4 + 378420q^5 + 2540160q^6 + \cdots,$$
$$\sum_{l\in\mathbb{Z}} N'(l)\,q^l = \left(\frac{1}{q}\prod_{n\geqslant 1}\frac{1}{(1-q^n)^{24}}\right)\left(q\frac{d}{dq}G_2 + G_2 + \frac{1}{24}\right)$$
$$= 2 + 57q + 880q^2 + 9735q^3 + 86160q^4 + 646850q^5 + 4269888q^6 + \cdots.$$

4.4. **Transport of integrals to Hilbert schemes.** For the proof of Proposition 4.7 we will use the general framework of monodromy operators of Markman [M08] (see also [O22a]) to transport the integrals to the Hilbert schemes.

Consider the Mukai lattice, which is the lattice $\Lambda = H^*(S,\mathbb{Z})$ endowed with the Mukai pairing

$$\langle x, y\rangle := -\int_S x^\vee y,$$

where, if we decompose an element $x \in \Lambda$ according to degree as (r,D,n), we write $x^\vee = (r,-D,n)$. Given a sheaf or a complex of sheaves E on S, its Mukai vector is defined by

$$v(E) := \sqrt{\mathrm{td}_S}\cdot \mathrm{ch}(E) \in \Lambda.$$

Let $M(v)$ be a proper smooth moduli space of stable sheaves on S with Mukai vector $v \in \Lambda$ (where stability is with respect to some fixed polarization). We assume that there exists a universal family $\mathbb{F}$ on $M(v)\times S$. If it does not exist, everything below can be made to work by working with the Chern character $\mathrm{ch}(\mathbb{F})$ of a quasi-universal family, see [M08] or [O22a]. Let π_M, π_S be the projections to $M(v)$ and S. One has the Mukai morphism $\theta_{\mathbb{F}} : \Lambda \to H^2(M(v))$ defined by

$$\theta_{\mathbb{F}}(x) = \left[\pi_{M*}\left(\mathrm{ch}(\mathbb{F})\cdot\sqrt{\mathrm{td}_S}\cdot x^\vee\right)\right]_{\deg=2},$$

where $[-]_{\deg=k}$ stands for extracting the degree k component and where (as we will also do below) have suppressed the pullback maps from the projection to S. The morphism restricts to an lattice isometry

$$\theta_{\mathbb{F}} : v^\perp \to H^2(M(v),\mathbb{Z}) \tag{4.10}$$

where on the right we consider the Beauville-Bogomolov-Fujiki form. Define the universal class

$$u_v = \exp\left(\frac{\theta_{\mathbb{F}}(v)}{\langle v,v\rangle}\right)\mathrm{ch}(\mathbb{F})\sqrt{\mathrm{td}_S},$$

which is independent of the choice of universal family $\mathbb{F}$. For $x \in \Lambda$, consider the normalized descendents:

$$B(x) := \pi_{M*}(u_v\cdot x^\vee),$$

and let $B_k(x) = [B(x)]_{\deg=2k}$ its degree $2k$ component.

Example 4.8. *For $v = (1, 0, 1-d)$, the moduli space becomes the punctual Hilbert scheme: $M(v) = S^{[n]}$. Then we have*

$$u_v = \exp\left(\frac{-\delta}{2d-2}\right)\operatorname{ch}(\mathcal{I}_{\mathcal{Z}})\sqrt{\operatorname{td}_S},$$

where we let $\delta = \pi_* \operatorname{ch}_3(\mathcal{O}_{\mathcal{Z}})$ (so that -2δ is the class of the locus of non-reduced subschemes).

We define the standard descendents on the Hilbert scheme by

$$\mathfrak{G}_d(\alpha) = \pi_{\mathrm{Hilb}\,*}(\pi_S^*(\alpha)\operatorname{ch}_d(\mathcal{O}_{\mathcal{Z}})) \in H^*(S^{[d]}),$$

where $\alpha \in H^*(S)$. One obtains that

$$B_1(\mathsf{p}) = -\frac{\delta}{2d-2},$$
$$B_2(\mathsf{p}) = \frac{1}{2}\frac{\delta^2}{(2d-2)^2} - \mathfrak{G}_2(\mathsf{p}).$$

For a divisor $D \in H^2(S)$ one finds

$$B_1(D) = \mathfrak{G}_2(D),$$
$$B_2(D) = \mathfrak{G}_3(D) - \frac{\delta}{2d-2}\mathfrak{G}_2(D).$$

And for the unit,

$$B_1(1) = -\frac{1}{2}\delta,$$
$$B_2(1) = \frac{3}{4}\frac{\delta^2}{2d-2} - \mathfrak{G}_2(\mathsf{p}) - \mathfrak{G}_4(1).$$

Example 4.9. *Let $\beta \in \operatorname{Pic}(S)$ be an effective class of square $\beta \cdot \beta = 2d-2$. For the Mukai vector $v = (0, \beta, 1)$ the moduli space is $M(v) = M_{S,\beta}$. Let $\mathbb{F}$ be the normalized universal family, i.e. which satisfies $\det \mathbf{R}\pi_{M*}\mathbb{F} = \mathcal{O}$. For $\alpha \in H^*(S)$, we define as before the descendents*

$$\sigma_d(\alpha) = \pi_{M*}(\pi_S^*(\alpha)\operatorname{ch}_d(\mathbb{F})).$$

By the normalization condition and GRR, we have:

$$c_1(\mathbf{R}\pi_{M*}\mathbb{F}) = \sigma_3(1) + 2\sigma_1(\mathsf{p}) = 0. \tag{4.11}$$

Moreover, by a direct computation, one also has:

$$B_1(\mathsf{p}) = \sigma_1(\mathsf{p}),$$
$$\theta_{\mathbb{F}}(v) = -\sigma_2(\beta) + \sigma_1(\mathsf{p}),$$

Using the vanishing $c_1(\mathbf{R}\pi_{M*}\mathbb{F}) = 0$ again yields

$$B_1(1+\mathsf{p}) = \left[\operatorname{ch}(\mathbf{R}\pi_{M*}\mathbb{F})\exp\left(\frac{\theta_{\mathbb{F}}(v)}{2d-2}\right)\right]_{\deg 2} = \frac{\theta_{\mathbb{F}}(v)}{2d-2} = \frac{1}{2d-2}(\sigma_1(\mathsf{p}) - \sigma_2(\beta)).$$

This shows

$$\sigma_2(\beta) = \sigma_1(\mathsf{p}) - (2d-2)B_1(1+\mathsf{p}),$$
$$\theta_{\mathbb{F}}(v) = (2d-2)B_1(1+\mathsf{p}).$$

By rewriting the B's in terms of the σ's using the formulae above and then inverting the relation, we obtain for all $D \in H^2(S)$ by a straightforward calculation the following:

$$\sigma_1(\mathsf{p}) = B_1(\mathsf{p}),$$
$$\sigma_2(D) = -(D\cdot\beta)B_1(1+\mathsf{p}) - B_1(D),$$

and

$$\sigma_2(\mathsf{p}) = B_2(\mathsf{p}) - B_1(\mathsf{p})B_1(1+\mathsf{p}),$$
$$\sigma_3(D) = -B_2(D) + B_1(D)B_1(1+\mathsf{p}) + \frac{1}{2}(D\cdot\beta)B_1(1+\mathsf{p})^2,$$
$$\sigma_4(1) = B_2(1-\mathsf{p}) + 2B_1(\mathsf{p})B_1(1+\mathsf{p}) - \frac{1}{2}B_1(1+\mathsf{p})^2$$
$$= B_2(1-\mathsf{p}) - \frac{1}{2}B_1(1+\mathsf{p})B_1(1-3\mathsf{p}).$$

For later, we also record some pairings with respect to the Beauville-Bogomolov-Fujiki form:

Lemma 4.10.

$$\sigma_3(1)\cdot\sigma_1(\mathsf{p}) = 0, \quad \sigma_1(\mathsf{p})\cdot\sigma_2(D) = D\cdot\beta, \quad \sigma_3(1)\cdot\sigma_2(D) = -2(D\cdot\beta).$$

Proof. By Eqn. (4.11) and since (4.10) is an isometry and moreover $B_1(-)|_{v^\perp} = \theta_{\mathbb{F}}|_{v^\perp}$ we have

$$\sigma_3(1)\cdot\sigma_1(\mathsf{p}) = -2\sigma_1(\mathsf{p})\cdot\sigma_1(\mathsf{p}) = -2B_1(\mathsf{p})\cdot B_1(\mathsf{p}) = 0.$$

Similarly,

$$\sigma_1(\mathsf{p})\cdot\sigma_2(D) = B_1(\mathsf{p})\cdot(-(D\cdot\beta)B_1(1+\mathsf{p}) - B_1(D)) = -(D\cdot\beta)(\mathsf{p}\cdot 1) = D\cdot\beta.$$

The last one uses again Eqn. (4.11). ☐

Using the descendents $B_k(x)$, one allows to move between any two moduli spaces of stable sheaves on S just by specifying a Mukai lattice isomorphism $g:\Lambda\otimes\mathbb{Q}\to\Lambda\otimes\mathbb{Q}$. We give the details in the case of our interest, see [M08, O22a] for the general case.

We want to connect the moduli spaces

$$M_{S,\beta} \rightsquigarrow S^{[d]}.$$

Define the isomorphism $g:\Lambda\otimes\mathbb{Q}\to\Lambda\otimes\mathbb{Q}$ by

$$1\mapsto(0,0,1), \quad \mathsf{p}\mapsto(1,-\beta,d-1), \quad (0,D,0)\mapsto(0,D,-(D\cdot\beta)),$$

for all $D\in H^2(S,\mathbb{Z})$. The isomorphism was constructed so that

$$(0,\beta,1)\mapsto(1,0,1-d), \quad 1\mapsto(0,0,1), \quad (2d-2,\beta,0)\mapsto(0,\beta,0), \quad g|_{\{1,\beta,\mathsf{p}\}^\perp} = \mathrm{id},$$

which shows that it is a lattice isomorphism. Then one has:

Theorem 4.11. (*Markman* [M08, Thm. 1.2], *reformulation as in* [O22a, Thm. 4]) *For any $k_i\geqslant 0$, $\alpha_i\in H^*(S)$ and any polynomial P,*

$$\int_{M_{S,\beta}} P(B_{k_i}(\alpha_i), c_j(T_{M_{S,\beta}})) = \int_{S^{[d]}} P(B_{k_i}(g\alpha_i), c_j(T_{S^{[n]}})).$$

4.5. **Proof of Proposition 4.7.** Let $\beta\in H_2(S,\mathbb{Z})$ be an effective curve class with $\beta^2 = 2d-2$. We begin with the first evaluation. The strategy is to use Theorem 4.11 and the formulae given in Examples 4.8, 4.9 to move between the standard descendents and Markman's B-classes. We obtain:

$$\begin{aligned}
&\int_{M_{S,\beta}} c_{2d-2}(T_{M_{S,\beta}})\sigma_2(\mathsf{p})\\
&=\int_{M_{S,\beta}} c_{2d-2}(T_{M_{S,\beta}})(B_2(\mathsf{p}) - B_1(\mathsf{p})B_1(1+\mathsf{p}))\\
&=\int_{S^{[d]}} c_{2d-2}(T_{S^{[d]}})(B_2(1-\beta+(d-1)\mathsf{p}) - B_1(1-\beta+(d-1)\mathsf{p})B_1(1-\beta+d\mathsf{p})).
\end{aligned}$$

Observe that:

$$\begin{aligned}
B_1(1-\beta+(d-1)\mathsf{p}) &= -\delta - \mathfrak{G}_2(\beta),\\
B_1(1-\beta+d\mathsf{p}) &= -\delta - \mathfrak{G}_2(\beta) - \frac{\delta}{2d-2},\\
B_2(1-\beta+(d-1)\mathsf{p}) &= \frac{\delta^2}{2d-2} - d\mathfrak{G}_2(\mathsf{p}) - \mathfrak{G}_4(1) - \left(\mathfrak{G}_3(\beta) - \frac{\delta}{2d-2}\mathfrak{G}_2(\beta)\right),
\end{aligned}$$

Using the vanishing in Proposition 4.6 (i) we find that

$$\begin{aligned}
&\int_{S^{[d]}} c_{2d-2}(T_{S^{[d]}})B_2(1-\beta+(d-1)\mathsf{p})\\
&=\int_{S^{[d]}} c_{2d-2}(T_{S^{[d]}})\left(\frac{\delta^2}{2d-2} - d\mathfrak{G}_2(\mathsf{p}) - \mathfrak{G}_4(1)\right)\\
&=-C(c_{2d-2}(T_{S^{[d]}})) - \int_{S^{[d]}} c_{2d-2}(T_{S^{[d]}})(d\mathfrak{G}_2(\mathsf{p}) + \mathfrak{G}_4(1)),
\end{aligned}$$

as well as

$$\int_{S^{[d]}} c_{2d-2}(T_{S^{[d]}})B_1(1-\beta+(d-1)\mathsf{p})B_1(1-\beta+d\mathsf{p}) = -C(c_{2d-2}(T_{S^{[d]}})).$$

Let us write

$$A_d = \int_{M_{S,\beta}} c_{2d-2}(T_{M_{S,\beta}})\sigma_2(\mathsf{p}),$$

which is well-defined since the above shows that the right hand side only depends on d. Taking generating series and using the evaluations of descendents on $S^{[d]}$ (in particular, the expression (4.8) and the differential equation (4.7)), we conclude

$$\begin{aligned}\sum_d A_d q^d &= -\sum_{d\geqslant 0} q^d \int_{S^{[d]}} c_{2d-2}(T_{S^{[d]}})(d\mathfrak{G}_2(\mathsf{p}) + \mathfrak{G}_4(1)) \\ &= -q\frac{d}{dq}\left(M(q)(1/24 + G_2)\right) + M(q)\left(2q\frac{d}{dq}G_2 + \left(\frac{1}{24} + G_2\right)(1+24G_2)\right) \\ &= M(q) q \frac{d}{dq} G_2,\end{aligned}$$

where we denote $M(q) = \prod_{n\geqslant 1}(1-q^n)^{-24}$. This proves the first evaluation (after shifting the generating series by q).

For the second case, one argues similarly, and obtains

$$\int_{M_{S,\beta}} c_{2d-2}(T_{M_{S,\beta}})\sigma_3(D) = -(D\cdot\beta)\left(\int_{S^{[d]}} c_{2d-2}(T_{S^{[d]}})\mathfrak{G}_2(\mathsf{p}) + C(c_{2d-2}(T_{S^{[d]}}))\right).$$

In the third case, one obtains

$$\int_{M_{S,\beta}} c_{2d-2}(T_{M_{S,\beta}})\sigma_4(1) = \int_{S^{[d]}} c_{2d-2}(T_{S^{[d]}})(\mathfrak{G}_4(1) + (d-1)\mathfrak{G}_2(\mathsf{p})). \quad \square$$

5. Product of $K3$ surfaces

In this section, we consider the product of two $K3$ surfaces S and T:

$$X = S \times T.$$

If the curve class $\beta \in H_2(S\times T, \mathbb{Z})$ is of non-trivial degree over both S and T, then one can construct two linearly independent cosections, which imply that the reduced invariants of X in this class vanish.[10] Because of that we always take β in the image of the natural inclusion

$$\iota_* : H_2(S,\mathbb{Z}) \hookrightarrow H_2(X,\mathbb{Z}),$$

where $\iota : S\times\{t\} \hookrightarrow X$ is the inclusion of a fiber. In §5.1, we first discuss the computations of GW/GV invariants. Then we completely determine all DT_4 invariants. By comparing them, we prove Conjecture 2.2 for $X = S\times T$.

5.1. **Gromov-Witten invariants.** For $\beta \in H_2(S,\mathbb{Z}) \subseteq H_2(X,\mathbb{Z})$, by the product formula in Gromov-Witten theory [B99], the reduced virtual classes satisfy

$$[\overline{M}_{g,n}(X,\beta)]^{\mathrm{vir}} = \begin{cases} [\overline{M}_{0,n}(S,\beta)]^{\mathrm{vir}} \times [T] & \text{if } g = 0 \\ [\overline{M}_{1,n}(S,\beta)]^{\mathrm{vir}} \times (c_2(T)\cap[T]) & \text{if } g = 1 \\ 0 & \text{if } g \geqslant 2. \end{cases} \tag{5.1}$$

The Gromov-Witten theory of $K3$ surfaces in low genus is well-known.

In genus 0, one defines BPS numbers $n_{0,\beta}(S)$ by the multiple cover formula

$$\deg[\overline{M}_{0,0}(S,\beta)]^{\mathrm{vir}} = \sum_{k\geqslant 1, k|\beta} \frac{1}{k^3}\cdot n_{0,\beta/k}(S). \tag{5.2}$$

By the Yau-Zaslow formula proven by Klemm, Maulik, Pandharipande and Scheidegger [KMPS], the invariant $n_{0,\beta}(S)$ only depends on the square β^2. By the evaluation for primitive curve classes due to Bryan and Leung [BL], one then has

$$n_{0,\beta}(S) = N_0\left(\frac{\beta^2}{2}\right), \tag{5.3}$$

where

$$\begin{aligned}\sum_{l\in\mathbb{Z}} N_0(l)\, q^l &= \frac{1}{q}\prod_{n\geqslant 1}\frac{1}{(1-q^n)^{24}} \\ &= q^{-1} + 24 + 324q + 3200q^2 + 25650q^3 + \cdots.\end{aligned} \tag{5.4}$$

[10] Of course, one may work with 2-reduced invariants but the moduli spaces becomes more difficult to handle. We leave the study of the 2-reduced theory to a future work.

In genus 1, by Pandharipande-Yin [PY, pp. 12, (8)], we have the multiple cover formula

$$\int_{[\overline{M}_{1,1}(S,\beta)]^{\mathrm{vir}}} \mathrm{ev}^*(\mathsf{p}) = \sum_{k\geqslant 1, k|\beta} k \cdot N_1\left(\frac{\beta^2}{2k^2}\right), \tag{5.5}$$

where $N_1(l)$ is defined as in (4.9) of the last section, that is

$$\begin{aligned}\sum_{l\in\mathbb{Z}} N_1(l)\, q^l &= \left(\frac{1}{q}\prod_{n\geqslant 1}\frac{1}{(1-q^n)^{24}}\right)\left(q\frac{d}{dq}G_2(q)\right)\\ &= 1 + 30q + 480q^2 + 5460q^3 + 49440q^4 + 378420q^5 + \cdots .\end{aligned}$$

We remark that although genus 2 Gromov-Witten invariants are zero (5.1), the corresponding Gopakumar-Vafa invariants are nontrivial (Proposition 5.1).

5.2. **Gopakumar-Vafa invariants.** Let $\gamma, \gamma' \in H^4(X)$ be cohomology classes and

$$\begin{aligned}\gamma &= A_1 \cdot 1 \otimes \mathsf{p} + D_1 \otimes D_2 + A_2 \cdot \mathsf{p} \otimes 1,\\ \gamma' &= A_1' \cdot 1 \otimes \mathsf{p} + D_1' \otimes D_2' + A_2' \cdot \mathsf{p} \otimes 1\end{aligned}$$

be their decompositions under the Künneth isomorphism:

$$H^4(X) \cong (H^0(S)\otimes H^4(T)) \oplus (H^2(S)\otimes H^2(T)) \oplus (H^4(S)\otimes H^0(T)).$$

Fix also a curve class

$$\alpha = \theta_1 \otimes \mathsf{p} + \mathsf{p}\otimes\theta_2 \in H^6(X) \cong (H^2(S)\otimes H^4(T)) \oplus (H^4(S)\otimes H^2(T)).$$

Proposition 5.1. *For any effective curve class $\beta \in H_2(S,\mathbb{Z}) \subseteq H_2(X,\mathbb{Z})$, we have*

$$\begin{aligned}n_{0,\beta}(\gamma,\gamma') &= (D_1\cdot\beta)\cdot(D_1'\cdot\beta)\cdot\int_T (D_2\cdot D_2')\cdot N_0\left(\frac{\beta^2}{2}\right),\\ n_{0,\beta}(\alpha) &= (\theta_1\cdot\beta)\, N_0\left(\frac{\beta^2}{2}\right).\end{aligned}$$

If β is primitive, we have

$$n_{1,\beta}(\gamma) = 24\, A_2\, N_1\left(\frac{\beta^2}{2}\right), \quad n_{2,\beta} = N_2\left(\frac{\beta^2}{2}\right),$$

where $N_1(l)$ is defined as in (4.9) *and*

$$\begin{aligned}\sum_{l\in\mathbb{Z}} N_2(l)\, q^l &= \left(\frac{1}{q}\prod_{n\geqslant 1}\frac{1}{(1-q^n)^{24}}\right)\left(24q\frac{d}{dq}G_2 - 24G_2 - 1\right)\\ &= 72q + 1920q^2 + 28440q^3 + 305280q^4 + 2639760q^5 + 19450368q^6 + \cdots .\end{aligned} \tag{5.6}$$

In particular, Conjecture 1.9 holds for $X = S\times T$.

Proof. By the divisor equation, we have

$$\begin{aligned}\big\langle \tau_0(\gamma)\tau_0(\gamma')\big\rangle^{\mathrm{GW}}_{0,\beta} &= (D_1\cdot\beta)\cdot(D_1'\cdot\beta)\cdot\int_T (D_2\cdot D_2')\cdot \deg([\overline{M}_{0,0}(S,\beta)]^{\mathrm{vir}}),\\ \big\langle \tau_0(\alpha)\big\rangle^{\mathrm{GW}}_{0,\beta} &= (\theta_1\cdot\beta)\cdot\deg([\overline{M}_{0,0}(S,\beta)]^{\mathrm{vir}}).\end{aligned}$$

The genus 0 formula hence follows from Eqn. (5.2) and the Yau-Zaslow formula (5.3). In genus 1, the product formula (5.1) and Eqn. (5.5) imply that for any effective class $\beta \in H_2(S,\mathbb{Z})$ we have:

$$\big\langle \tau_0(\gamma)\big\rangle^{\mathrm{GW}}_{1,\beta} = A_2\, e(T)\int_{[\overline{M}_{1,1}(S,\beta)]^{\mathrm{vir}}} \mathrm{ev}^*(\mathsf{p}) = A_2\, e(T)\sum_{k\geqslant 1, k|\beta} k\cdot N_1\left(\frac{\beta^2}{2k^2}\right). \tag{5.7}$$

Moreover, by the first part we have

$$\big\langle \tau_0(\gamma)\tau_0(c_2(X))\big\rangle^{\mathrm{GW}}_{0,\beta} = 0.$$

Hence taking these formulae for primitive β yields the result.

For the genus 2 Gopakumar-Vafa invariant, let β be primitive. Observe that we have

$$\big\langle \varnothing\big\rangle^{\mathrm{GW}}_{2,\beta} = 0, \quad n_{1,\beta}(c_2(X)) = 24^2 N_1(\beta^2/2), \quad \big\langle \tau_0(c_2(X))^2\big\rangle^{\mathrm{GW}}_{0,\beta} = 0.$$

The nodal invariant is computed as follows:

$$\begin{aligned} N_{\mathrm{nodal},\beta} &= \frac{1}{2}\left[\big\langle \tau_0(\Delta_X)\big\rangle^{\mathrm{GW}}_{0,\beta} - \big\langle \tau_1(c_2(T_X))\big\rangle^{\mathrm{GW}}_{0,\beta}\right] \\ &= \frac{1}{2}\left[24\int_{[\overline{M}_{0,2}(S,\beta)]^{\mathrm{vir}}} (\mathrm{ev}_1\times \mathrm{ev}_2)^*(\Delta_S) - 24\int_{[\overline{M}_{0,1}(S,\beta)]^{\mathrm{vir}}}\psi_1\right] \\ &= \frac{1}{2}\left[24(\beta\cdot\beta)N_0(\beta^2/2) + 2\cdot 24 N_0(\beta^2/2)\right]. \end{aligned}$$

If β_h is a primitive curve class of square $\beta_h^2 = 2h-2$, we conclude:

$$\begin{aligned} \sum_{h\geqslant 0} N_{\mathrm{nodal},\beta_h} q^{h-1} &= \frac{1}{2}\left[48 q\frac{d}{dq}\left(\frac{1}{\Delta(q)}\right) + 48\frac{1}{\Delta(q)}\right] \\ &= 24^2 G_2(q)\frac{1}{\Delta(q)} + 24\frac{1}{\Delta(q)}, \end{aligned}$$

where we used $\Delta(q) = q\prod_{n\geqslant 1}(1-q^n)^{24}$ and the identity (ref. Eqn. (4.7)):

$$q\frac{d}{dq}\left(\frac{1}{\Delta(q)}\right) = \frac{24 G_2(q)}{\Delta(q)}.$$

Using the definition of $n_{2,\beta}$, we conclude that:

$$\begin{aligned} \sum_{h\geqslant 0} n_{2,\beta_h} q^{h-1} &= \left(24\frac{1}{\Delta(q)} q\frac{d}{dq} G_2(q)\right) - \left(24 G_2(q)\frac{1}{\Delta(q)} + \frac{1}{\Delta(q)}\right) \\ &= \frac{1}{\Delta(q)}\left(24 q\frac{d}{dq}G_2(q) - 24 G_2(q) - 1\right). \end{aligned}$$

This is exactly the desired result. □

We will also need the following later (in the appendix):

Lemma 5.2. *For any effective curve class $\beta \in H_2(S,\mathbb{Z}) \subseteq H_2(X,\mathbb{Z})$, we have*

$$\big\langle \tau_1(\gamma)\big\rangle^{\mathrm{GW}}_{0,\beta} = -2A_1 \sum_{k\geqslant 1, k|\beta} \frac{1}{k^3}\cdot N_0\left(\frac{(\beta/k)^2}{2}\right).$$

Proof. Consider a divisor $D = \mathrm{pr}_1^*(\alpha) \in H^2(X)$ with $d := \alpha\cdot\beta \neq 0$. By Lemma 1.1 and Eqn. (5.1) we have

$$\begin{aligned} \big\langle \tau_1(\gamma)\big\rangle^{\mathrm{GW}}_{0,\beta} &= -\frac{2}{d}\big\langle \tau_0(\gamma\cdot D)\big\rangle^{\mathrm{GW}}_{0,\beta} \\ &= -\frac{2}{d}\left(A_1\left\langle \tau_0(\alpha\otimes \mathsf{p})\right\rangle^{\mathrm{GW}}_{0,\beta} + (D_1\cdot\alpha)\left\langle \tau_0(\mathsf{p}\otimes D_2)\right\rangle^{\mathrm{GW}}_{0,\beta}\right) \\ &= -\frac{2}{d}A_1\int_{[\overline{M}_{0,1}(S,\beta)]^{\mathrm{vir}}} \mathrm{ev}^*(\alpha) \\ &= -2A_1\, \deg[\overline{M}_{0,0}(S,\beta)]^{\mathrm{vir}}. \end{aligned}$$

By Eqn. (5.2) and the Yau-Zaslow formula (5.3), we obtain the claim. □

5.3. DT_4 **virtual classes.** The moduli space M_β of one dimensional stable sheaves on X satisfies (e.g. [CMT18, Lem. 2.2]):

$$M_\beta \cong M_{S,\beta}\times T, \tag{5.8}$$

where $M_{S,\beta}$ is the moduli space of one dimensional stable sheaves F on S with $[F]=\beta$ and $\chi(F)=1$. By a result of Mukai [M], $M_{S,\beta}$ is a smooth projective holomorphic symplectic variety of dimension β^2+2. In order to determine the DT_4 virtual class of M_β, we first recall:

Definition 5.3. ([Sw, Ex. 16.52, pp. 410], [EG, Lem. 5]) *Let E be a $\mathrm{SO}(2n,\mathbb{C})$-bundle with a non-degenerate symmetric bilinear form Q on a connected scheme M. Denote E_+ to be its positive real form*[11]*. The half Euler class of (E,Q) is*

$$e^{\frac{1}{2}}(E,Q) := \pm\, e(E_+) \in H^{2n}(M,\mathbb{Z}),$$

where the sign depends on the choice of orientation of E_+.

[11]This means a real half dimensional subbundle such that Q is real and positive definite on it. By homotopy equivalence $\mathrm{SO}(m,\mathbb{C}) \sim \mathrm{SO}(m,\mathbb{R})$, it exists and is unique up to isomorphisms.

Definition 5.4. ([EG], [KiP, Def. 8.7]) *Let E be a $\mathrm{SO}(2n,\mathbb{C})$-bundle with a non-degenerate symmetric bilinear form Q on a connected scheme M. An isotropic cosection of (E,Q) is a map*

$$\phi : E \to \mathcal{O}_M,$$

such that the composition

$$\phi\circ\phi^\vee : \mathcal{O}_M \to E^\vee \overset{Q}{\cong} E \to \mathcal{O}_M$$

is zero. If ϕ is furthermore surjective, we define the (reduced) half Euler class:

$$e^{\frac{1}{2}}_{\mathrm{red}}(E,Q) := e^{\frac{1}{2}}\left((\phi^\vee\mathcal{O}_M)^\perp/(\phi^\vee\mathcal{O}_M), \bar{Q}\right) \in H^{2n-2}(M,\mathbb{Z}),$$

as the half Euler class of the isotropic reduction. Here $\bar{Q}$ denotes the induced non-degenerate symmetric bilinear form on $(\phi^\vee\mathcal{O}_M)^\perp/(\phi^\vee\mathcal{O}_M)$.

We show reduced half Euler classes are independent of the choice of surjective isotropic cosection.

Lemma 5.5. *Let E be a $\mathrm{SO}(2n,\mathbb{C})$-bundle with a non-degenerate symmetric bilinear form Q on a connected scheme M and*

$$\phi : E \to \mathcal{O}_M$$

be a surjective isotropic cosection. Then we can write the positive real form E_+ of E as

$$E_+ = \mathcal{E}_+ \oplus \underline{\mathbb{R}}^2$$

such that

$$e^{\frac{1}{2}}_{\mathrm{red}}(E,Q) = \pm\, e(\mathcal{E}_+).$$

Moreover, it is independent of the choice of surjective cosection.

In particular, when $E = \mathcal{O}^{\oplus 2}\oplus V$ such that $Q = \begin{pmatrix} 0 & 1 \\ 1 & 0\end{pmatrix} \oplus Q|_V$, we have

$$e^{\frac{1}{2}}_{\mathrm{red}}(E,Q) = \pm\, e^{\frac{1}{2}}(V, Q|_V).$$

Proof. Let $E_- := \sqrt{-1}\cdot E_+$, then $E = E_+\oplus E_-$. Since ϕ is surjective, $\phi^\vee$ determines a trivial subbundle $\mathcal{O}_M$ of E. In the diagram:

$$\begin{array}{ccc} \mathcal{O}_M & \xrightarrow{\phi^\vee} & E = E_+\oplus E_- \\ & \searrow & \downarrow{\scriptstyle \pi_\pm} \\ & & E_\pm, \end{array}$$

for $v\in\phi^\vee(\mathcal{O}_M)$, write $v = v_+ + v_-$ based on above decomposition. The isotropic condition gives

$$0 = Q(v,v) = Q(v_+,v_+) + 2\,Q(v_+,v_-) + Q(v_-,v_-).$$

If $v_+ = 0$, then $Q(v_-,v_-) = 0$ which implies $v_- = 0$ as Q on E_- is negative definite. Therefore the composition $\pi_\pm\circ\phi^\vee$ determines a trivial subbundle $\underline{\mathbb{R}}\subset E_\pm$.

We write $(\phi^\vee\mathcal{O}_M)^\perp = V_+\oplus V_-$ for $V_\pm = E_\pm \bigcap (\phi^\vee\mathcal{O}_M)^\perp$, which fits in the diagram

$$\begin{array}{ccccc} \phi^\vee(\mathcal{O}_M) & \xrightarrow{\subset} & (\phi^\vee\mathcal{O}_M)^\perp & \xrightarrow{\subset} & E \\ \downarrow{\scriptstyle =} & & \downarrow{\scriptstyle =} & & \downarrow{\scriptstyle =} \\ \underline{\mathbb{R}}\oplus\underline{\mathbb{R}} & & V_+\oplus V_- & & E_+\oplus E_-. \end{array}$$

Then $\mathrm{rank}_{\mathbb{R}} V_+ + \mathrm{rank}_{\mathbb{R}} V_- = 4n-2$ and $\mathrm{rank}_{\mathbb{R}} V_\pm \leqslant \mathrm{rank}_{\mathbb{R}} E_\pm$. As $(\phi^\vee\mathcal{O}_M)^\perp/(\phi^\vee\mathcal{O}_M)$ has an induced non-degenerate symmetric bilinear form, so

$$\mathrm{rank}_{\mathbb{R}} V_+ = \mathrm{rank}_{\mathbb{R}} V_- = 2n-1.$$

Let $\mathcal{E}_+ := V_+/\underline{\mathbb{R}}$, by the metric $Q|_{V_+}$ on V_+, we may write

$$V_+ = \mathcal{E}_+\oplus\underline{\mathbb{R}}. \tag{5.9}$$

Under the identification $E^\vee \overset{Q}{\cong} E$, we have

$$\begin{aligned} \mathrm{Ker}(\phi) &= \left\{ v\in E \,|\, \phi(Q(v,-)) = 0 \right\} \\ &= \left\{ v\in E \,|\, Q(v,\phi^\vee\mathcal{O}_M) = 0 \right\} \\ &= (\phi^\vee\mathcal{O}_M)^\perp. \end{aligned}$$

Therefore $E/(\phi^\vee\mathcal{O}_M)^\perp \overset{\phi}{\cong} \mathcal{O}_M$. By using the metric $Q|_{E_+}$ on E_+, we may write

$$E_+ = V_+ \oplus \underline{\mathbb{R}}.$$

Combining with Eqn. (5.9), we have

$$E_+ = \mathcal{E}_+ \oplus \underline{\mathbb{R}}^2.$$

By definition, the reduced half Euler class is the Euler class of $\mathcal{E}_+$.

Given two surjective cosections ϕ_1, ϕ_2, if $\phi_1^\vee\mathcal{O}_M = \phi_2^\vee\mathcal{O}_M$, then the bundle $\mathcal{E}_+$ they determine are the same, so are the reduced half Euler classes. If $\phi_1^\vee\mathcal{O}_M \neq \phi_2^\vee\mathcal{O}_M$, we divide into two cases: (1) when $\phi_2^\vee\mathcal{O}_M \subseteq (\phi_1^\vee\mathcal{O}_M)^\perp$ (which automatically implies $\phi_1^\vee\mathcal{O}_M \subseteq (\phi_2^\vee\mathcal{O}_M)^\perp$), it is easy to see the corresponding $\mathcal{E}_+$ has a trivial subbundle $\underline{\mathbb{R}}$, so both reduced half Euler classes vanish. (2) when $\phi_2^\vee\mathcal{O}_M \nsubseteq (\phi_1^\vee\mathcal{O}_M)^\perp$ (hence also $\phi_1^\vee\mathcal{O}_M \nsubseteq (\phi_2^\vee\mathcal{O}_M)^\perp$), then we have

$$E \cong (\phi_1^\vee\mathcal{O}_M)^\perp \oplus \phi_2^\vee\mathcal{O}_M \cong (\phi_2^\vee\mathcal{O}_M)^\perp \oplus \phi_1^\vee\mathcal{O}_M.$$

Taking quotient by $\phi_1^\vee\mathcal{O}_M \oplus \phi_2^\vee\mathcal{O}_M$, we obtain

$$(\phi_1^\vee\mathcal{O}_M)^\perp/\phi_1^\vee\mathcal{O}_M \cong (\phi_2^\vee\mathcal{O}_M)^\perp/\phi_2^\vee\mathcal{O}_M,$$

whose half Euler classes are the same. Therefore we know the reduced half Euler class is independent of the choice of surjective isotropic cosection. The last statement when $E = \mathcal{O}^{\oplus 2} \oplus V$ such that $Q = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \oplus Q|_V$ follows from this. □

Recall a $\mathrm{Sp}(2r,\mathbb{C})$-bundle (or symplectic vector bundle) is a complex vector bundle of rank $2r$ with a non-degenerate anti-symmetric bilinear form. One class of quadratic vector bundles is given by tensor product of two symplectic vector bundles V_1, V_2. Their half Euler classes can be computed using Chern classes of V_1, V_2. For our purpose, we restrict to the following case.

Lemma 5.6. *Let (V_1,ω_1), (V_2,ω_2) be a $\mathrm{Sp}(2r,\mathbb{C})$ (resp. $\mathrm{Sp}(2,\mathbb{C})$-bundle) on a connected scheme M. Then*

$$(V_1 \otimes V_2, \omega_1 \otimes \omega_2)$$

defines a $SO(4r,\mathbb{C})$-bundle whose half Euler class satisfies

$$e^{\frac{1}{2}}(V_1 \otimes V_2, \omega_1 \otimes \omega_2) = \pm\big(e(V_1) - c_{2r-2}(V_1)\cdot e(V_2)\big).$$

Proof. Consider the universal $\mathrm{Sp}(2r,\mathbb{C})$-bundle $\mathcal{V}_1$ (resp. $\mathrm{Sp}(2,\mathbb{C})$-bundle $\mathcal{V}_2$) on the classifying space $\mathrm{BSp}(2r,\mathbb{C})$ (resp. $\mathrm{BSp}(2,\mathbb{C})$). Their tensor product gives a $\mathrm{SO}(4r,\mathbb{C})$-bundle on $\mathrm{BSp}(2r,\mathbb{C}) \times \mathrm{BSp}(2,\mathbb{C})$, whose half Euler class is denoted by $e^{\frac{1}{2}}(\mathcal{V}_1 \otimes \mathcal{V}_2)$.

By the property of half Euler class (e.g. [EG, Prop. 2]):

$$e^{\frac{1}{2}}(\mathcal{V}_1 \otimes \mathcal{V}_2)^2 = e(\mathcal{V}_1 \otimes \mathcal{V}_2) = c_{2r}(\mathcal{V}_1)^2 - 2c_{2r}(\mathcal{V}_1)c_{2r-2}(\mathcal{V}_1)c_2(\mathcal{V}_2),$$

where we use the fact that the odd Chern classes of $\mathcal{V}_i$ vanish in the second equality. Note that above expression is the same as the square of $c_{2r}(\mathcal{V}_1) - c_{2r-2}(\mathcal{V}_1)c_2(\mathcal{V}_2)$. Since $H^*(\mathrm{BSp}(2r,\mathbb{C}) \times \mathrm{BSp}(2,\mathbb{C}))$ is the tensor product of two polynomial rings (e.g. [Sw, Thm. 16.10]), hence it is an integral domain. Therefore

$$e^{\frac{1}{2}}(\mathcal{V}_1 \otimes \mathcal{V}_2) = \pm\,(c_{2r}(\mathcal{V}_1) - c_{2r-2}(\mathcal{V}_1)c_2(\mathcal{V}_2)).$$

Since this construction is universal, we are done. □

Finally, we can determine the (reduced) virtual class of M_β.

Theorem 5.7. *For certain choice of orientation, we have*

$$[M_\beta]^{\mathrm{vir}} = e(M_{S,\beta})\cdot[T] - e(T)\cdot c_{\beta^2}(M_{S,\beta}). \tag{5.10}$$

Proof. Under the isomorphism (5.8):

$$M_\beta \cong M_{S,\beta} \times T,$$

a universal family $\mathbb{F}$ of M_β satisfies

$$\mathbb{F} = \mathbb{F}_S \boxtimes \mathcal{O}_{\Delta_T}, \tag{5.11}$$

where $\mathbb{F}_S$ is a universal sheaf of $M_{S,\beta}$ and Δ_T denotes the diagonal in $T \times T$.

Then the obstruction sheaf of M_β:

$$\mathcal{E}xt^2_{\pi_{M_\beta}}(\mathbb{F},\mathbb{F}) \cong \mathcal{E}xt^2_{\pi_{M_{S,\beta}}}(\mathbb{F}_S,\mathbb{F}_S) \oplus \mathcal{E}xt^1_{\pi_{M_{S,\beta}}}(\mathbb{F}_S,\mathbb{F}_S)\boxtimes T_T \oplus \mathcal{E}xt^0_{\pi_{M_{S,\beta}}}(\mathbb{F}_S,\mathbb{F}_S)\boxtimes \wedge^2 T_T$$

is a vector bundle with two trivial subbundles $\mathcal{E}xt^2_{\pi_{M_{S,\beta}}}(\mathbb{F}_S,\mathbb{F}_S)$, and $\mathcal{E}xt^0_{\pi_{M_{S,\beta}}}(\mathbb{F}_S,\mathbb{F}_S)\boxtimes \wedge^2 T_T$. By Lemmata 5.5, 5.6, we are done. □

5.4. DT_4 **invariants and proof of conjectures.** In this section, we determine all DT_4 invariants of $S \times T$. Let $\gamma, \gamma' \in H^4(X)$ be cohomology classes and decompose them as

$$\gamma = A_1 \cdot 1 \otimes \mathsf{p} + D_1 \otimes D_2 + A_2 \cdot \mathsf{p} \otimes 1,$$
$$\gamma' = A_1' \cdot 1 \otimes \mathsf{p} + D_1' \otimes D_2' + A_2' \cdot \mathsf{p} \otimes 1,$$

according to the Künneth decomposition:

$$H^4(X) \cong (H^0(S) \otimes H^4(T)) \oplus (H^2(S) \otimes H^2(T)) \oplus (H^4(S) \otimes H^0(T)).$$

Fix also a divisor class

$$\theta = \theta_1 + \theta_2 \in H^2(X) \cong H^2(S) \oplus H^2(T),$$

and a curve class

$$\alpha = \theta_1 \otimes \mathsf{p} + \mathsf{p} \otimes \theta_2 \in H^6(X) \cong (H^2(S) \otimes H^4(T)) \oplus (H^4(S) \otimes H^2(T)).$$

Theorem 5.8. *Let $\beta \in H_2(S, \mathbb{Z}) \subseteq H_2(X, \mathbb{Z})$ be any effective curve class. With respect to the choice of orientation* (5.10)*, we have*

(i) $$\big\langle \tau_0(\alpha) \big\rangle_\beta^{\mathrm{DT}_4} = (\theta_1 \cdot \beta)\, N_0\left(\frac{\beta^2}{2}\right),$$

(ii) $$\big\langle \tau_1(\gamma) \big\rangle_\beta^{\mathrm{DT}_4} = A_1\, N_0\left(\frac{\beta^2}{2}\right) - A_2\, e(T)\, N_1\left(\frac{\beta^2}{2}\right),$$

(iii) $$\big\langle \tau_2(\theta) \big\rangle_\beta^{\mathrm{DT}_4} = (\theta_1 \cdot \beta)\, N''\left(\frac{\beta^2}{2}\right),$$

(iv) $$\big\langle \tau_3(1) \big\rangle_\beta^{\mathrm{DT}_4} = N''\left(\frac{\beta^2}{2}\right),$$

(v) $$\big\langle \tau_0(\gamma), \tau_0(\gamma') \big\rangle_\beta^{\mathrm{DT}_4} = (D_1 \cdot \beta) \cdot (D_1' \cdot \beta) \cdot \left(\int_T D_2 \cup D_2'\right) \cdot N_0\left(\frac{\beta^2}{2}\right),$$

(vi) $$\big\langle \tau_0(\gamma), \tau_1(\theta) \big\rangle_\beta^{\mathrm{DT}_4} = (D_1 \cdot \beta) \left(\int_T D_2 \cup \theta_2\right) N_0\left(\frac{\beta^2}{2}\right) - 24 A_2 (\theta_1 \cdot \beta) N_1\left(\frac{\beta^2}{2}\right),$$

(vii) $$\big\langle \tau_0(\gamma), \tau_2(1) \big\rangle_\beta^{\mathrm{DT}_4} = 0,$$

(viii) $$\big\langle \tau_1(\theta), \tau_2(1) \big\rangle_\beta^{\mathrm{DT}_4} = 48 (\theta_1 \cdot \beta) N_1\left(\frac{\beta^2}{2}\right),$$

where $N_0(l)$, $N_1(l)$ are defined in Eqns. (5.4), (4.9) *respectively and*

$$\sum_{l \in \mathbb{Z}} N''(l)\, q^l = \frac{1}{q} \prod_{n \geqslant 1} (1 - q^n)^{-24} \left(24 q \frac{d}{dq} G_2(q) + 24 G_2(q) - 1\right)$$
$$= -2q^{-1} + 720 q + 14720 q^2 + 182340 q^3 + 1715328 q^4 + \cdots .$$

The above theorem immediately implies the following:

Corollary 5.9. *Conjecture 2.2 holds for the product $X = S \times T$ and $\beta \in H_2(S, \mathbb{Z}) \subseteq H_2(X, \mathbb{Z})$.*

Proof. This follows by inspection using Theorem 5.8 on DT_4 invariants and Proposition 5.1 and Lemma 5.2 for the GV/GW invariants respectively. □

Another remarkable consequence of Theorem 5.8 is that all DT_4 invariants of $S \times T$ depend upon the curve class β only via the square β^2 and not the divisibility. More precisely, given pairs (S, β) and (S', β') of a $K3$ surface and an effective curve class such that $\beta^2 = \beta'^2$, let

$$\varphi : H^2(S, \mathbb{R}) \to H^2(S', \mathbb{R})$$

be any *real* isometry such that $\varphi(\beta) = \beta'$. Extend φ to the full cohomology by setting $\varphi(1) = 1$ and $\varphi(\mathsf{p}_S) = \mathsf{p}_{S'}$ where $\mathsf{p}_S \in H^4(S, \mathbb{Z})$ is the point class.

Corollary 5.10. *With respect to the choice of orientation* (5.10)*, we have*

$$\big\langle \tau_{k_1}(\gamma_1), \ldots, \tau_{k_n}(\gamma_n) \big\rangle_\beta^{\mathrm{DT}_4, S \times T} = \big\langle \tau_{k_1}((\varphi \otimes id)\gamma_1), \ldots, \tau_{k_n}((\varphi \otimes id)\gamma_n) \big\rangle_{\beta'}^{\mathrm{DT}_4, S' \times T}$$

for any $\gamma_i \in H^*(X)$ and $k_i \geqslant 0$.

This raises the question whether a similar independence of the divisibility holds for Donaldson-Thomas invariants of holomorphic symplectic 4-folds more generally.

5.5. **Proof of Theorem 5.8.** We split the proof in two parts.

Proof of Theorem 5.8 Part (i), (v). We begin with part (v). By Eqn. (5.11), the primary insertion becomes

$$\tau_0(\gamma) = (D_1 \cdot \beta) \otimes D_2 + A_2\, \pi_{M_{S,\beta}*}(\pi_S^* \mathsf{p} \cdot \mathrm{ch}_1(\mathbb{F}_S)) \otimes 1,$$

where π_S, $\pi_{M_{S,\beta}}$ are projections to each factor of $S \times M_{S,\beta}$. Therefore

$$\tau_0(\gamma) \cdot \tau_0(\gamma') = (D_1 \cdot \beta) \cdot (D_1' \cdot \beta) \otimes (D_2 \cdot D_2') + A_2 A_2' \left(\pi_{M_{S,\beta}*}\left(\pi_S^* \mathsf{p} \cdot \mathrm{ch}_1(\mathbb{F}_S)\right)\right)^2 \otimes 1 + \text{others},$$

where "others" lie in $H^2(M_{S,\beta}) \otimes H^2(T)$. Combining with Theorem 5.7, we get

$$\begin{aligned} \left\langle \tau_0(\gamma), \tau_0(\gamma') \right\rangle_\beta^{\mathrm{DT}_4} =& (D_1 \cdot \beta) \cdot (D_1' \cdot \beta) \cdot e(M_{S,\beta}) \int_T (D_2 \cdot D_2') \\ & - A_2\, A_2'\, e(T) \int_{M_{S,\beta}} \left(\pi_{M_{S,\beta}*}\left(\pi_S^* \mathsf{p} \cdot \mathrm{ch}_1(\mathbb{F}_S)\right)\right)^2 \cdot c_{\beta^2}(M_{S,\beta}). \end{aligned} \tag{5.12}$$

There exists a Hilbert-Chow map

$$\mathrm{HC} : M_{S,\beta} \to |\beta| = \mathbb{P}^{\frac{1}{2}\beta^2+1},$$

to the linear system $|\beta|$ and $\mathrm{ch}_1(\mathbb{F}_S) = (\mathrm{id}_S \times \mathrm{HC})^*[\mathcal{C}]$, where $\mathcal{C}$ is the universal curve of the linear system:

$$\begin{array}{ccc} \mathcal{C} \hookrightarrow S \times |\beta| & \xrightarrow{\ q\ } & |\beta| \\ \big\downarrow{\scriptstyle p} & & \\ S. & & \end{array}$$

Since $[\mathcal{C}] = p^*\beta + q^*h$ for the hyperplane class h of $|\beta|$, we have

$$\begin{aligned} \pi_{M_{S,\beta}*}\left(\pi_S^* \mathsf{p} \cdot \mathrm{ch}_1(\mathbb{F}_S)\right) &= \pi_{M_{S,\beta}*}\left(\pi_S^* \mathsf{p} \cdot (\mathrm{id}_S \times \mathrm{HC})^*[\mathcal{C}]\right) \\ &= \mathrm{HC}^* q_*([\mathcal{C}] \cdot p^* \mathsf{p}) \\ &= \mathrm{HC}^*(h). \end{aligned}$$

By Theorem 4.1, we have

$$\begin{aligned} &\int_{M_{S,\beta}} \left(\pi_{M_{S,\beta}*}\left(\pi_S^* \mathsf{p} \cdot \mathrm{ch}_1(\mathbb{F}_S)\right)\right)^2 \cdot c_{\beta^2}(M_{S,\beta}) \\ &= C(c_{\beta^2}(M_{S,\beta})) \cdot \mathsf{q}(\mathrm{HC}^*(h)) \\ &= C(c_{\beta^2}(M_{S,\beta})) \left(C(1)^{-1} \cdot \int_M \left(\mathrm{HC}^*(h^{\beta^2+2})\right)\right)^{\frac{1}{\beta^2+2}} \\ &= 0. \end{aligned}$$

Therefore Eqn. (5.12) becomes

$$\left\langle \tau_0(\gamma), \tau_0(\gamma') \right\rangle_\beta^{\mathrm{DT}_4} = (D_1 \cdot \beta) \cdot (D_1' \cdot \beta) \cdot (D_2 \cdot D_2') \cdot e(M_{S,\beta}).$$

Finally, since $M_{S,\beta}$ is deformation equivalent to $S^{[d]}$ ($\beta^2 = 2d - 2$) (e.g. [Y09, Cor. 3.5, pp. 136]), they have the same Euler numbers:

$$e(M_{S,\beta}) = e(S^{[d]}),$$

which is given by $N_0(\beta^2/2)$ due to Göttsche [G90]. This proves (v).

For (i), we similarly have

$$\begin{aligned} \tau_0(\alpha) &= \pi_{M_{S,\beta}*}(\pi_S^*\theta_1 \cdot \mathrm{ch}_1(\mathbb{F}_S)) \otimes \mathsf{p} + \pi_{M_{S,\beta}*}(\pi_S^* \mathsf{p} \cdot \mathrm{ch}_1(\mathbb{F}_S)) \otimes \theta_2, \\ \left\langle \tau_0(\alpha) \right\rangle_\beta^{\mathrm{DT}_4} &= (\theta_1 \cdot \beta) \cdot e(M_{S,\beta}) = (\theta_1 \cdot \beta) \cdot N_0(\beta^2/2). \end{aligned}$$

□

Proof of Theorem 5.8 Parts (ii-iv) and (vi-viii). We first express the DT_4 descendent invariants as integrals on $M_{S,\beta}$. Let $\mathbb{F}_S^{\mathrm{norm}}$ be the normalized universal sheaf on $M_{S,\beta} \times S$, i.e.

$$\det(\pi_{M_{S,\beta}*}\mathbb{F}_S^{\mathrm{norm}}) \cong \mathcal{O}_{M_{S,\beta}},$$

where $\pi_{M_{S,\beta}} : M_{S,\beta} \times S \to M_{S,\beta}$ is the projection. By Eqn. (5.11), we have

$$\mathbb{F} = \mathbb{F}_S \boxtimes \mathcal{O}_{\Delta_T}.$$

Hence the family $\mathbb{F}$ is normalized if and only if $\mathbb{F}_S$ is so. Moreover, for the diagonal embedding $\Delta : T \to T \times T$, by GRR, we have

$$\mathrm{ch}(\mathcal{O}_\Delta) = \Delta_*(1 - 2\mathsf{p}).$$

We obtain

$$\tau_1(\gamma) = A_1\, \pi_{M_{S,\beta}*}(\mathrm{ch}_2(\mathbb{F}_S^{\mathrm{norm}})) \otimes \mathsf{p} + A_2\, \pi_{M_{S,\beta}*}(\pi_S^*\mathsf{p} \cdot \mathrm{ch}_2(\mathbb{F}_S^{\mathrm{norm}})) \otimes 1 \\ + \pi_{M_{S,\beta}*}(\pi_S^*(D_1) \cdot \mathrm{ch}_2(\mathbb{F}_S^{\mathrm{norm}})) \otimes D_2.$$

By base change to a point, we have

$$\pi_{M_{S,\beta}*}(\mathrm{ch}_2(\mathbb{F}_S^{\mathrm{norm}})) = 1.$$

Combining with Theorem 5.7, we obtain that

$$\big\langle \tau_1(\gamma) \big\rangle_\beta^{\mathrm{DT}_4} = A_1\, e(M_{S,\beta}) - A_2\, e(T) \int_{M_{S,\beta}} c_{\beta^2}(M_{S,\beta}) \cdot \pi_{M_{S,\beta}*}\left(\mathrm{ch}_2(\mathbb{F}_S^{\mathrm{norm}})\, \pi_S^*\mathsf{p}\right).$$

Part (ii) now follows from Proposition 4.7(i).

Similarly, for (iii) we have

$$\mathrm{ch}_5(\mathbb{F}_{\mathrm{norm}}) = \mathrm{ch}_3(\mathbb{F}_S^{\mathrm{norm}}) \cdot \Delta_* 1 - 2\, \mathrm{ch}_1(\mathbb{F}_S^{\mathrm{norm}}) \cdot \Delta_*(\mathsf{p}).$$

Hence

$$\begin{aligned} \tau_2(\theta) &= \pi_{M*}(\mathrm{ch}_5(\mathbb{F}_{\mathrm{norm}})\pi_X^*\theta) \\ &= \pi_{M_{S,\beta}*}(\mathrm{ch}_3(\mathbb{F}_S^{\mathrm{norm}})\pi_S^*\theta_1) + \pi_{M_{S,\beta}*}(\mathrm{ch}_3(\mathbb{F}_S^{\mathrm{norm}})) \boxtimes \theta_2 - 2\pi_{M_{S,\beta}*}(\mathrm{ch}_1(\mathbb{F}_S^{\mathrm{norm}})\pi_S^*\theta_1) \boxtimes \mathsf{p} \\ &= \pi_{M_{S,\beta}*}(\mathrm{ch}_3(\mathbb{F}_S^{\mathrm{norm}})\pi_S^*\theta_1) + \pi_{M_{S,\beta}*}(\mathrm{ch}_3(\mathbb{F}_S^{\mathrm{norm}})) \boxtimes \theta_2 - 2(\theta_1 \cdot \beta) \boxtimes \mathsf{p}, \end{aligned}$$

where the last equality is by base change to a point. Using Theorem 5.7, we obtain

$$\big\langle \tau_2(\theta) \big\rangle_\beta^{\mathrm{DT}_4} = -2(\theta_1 \cdot \beta)\, e(M_{S,\beta}) - 24 \int_{M_{S,\beta}} c_{\beta^2}(M_{S,\beta}) \cdot \pi_{M_{S,\beta}*}\left(\mathrm{ch}_3(\mathbb{F}_S^{\mathrm{norm}})\, \pi_S^*\theta_1\right).$$

Thus with Proposition 4.7, we obtain that

$$\big\langle \tau_2(\theta) \big\rangle_\beta^{\mathrm{DT}_4} = (\theta_1 \cdot \beta) \left(-2N_0 \left(\frac{\beta^2}{2} \right) + 24N' \left(\frac{\beta^2}{2} \right) \right).$$

For part (iv), one similarly establishes:

$$\begin{aligned} \big\langle \tau_3(1) \big\rangle_\beta^{\mathrm{DT}_4} &= -2\, e(M_{S,\beta}) - 24 \int_{M_{S,\beta}} c_{\beta^2}(M_{S,\beta}) \cdot \pi_{M_{S,\beta}*}(\mathrm{ch}_4(\mathbb{F}_S^{\mathrm{norm}})) \\ &= -2N_0 \left(\frac{\beta^2}{2} \right) + 24N' \left(\frac{\beta^2}{2} \right), \end{aligned}$$

For (vi), we compute using Lemma 4.10 that

$$\begin{aligned} \big\langle \tau_0(\gamma), \tau_1(\theta) \big\rangle_\beta^{\mathrm{DT}_4} &= (D_1 \cdot \beta)(D_2 \cdot \theta_2)\, e(M_{S,\beta}) \\ &\quad - 24A_2 \int_{M_{S,\beta}} c_{\beta^2}(M_{S,\beta}) \cdot \pi_{M_{S,\beta}*}\left(\mathrm{ch}_1(\mathbb{F}_S^{\mathrm{norm}})\, \pi_S^*\mathsf{p}\right) \cdot \pi_{M_{S,\beta}*}\left(\mathrm{ch}_2(\mathbb{F}_S^{\mathrm{norm}})\, \pi_S^*\theta_1\right) \\ &= (D_1 \cdot \beta)(D_2 \cdot \theta_2)\, e(M_{S,\beta}) - 24A_2(\theta_1 \cdot \beta)\, C(c_{\beta^2}(T_{M_{S,\beta}})). \end{aligned}$$

Since $M_{S,\beta}$ and $S^{[d]}$ are deformation equivalent they share the same Fujiki constants:

$$C(c_{\beta^2}(T_{M_{S,\beta}})) = C(c_{2d-2}(T_{S^{[d]}})) = N_1(\beta^2/2),$$

where $\beta^2 = 2d-2$. This implies the claim. Finally for (vii) and (viii), we similarly find:

$$\begin{aligned} \big\langle \tau_0(\gamma), \tau_2(1) \big\rangle_\beta^{\mathrm{DT}_4} &= -24A_2 \int_{M_{S,\beta}} c_{\beta^2}(M_{S,\beta}) \cdot \pi_{M_{S,\beta}*}\left(\mathrm{ch}_1(\mathbb{F}_S^{\mathrm{norm}})\, \pi_S^*\mathsf{p}\right) \cdot \pi_{M_{S,\beta}*}\left(\mathrm{ch}_3(\mathbb{F}_S^{\mathrm{norm}})\right) \\ &= 0, \\ \big\langle \tau_1(\theta), \tau_2(1) \big\rangle_\beta^{\mathrm{DT}_4} &= -24 \int_{M_{S,\beta}} c_{\beta^2}(M_{S,\beta}) \cdot \pi_{M_{S,\beta}*}\left(\mathrm{ch}_2(\mathbb{F}_S^{\mathrm{norm}})\, \pi_S^*\theta_1\right) \cdot \pi_{M_{S,\beta}*}\left(\mathrm{ch}_3(\mathbb{F}_S^{\mathrm{norm}})\right) \\ &= -24 \cdot (-2) \cdot (\theta_1 \cdot \beta)\, C(c_{\beta^2}(T_{M_{S,\beta}})) \\ &= 48(\theta_1 \cdot \beta) N_1(\beta^2/2). \end{aligned}$$

□

6. Cotangent bundle of $\mathbb{P}^2$

We consider the geometry $X = T^*\mathbb{P}^2$. There is a natural identification of curve classes:

$$H_2(X, \mathbb{Z}) = H_2(\mathbb{P}^2, \mathbb{Z}) = \mathbb{Z}[\ell],$$

where $\ell \subset \mathbb{P}^2$ is a line.

6.1. **GW and GV invariants.** Let $H \in H^2(T^*\mathbb{P}^2)$ be the pullback of hyperplane class. We identify $H_2(T^*\mathbb{P}^2, \mathbb{Z}) \equiv \mathbb{Z}$ by its degree against H.

Proposition 6.1.

$$\big\langle \tau_0(H^2), \tau_0(H^2) \big\rangle_{0,d}^{\mathrm{GW}} = \frac{(-1)^{d-1}}{d}, \quad \big\langle \tau_0(H^2) \big\rangle_{1,d}^{\mathrm{GW}} = \frac{(-1)^{d-1}}{8} d, \quad \big\langle \varnothing \big\rangle_{2,d}^{\mathrm{GW}} = \frac{(-1)^{d-1}}{128} d^3.$$

Proof. This follows by a direct calculation using Graber-Pandharipande virtual localization formula [GP]. We refer to [PZ, §3] for a computation with parallel features. □

Based on Definition 1.5, 1.6, 1.7, we then obtain the following:

Corollary 6.2.

$$n_{0,d}(H^2, H^2) = \begin{cases} 1 & \text{if } d = 1, \\ -1 & \text{if } d = 2, \\ 0 & \text{otherwise.} \end{cases}$$

$$n_{1,1}(H^2) = 0, \quad n_{2,1} = 0.$$

In particular, Conjecture 1.9 holds for $T^*\mathbb{P}^2$.

Proof. In genus 0 and 1, this follows from a direct calculation using the definition and that

$$c_2(T_X)|_{\mathbb{P}^2} = -3H^2.$$

In genus 2, it remains to determine the nodal invariant $N_{\mathrm{nodal},\beta}$. In $H^*(\mathbb{P}^2 \times \mathbb{P}^2)$, we have

$$\Delta_X|_{\mathbb{P}^2 \times \mathbb{P}^2} = \Delta_{\mathbb{P}^2} \cdot \mathrm{pr}_1^*(c_2(\Omega_{\mathbb{P}^2})) = 3\mathrm{pr}_1^*(H^2)\mathrm{pr}_2^*(H^2).$$

Using Lemma 1.1 and Eqn. (1.2) we find that

$$\begin{aligned} N_{\mathrm{nodal},1} &= \frac{1}{2}\left[3 - \left(\big\langle \tau_1(c_2(T_X)) \big\rangle_{0,1}^{\mathrm{GW}} + \big\langle \tau_3(1) \big\rangle_{0,1}^{\mathrm{GW}}\right)\right] \\ &= \frac{1}{2}\left[3 - (-3+6)\right] = 0. \end{aligned}$$

The vanishing $n_{2,1} = 0$ follows now from a direct calculation. □

6.2. DT_4 **invariants.** Let $M_{T^*\mathbb{P}^2,d}$ (resp. $M_{\mathbb{P}^2,d}$) be the moduli scheme of compactly supported one dimensional stable sheaves F on $T^*\mathbb{P}^2$ (resp. $\mathbb{P}^2$) with $[F] = d[\ell]$ ($d \geqslant 1$) and $\chi(F) = 1$.

Lemma 6.3. *Let $\iota : \mathbb{P}^2 \to T^*\mathbb{P}^2$ be the zero section. Then the pushforward map*

$$\iota_* : M_{\mathbb{P}^2,d} \to M_{T^*\mathbb{P}^2,d} \tag{6.1}$$

is an isomorphism.

Proof. The map ι_* is obviously injective. We show that ι_* is also surjective. As $T^*\mathbb{P}^2$ admits a birational contraction $T^*\mathbb{P}^2 \to Y$ which contracts the zero section $\mathbb{P}^2 \hookrightarrow T^*\mathbb{P}^2$ to $0 \in Y$ and Y is affine, any one dimensional sheaf on $T^*\mathbb{P}^2$ is set theoretically supported on the zero section. It is enough to show that any one dimensional stable sheaf F on $T^*\mathbb{P}^2$ is scheme theoretically supported on the zero section.

Recall the following fact as stated in [CMT18, Lem. 2.2]: let $g\colon Z \to T$ be a morphism of $\mathbb{C}$-schemes, and take a closed point $t \in Z$. Let $Z_t \subset Z$ be the scheme theoretic fiber of g at t. Suppose that $F \in \mathrm{Coh}(Z)$ is set theoretically supported on Z_t and satisfies $\mathrm{End}(F) = \mathbb{C}$. Then F is scheme theoretically supported on Z_t.

It should be well-known (and easy) that the scheme theoretic fiber of $T^*\mathbb{P}^2 \to Y$ at $0 \in Y$ is the reduced zero section, then surjectivity of ι_* follows from the above fact. As we cannot find its reference, we give another argument here. Consider the closed embedding $T^*\mathbb{P}^2 \subset \mathcal{O}_{\mathbb{P}^2}(-1)^{\oplus 3}$ induced by the Euler sequence on $\mathbb{P}^2$. Note that $\mathcal{O}_{\mathbb{P}^2}(-1)^{\oplus 3}$ is an open subscheme of $[\mathbb{C}^6/\mathbb{C}^*]$, where $\mathbb{C}^*$ on $\mathbb{C}^6$ by

$$t(x_1, x_2, x_3, y_1, y_2, y_3) = (tx_1, tx_2, tx_3, t^{-1}y_1, t^{-1}y_2, t^{-1}y_3),$$

and corresponds to $(x_1, x_2, x_3) \neq (0,0,0)$. The stack $[\mathbb{C}^6/\mathbb{C}^*]$ admits a good moduli space

$$[\mathbb{C}^6/\mathbb{C}^*] \to T := \mathrm{Spec}\, \mathbb{C}[x_i, y_i]^{\mathbb{C}^*} = \mathrm{Spec}\, \mathbb{C}[x_i y_j : 1 \leqslant i, j \leqslant 3].$$

One can easily calculates that the scheme theoretic fiber of the above morphism restricted to $(x_1, x_2, x_3) \neq (0,0,0)$ is $(y_1 = y_2 = y_3 = 0)$. It follows that the scheme theoretic fiber of $T^*\mathbb{P}^2 \subset \mathcal{O}_{\mathbb{P}^2}(-1)^{\oplus 3} \to T$ at $0 \in T$ is the reduced zero section $\mathbb{P}^2$. As T is affine, any one dimensional stable sheaf is set theoretically supported on the (scheme theoretic) fiber of $0 \in T$. Using the above fact, it is also scheme theoretically supported on it. Therefore ι_* is surjective.

Since $M_{\mathbb{P}^2,d}$ is smooth and ι_* is bijective on closed points, it remains to show that ι_* induces an isomorphisms on tangent spaces. For a one dimensional stable sheaf F on $\mathbb{P}^2$, the tangent space of $M_{T^*\mathbb{P}^2,d}$ at $\iota_* F$ is

$$\mathrm{Ext}^1_{T^*\mathbb{P}^2}(\iota_*F, \iota_*F) \cong \mathrm{Ext}^1_{\mathbb{P}^2}(F,F) \oplus \mathrm{Hom}(F, F \otimes T^*\mathbb{P}^2).$$

By the Euler sequence and stability, we have

$$\mathrm{Hom}(F, F\otimes T^*\mathbb{P}^2) \subset \mathrm{Hom}(F, F \otimes \mathcal{O}_{\mathbb{P}^2}(-1)^{\oplus 3}) = 0.$$

Therefore ι_* induce an isomorphism of tangent spaces. □

Then the following result is straightforward.

Lemma 6.4. *Under the isomorphism* (6.1)*, we have*

$$[M_{T^*\mathbb{P}^2,d}]^{\mathrm{vir}} = \mathrm{PD}\left(e^{\frac{1}{2}}_{\mathrm{red}}\left(\mathcal{E}xt^1_{\pi_M}(\mathbb{F}, \mathbb{F}\boxtimes T^*\mathbb{P}^2), Q\right)\right) \in H_4(M_{\mathbb{P}^2,d}, \mathbb{Z}).$$

Here PD *denotes the Poincaré dual,* $e^{\frac{1}{2}}_{\mathrm{red}}$ *is the reduced half Euler class as in Definition 5.4,* $\mathbb{F}$ *denotes a universal sheaf of* $M_{\mathbb{P}^2,d}$ *and* $\pi_M : M_{\mathbb{P}^2,d} \times \mathbb{P}^2 \to M_{\mathbb{P}^2,d}$ *is the projection.*

Proposition 6.5. *For certain choice of orientation, we have*

$$\langle\tau_0(H^2), \tau_0(H^2)\rangle^{\mathrm{DT}_4}_{[\ell]} = 1, \quad \langle\tau_0(H^2), \tau_0(H^2)\rangle^{\mathrm{DT}_4}_{2[\ell]} = -1, \quad \langle\tau_0(H^2), \tau_0(H^2)\rangle^{\mathrm{DT}_4}_{3[\ell]} = 0,$$

$$\langle\tau_1(H^2)\rangle^{\mathrm{DT}_4}_{[\ell]} = -\frac{1}{2}, \quad \langle\tau_1(H^2)\rangle^{\mathrm{DT}_4}_{2[\ell]} = \frac{1}{2}, \quad \langle\tau_1(H^2)\rangle^{\mathrm{DT}_4}_{3[\ell]} = 0,$$

$$\langle\tau_2(H)\rangle^{\mathrm{DT}_4}_{[\ell]} = -\frac{1}{4}, \quad \langle\tau_2(H)\rangle^{\mathrm{DT}_4}_{2[\ell]} = -\frac{1}{4}, \quad \langle\tau_2(H)\rangle^{\mathrm{DT}_4}_{3[\ell]} = 0,$$

$$\langle\tau_3(1)\rangle^{\mathrm{DT}_4}_{[\ell]} = -\frac{1}{8}, \quad \langle\tau_3(1)\rangle^{\mathrm{DT}_4}_{2[\ell]} = \frac{1}{8}, \quad \langle\tau_3(1)\rangle^{\mathrm{DT}_4}_{3[\ell]} = 0.$$

In particular, for $X = T^*\mathbb{P}^2$*, we have*

- *Conjecture 2.2* (i) *holds when* $d \leqslant 3$.
- *Conjecture 2.2* (ii), (iii) *hold.*

Proof. We present the proof of $d = 2$ case (the $d = 1$ case follows similarly). The support map

$$M_{\mathbb{P}^2,2} \xrightarrow{\cong} |\mathcal{O}_{\mathbb{P}^2}(2)| \cong \mathbb{P}^5, \quad F \mapsto \mathrm{supp}(F)$$

is an isomorphism. The normalized universal sheaf satisfies $\mathbb{F}_{\mathrm{norm}} = \mathcal{O}_{\mathcal{C}}$ for the universal $(1,2)$-divisor $\mathcal{C} \hookrightarrow \mathbb{P}^5 \times \mathbb{P}^2$. Let $\pi_M : M_{\mathbb{P}^2,2} \times \mathbb{P}^2 \to M_{\mathbb{P}^2,2}$ be the projection. Bott's formula implies

$$\mathbf{R}\mathcal{H}om_{\pi_M}(\mathcal{O}, \mathcal{O}(-\mathcal{C}) \boxtimes T^*\mathbb{P}^2) \cong \mathcal{O}_{\mathbb{P}^5}(-1)[-2]^{\oplus 3},$$
$$\mathbf{R}\mathcal{H}om_{\pi_M}(\mathcal{O}, \mathcal{O}(\mathcal{C}) \boxtimes T^*\mathbb{P}^2) \cong \mathcal{O}_{\mathbb{P}^5}(-1)^{\oplus 3},$$
$$\mathbf{R}\mathcal{H}om_{\pi_M}(\mathcal{O}, \mathcal{O} \boxtimes T^*\mathbb{P}^2) \cong \mathcal{O}_{\mathbb{P}^5}[-1].$$

Therefore, we have

$$\begin{aligned}
&\mathbf{R}\mathcal{H}om_{\pi_M}(\mathcal{O}_{\mathcal{C}}, \mathcal{O}_{\mathcal{C}} \boxtimes T^*\mathbb{P}^2)[1] \\
&\cong \mathbf{R}\mathcal{H}om_{\pi_M}(\mathcal{O}(-\mathcal{C}) \to \mathcal{O}, (\mathcal{O}(-\mathcal{C}) \to \mathcal{O}) \boxtimes T^*\mathbb{P}^2)[1] \\
&\cong \mathcal{O}_{\mathbb{P}^5}(-1)^{\oplus 3} \oplus \mathcal{O}_{\mathbb{P}^5}(1)^{\oplus 3} \oplus \mathcal{O}_{\mathbb{P}^5} \oplus \mathcal{O}_{\mathbb{P}^5}.
\end{aligned}$$

By Grothendieck-Verdier duality, it is easy to see

$$\mathcal{O}_{\mathbb{P}^5}(-1)^{\oplus 3} \oplus \mathcal{O}_{\mathbb{P}^5}$$

is a maximal isotropic subbundle of $\mathbf{R}\mathcal{H}om_{\pi_M}(\mathcal{O}_{\mathcal{C}}, \mathcal{O}_{\mathcal{C}} \boxtimes T^*\mathbb{P}^2)[1]$. Hence the reduced virtual class satisfies

$$[M_{T^*\mathbb{P}^2,2}]^{\mathrm{vir}} = \pm e(\mathcal{O}_{\mathbb{P}^5}(-1)^{\oplus 3}) \cap [\mathbb{P}^5] \in H_4(\mathbb{P}^5).$$

Let $h \in H^2(\mathbb{P}^5)$ denote the hyperplane class. It is straightforward to check

$$\tau_0(H^2) = [h], \ \tau_1(H^2) = -\frac{1}{2}h^2, \ \tau_2(H) = \frac{1}{4}h^2, \ \tau_3(1) = -\frac{1}{8}h^2.$$

By integration again the virtual class, we have the desired result for $d = 2$ case.

The $d = 3$ case can be computed by a torus localization as in [CKM19, CKM20]. One sees that for any torus fixed point, the reduced obstruction space has a trivial factor[12] which implies the vanishing of (reduced) invariants. □

[12] We thank Sergej Monavari for his observation and help on this.

7. Hilbert scheme of two points on a $K3$ surface

Let S be a $K3$ surface. There are three fundamental conjectures which govern the Gromov-Witten invariants of the Hilbert scheme of points $S^{[n]}$:

(i) Multiple cover conjecture (proposed in [O21a], and proven partially in [O21c]) which expresses Gromov-Witten invariants for imprimitive curve classes as an explicit linear combination of primitive invariants,
(ii) Quasi-Jacobi form property (proposed in [O18, O22b]),
(iii) Holomorphic anomaly equation (proposed in [O22b], see also [O21b] for a progress report).

For the Hilbert scheme of two points $S^{[2]}$ these conjectures have been established in genus 0 by [O18, O21c, O22b]. Together with [O18] they yield a complete evaluation of all genus 0 Gromov-Witten invariants of $S^{[2]}$, that is for all curve classes and all insertions. We consider here the case of genus 1 and genus 2 Gromov-Witten invariants of $S^{[2]}$ for primitive curve classes. The strategy is to assume both the quasi-Jacobi form property (ii) and the holomorphic anomaly equation (iii). Under this assumption, the natural generating series of genus 1 and 2 Gromov-Witten invariants are given in terms of Jacobi forms and are determined up to finitely many coefficients. Using our earlier computations in ideal geometries we are able to uniquely fix these finitely many coefficients. Modulo the above conjectures, this leads to a complete evaluation of Gopakumar-Vafa invariants for $S^{[2]}$ in all genera.

7.1. **Quasi-Jacobi forms.** To state the result we will work with quasi-Jacobi forms. We refer to [Lib, vIOP] for an introduction to quasi-Jacobi forms, and to [O18, App. B] for the variable conventions that we follow here. We work here entirely on the level of (q, y)-series. We need the following series:

$$E_k(q) = 1 - \frac{2k}{B_k} \sum_{n\geqslant 1} \sum_{d|n} d^{k-1} q^n, \qquad \Delta(q) = q \prod_{n\geqslant 1} (1-q^n)^{24},$$

$$\Theta(y,q) = (y^{1/2} + y^{-1/2}) \prod_{m\geqslant 1} \frac{(1+yq^m)(1+y^{-1}q^m)}{(1-q^m)^2},$$

$$\wp(y,q) = \frac{1}{12} - \frac{y}{(1+y)^2} + \sum_{d\geqslant 1} \sum_{m|d} m((-y)^m - 2 + (-y)^{-m}) q^d.$$

Sometimes it will also be convenient to use the following alternative convention of Eisenstein series:

$$G_k(q) = -\frac{B_k}{2\cdot k} E_k = -\frac{B_k}{2\cdot k} + \sum_{n\geqslant 1}\sum_{d|n} d^{k-1} q^n.$$

The algebra of quasi-Jacobi forms is then the subring of

$$\mathbb{C}\left[\Theta,\, \frac{1}{\Theta} y \frac{d}{dy}\Theta,\, G_2,\, G_4,\, \wp,\, y\frac{d}{dy}\wp\right]$$

consisting of all series which define holomorphic functions $\mathbb{C}\times\mathbb{H} \to \mathbb{C}$ in (z,τ) where $y = e^{2\pi i(z+1/2)}$ and $q = e^{2\pi i \tau}$. A key fact is that the generator $G_2(q)$ is algebraically independent in the algebra of quasi-Jacobi forms from the other generators. Hence for any quasi-Jacobi form $F(y,q)$ we can speak of its 'holomorphic anomaly', which is defined by $\frac{d}{dG_2} F(y,q)$, see [vIOP].

7.2. **Curve classes.** Since $X := S^{[2]}$ is irreducible hyperkähler, recall from Section 4.1 the integral, even, non-degenerate Beauville-Bogomolov-Fujiki form

$$\mathsf{q} : H^2(X,\mathbb{Z}) \to \mathbb{Z}.$$

Since q is non-degenerate, we obtain an inclusion of finite index

$$H^2(X,\mathbb{Z}) \hookrightarrow H^2(X,\mathbb{Z})^* \cong H_2(X,\mathbb{Z}), \quad D \mapsto (D,-),$$

where we write $(-,-)$ for the induced inner product on $H^2(X,\mathbb{Z})$. By extending q, we hence obtain a $\mathbb{Q}$-valued non-degenerate quadratic form

$$\mathsf{q} : H_2(X,\mathbb{Z}) \to \mathbb{Q}, \quad \beta \mapsto (\beta,\beta).$$

Given a class $\beta \in H_2(X,\mathbb{Z})$, we write

$$h_\beta = (\beta,-) \in (H_2(X,\mathbb{Q})^*) \cong H^2(X,\mathbb{Q})$$

for its dual with respect to the Beauville-Bogomolov-Fujiki form $(-,-)$. We have

$$(h_\beta, h_\beta) = (\beta,\beta).$$

Let also

$$c_{BB} \in H^2(X) \otimes H^2(X)$$

be the inverse of the Beauville-Bogomolov-Fujiki form, i.e. the image of $\mathsf{q} \in H^2(X)^* \otimes H^2(X)^*$ under the natural isomorphism $H^2(X,\mathbb{Q})^* \cong H^2(X,\mathbb{Q})$ induced by q.

We will also require the following definition:

Definition 7.1. *Let $F(y,q)$ be a quasi-Jacobi form of index 1 which satisfies the transformation law of Jacobi forms for the elliptic transformation $z \mapsto z+\tau$ (in generators this means it is independent of $\frac{1}{\Theta} y \frac{d}{dy}\Theta$; we will only encounter such kind here).*

For any class $\beta \in H_2(X,\mathbb{Z})$, the β-coefficient of $F(y,q)$,

$$F_\beta \in \mathbb{Q},$$

is defined to be the coefficient of $q^d y^k$ for any $d,k \in \mathbb{Z}$ such that $(\beta,\beta) = 2d - k^2/2$.

Remark 7.2. *The choice of d,k is not unique, but the coefficient F_β is independent of the choice by the elliptic transformation law of Jacobi forms* [EZ].

7.3. **Gromov-Witten invariants.** We first recall the genus 0 Gromov-Witten invariants of $S^{[2]}$ which are completely determined by the following two quasi-Jacobi forms:

$$F(y,q) := \frac{\Theta(y,q)^2}{\Delta(q)},$$
$$G(y,q) := \frac{\Theta(y,q)^2}{\Delta(q)}(-\wp(y,q) + \frac{1}{12}E_2(q)).$$

The first coefficients read:

$$\begin{aligned} F(y,q) &= \left(y^{-1}+2+y\right)q^{-1} + \left(2y^{-2}+32y^{-1}+60+32y+2y^2\right) \\ &\quad + (y^{-3}+60y^{-2}+555y^{-1}+992+555y+60y^2+1y^2)q+\cdots, \\ G(y,q) &= q^{-1} + (4y+30+4y^{-1}) + (30y^{-2}+120y^{-1}+504+120y+30y^3)q+\cdots. \end{aligned}$$

The following completely determines all primary Gromov-Witten invariants of $X = S^{[2]}$ in primitive curve classes (see [O21c] for the imprimitive case):

Theorem 7.3 ([O18, O21a])**.** *Let $\beta \in H_2(X,\mathbb{Z})$ be a primitive curve class. We have*

$$\begin{aligned} \mathrm{ev}_*[\overline{M}_{0,1}(X,\beta)]^{\mathrm{vir}} &= G_\beta h_\beta, \\ \mathrm{ev}_*\left(\psi_1 \cdot [\overline{M}_{0,1}(X,\beta)]^{\mathrm{vir}}\right) &= \frac{1}{2}F_\beta h_\beta^2 - \frac{1}{15}\left(G_\beta + \frac{1}{4}(\beta,\beta)F_\beta\right)c_2(X), \\ \mathrm{ev}_*\left(\psi_1^2 \cdot [\overline{M}_{0,1}(X,\beta)]^{\mathrm{vir}}\right) &= -3F_\beta \cdot \beta, \\ \mathrm{ev}_*\left(\psi_1^3 \cdot [\overline{M}_{0,1}(X,\beta)]^{\mathrm{vir}}\right) &= 6F_\beta\,[\mathsf{p}], \end{aligned}$$

as well as:

$$\begin{aligned} \mathrm{ev}_*[\overline{M}_{0,2}(X,\beta)]^{\mathrm{vir}} &= \frac{1}{4}F_\beta(h_\beta^2 \otimes h_\beta^2) + G_\beta\big(h_\beta\otimes\beta + \beta\otimes h_\beta + (h_\beta\otimes h_\beta)\cdot c_{BB}\big) \\ &+ \left(-\frac{1}{30}(h_\beta^2\otimes c_2(X) + c_2(X)\otimes h_\beta^2) + \frac{1}{900}(\beta,\beta)c_2(X)\otimes c_2(X)\right)\left(G_\beta + \frac{1}{4}(\beta,\beta)F_\beta\right). \end{aligned}$$

Modulo conjectures we have the following evaluation of genus 1, 2 Gromov-Witten invariants:

Theorem 7.4. *Assume Conjectures A and C of* [O22b]. *Then for any primitive curve class $\beta \in H_2(X,\mathbb{Z})$, in genus 1, we have:*

$$\mathrm{ev}_*[\overline{M}_{1,1}(X,\beta)]^{\mathrm{vir}} = \frac{1}{2}\mathcal{A}_\beta h_\beta^2 + \mathcal{B}_\beta c_2(T_X),$$

where

$$\begin{aligned} \mathcal{A} &= \frac{\Theta^2}{\Delta}\left(\frac{1}{4}\wp E_2 + \frac{3}{32}E_2^2 + \frac{1}{96}E_4\right), \\ \mathcal{B} &= \frac{\Theta^2}{\Delta}\left(-\frac{5}{46}\wp^3 + \frac{5\wp E_2^2}{384} + \frac{5E_2^3}{1536} - \frac{\wp E_4}{2944} + \frac{5E_2E_4}{4608} + \frac{5}{184}\left(y\frac{d}{dy}\wp\right)^2 - \frac{5E_6}{39744}\right). \end{aligned}$$

In genus 2, we have

$$\langle\varnothing\rangle_{2,\beta}^{\mathrm{GW}} = I_\beta,$$

where

$$I(y,q) = \frac{\Theta^2}{\Delta}\left(\frac{5\wp E_2^3}{384} + \frac{25E_2^4}{6144} + \frac{5\wp E_2E_4}{384} + \frac{7E_2^2E_4}{3072} - \frac{13E_4^2}{18432} - \frac{\wp E_6}{96} + \frac{E_2E_6}{1152}\right).$$

The first coefficients of $\mathcal{A}$ and $\mathcal{B}$ and I are as follows:

$$\begin{aligned}
\mathcal{A}(y,q) &= \frac{(y+y^{-1})}{8}q^{-1} + \left(\frac{1}{8}y^3 + \frac{315}{8}y + 160 + \frac{315}{8}y^{-1} + \frac{1}{8}y^{-3}\right)q + \cdots,\\
\mathcal{B}(y,q) &= \frac{(y+y^{-1})}{192}q^{-1} + 1\\
&\quad + \left(\frac{1}{192}y^{-3} + y^{-2} + \frac{385}{64}y^{-1} + \frac{110}{3} + \frac{385}{64}y + y^2 + \frac{1}{192}y^3\right)q + \cdots,\\
I(y,q) &= \frac{(y+y^{-1})}{128}q^{-1} - \frac{15}{2}\\
&\quad + \left(\frac{1}{128}y^{-3} - \frac{15}{2}y^{-2} - \frac{11445}{128}y^{-1} - 485 - \frac{11445}{128}y - \frac{15}{2}y^2 + \frac{1}{128}y^3\right)q + O(q^2).
\end{aligned}$$

7.4. **Proof of Theorem 7.4: Holomorphic anomaly equations.** The global Torelli theorem for hyperkähler varieties implies that the Hilbert scheme $S^{[2]}$ has a large monodromy group, we refer to [M11] for an introduction. In our case, as [OSY, §2.7] or [O21a] the monodromy implies that for a primitive curve class, we have

$$\begin{aligned}
\mathrm{ev}_*[\overline{M}_{1,1}(X,\beta)]^{\mathrm{vir}} &= \frac{1}{2}\mathcal{A}_\beta h_\beta^2 + \mathcal{B}_\beta c_2(T_X), \qquad (7.1)\\
\langle \varnothing \rangle^{\mathrm{GW}}_{2,\beta} &= I_\beta,
\end{aligned}$$

for some constants $\mathcal{A}_\beta, \mathcal{B}_\beta, I_\beta \in \mathbb{Q}$ which only depend on the square (β,β) of the class.

To determine these constants, we can work with an elliptic $K3$ surface $S \to \mathbb{P}^1$ with section. The Hilbert scheme in this case has an induced Lagrangian fibration $S^{[2]} \to \mathbb{P}^2$ with section. Let B, F be the section and fiber class of S respectively, and let $A \in H_2(S^{[2]},\mathbb{Z})$ be the class of the locus of non-reduced subschemes supported at a single point. There exists a natural isomorphism

$$H_2(S^{[2]},\mathbb{Z}) = H_2(S,\mathbb{Z}) \oplus \mathbb{Z}A$$

given by the Nakajima basis [O18, §0.2]. For $h \geqslant 0$ and $k \in \mathbb{Z}$, we consider the classes

$$\beta_{h,k} = B + hF + kA,$$

which are of square

$$(\beta_{h,k}, \beta_{h,k}) = 2h - 2 - \frac{k^2}{2}.$$

The set of these squares contains all possible squares of curve classes $\beta \in H_2(X,\mathbb{Z})$, we see that any (X,β) can be deformed to $(S^{[2]}, \beta_{h,k})$ for some h,k. We form the generating series

$$\mathsf{F}_g(\gamma_1,\ldots,\gamma_n) = \sum_{d\geqslant 0}\sum_{k\in\mathbb{Z}} \langle \tau_0(\gamma_1),\ldots,\tau_0(\gamma_n)\rangle^{S^{[2]}}_{g,B+dF+kA} q^{d-1}y^k.$$

The F_g's are conjectured to be quasi-Jacobi forms and that their formal derivatives $\frac{d}{dG_2}\mathsf{F}_g$ are determined by a holomorphic anomaly equation [O22b].

Below we will freely use the language of Nakajima operators

$$\mathfrak{q}_i(\alpha) : H^*(S^{[m]}) \to H^*(S^{[m+i]})$$

for all $i \in \mathbb{Z}$ and $\alpha \in H^*(S)$, where we follow the conventions of [NOY]. Given $\gamma_1,\ldots,\gamma_k \in H^*(S)$ and $n_1,\ldots,n_k \geqslant 1$, we will write

$$\gamma_1[n_1]\cdots\gamma_k[n_k] := \mathfrak{q}_{n_1}(\gamma_1)\cdots\mathfrak{q}_{\ell_k}(n_k)1 \ \in H^*(S^{[\sum_i n_i]}),$$

where the unit $1 \in H^*(S^{[0]})$ is also sometimes called the *vacuum*.

Proof of Theorem 7.4: Genus 1 case. By [O22b, Conj. C], we have for any $\gamma \in H^4(X)$ the following holomorphic-anomaly equation:

$$\begin{aligned}
\frac{d}{dG_2}\mathsf{F}_1(\gamma) &= \mathsf{F}_0(\gamma, U) - 2\mathsf{F}_1(\lambda_1; U(\gamma))\\
&= y\frac{d}{dy}\mathsf{F}_0(\gamma, F[2]) + 2\mathsf{F}_0(\gamma, F[1]W[1] + 1_S[1]\mathsf{p}[1])\\
&\quad + 2q\frac{d}{dq}\mathsf{F}_0(\gamma, F[1]^2) - 2\mathsf{F}_1(\lambda_1; U(\gamma)),
\end{aligned}$$

where

$$W = B + F$$

and

$$\begin{aligned} U &= -\frac{1}{4}\mathfrak{q}_2\mathfrak{q}_{-2}(F_1+F_2) - \mathfrak{q}_1\mathfrak{q}_{-1}(F_1+F_2) \\ &= -\frac{1}{4}\mathfrak{q}_2\mathfrak{q}_{-2}(F_1+F_2) + \mathfrak{q}_1\mathfrak{q}_1\mathfrak{q}_{-1}\mathfrak{q}_{-1}((F_1+F_4)\Delta_{23}) \\ &= -\frac{1}{4}\mathfrak{q}_2\mathfrak{q}_2'(F_1+F_2)(1\otimes 1) + \mathfrak{q}_1\mathfrak{q}_1\mathfrak{q}_1'\mathfrak{q}_1'((F_1+F_4)\Delta_{23})(1\otimes 1), \end{aligned}$$

which is viewed here both as a morphism $H^*(S^{[2]}) \to H^*(S^{[2]})$ and by Poincarë duality in the last line as a class in $H^*(S^{[2]} \times S^{[2]})$ (we let $\mathfrak{q}_i'$ denote the Nakajima operator acting on the second factor of $H^*(S^{[n]}) \otimes H^*(S^{[m]})$). Moreover, $\mathsf{F}_g(\lambda_1; \ldots)$ stands for the (obvious) generating series where we integrate also over the tautological class λ_1, see [O22b].

We consider the invariant $\mathsf{F}_1(\mathfrak{q}_1(F)^2 1)$. Using the holomorphic anomaly equation above, the known results in genus 0 (Theorem 7.3) and the discussion in [O18, OP] on how to reduce the series $\mathsf{F}_1(\lambda_1; ...)$ to genus 0 invariants, we have:

$$\frac{d}{dG_2}\mathsf{F}_1(\mathfrak{q}_1(F)^2) = \frac{\Theta^2}{\Delta}\left(-6\wp + 108 G_2\right).$$

Integrating with respect to G_2 yields

$$\mathsf{F}_1(\mathfrak{q}_1(F)^2) = \frac{\Theta^2}{\Delta}\left(aE_4 - 6\wp G_2 + 54 G_2^2\right),$$

where $\wp^2$ does not appear, because it would yield the only pole on the left hand side (contradicting Conjecture A of [O22b] or also monodromy invariance). By Proposition 6.1, we have

$$\mathrm{Coeff}_{q^{-1}y^{-1}}(\mathsf{F}_1(\mathfrak{q}_1(F)^2 1)) = \big\langle \tau_0(F[1]^2)\big\rangle^{\mathrm{GW}}_{g=1,B-A} = \big\langle \tau_0(H^2)\big\rangle^{\mathrm{GW}, T^*\mathbb{P}^2}_{g=1,1} = \frac{1}{8}.$$

Solving for a one finds $a = 1/96$, and hence

$$\mathsf{F}_1(\mathfrak{q}_1(F)^2 1) = \frac{\Theta^2}{\Delta}\left(\frac{1}{4}\wp E_2 + \frac{3}{32}E_2^2 + \frac{1}{96}E_4\right).$$

Similarly, we have

$$\frac{d}{dG_2}\mathsf{F}_1(c_2(X)) = \frac{\Theta^2}{\Delta}\left(-105\wp E_2 + \frac{135}{8}E_2^2 - \frac{5}{8}E_4\right),$$

where we used that $U(c_2(X)) = 30\mathfrak{q}_1(F)\mathfrak{q}_1(1)1$. This yields

$$\mathsf{F}_1(c_2(X)) = \frac{\Theta^2}{\Delta}\left(\frac{35}{16}\wp E_2^2 - \frac{15}{64}E_2^3 + \frac{5}{192}E_2E_4 + aE_4\wp + bE_6\right),$$

where, since there are no poles on the left hand side, the poles in $(D_z\wp)^2$ and $\wp^3$ cancel and give the Eisenstein series E_6. By Proposition 6.1 and since the pair $(S^{[2]}, B+F+A)$ is deformation equivalent to $(S^{[2]}, A)$ and we have seen in Lemma 3.3 that the genus 1 invariants vanishing in this case, we have:

$$\big\langle c_2(X)\big\rangle^{\mathrm{GW}}_{1,B-A} = -3\big\langle \tau_0(H^2)\big\rangle^{\mathrm{GW},T^*\mathbb{P}^2}_{g=1,1} = -\frac{3}{8},$$
$$\big\langle c_2(X)\big\rangle^{\mathrm{GW}}_{1,B+F+A} = 0.$$

Solving with these conditions for a and b, we obtain

$$\mathsf{F}_1(c_2(X)) = \frac{\Theta^2}{\Delta}\left(\frac{35\wp E_2^2}{16} - \frac{15E_2^3}{64} - \frac{47\wp E_4}{16} + \frac{5E_2E_4}{192} - \frac{5E_6}{48}\right) = -3/8(y^{-1}+y)q^{-1} + 828 + O(q).$$

Finally, by Lemma 7.5 below and the definition of $\mathcal{A}_\beta, \mathcal{B}_\beta$ in (7.1), the functions

$$\mathcal{A} = \sum_{d,k}\mathcal{A}_{\beta_{h,k}}q^{h-1}y^k, \qquad \mathcal{B} = \sum_{d,k}\mathcal{B}_{\beta_{h,k}}q^{h-1}y^k$$

satisfy:

$$\mathsf{F}_1(\mathfrak{q}_1(F)^2 1) = \mathcal{A}, \quad \mathsf{F}_1(c_2(X)) = 30\left(q\frac{d}{dq} - \frac{1}{4}\left(y\frac{d}{dy}\right)^2\right)\mathcal{A} + 828\mathcal{B}.$$

This proves the claim by solving for $\mathcal{A}$ and $\mathcal{B}$.

We remark that determining $\mathsf{F}_1(\mathfrak{q}_1(F)^2 1)$ only required a single geometric constraint, namely the computation for class $B-A$. However, the formula also matches the vanishings obtained from computations in the ideal geometry (which applies to classes $\beta \in \{B, B+F+A\}$). For $\mathsf{F}_1(c_2(X))$ the system is likewise overdetermined: we only used 2 of the 3 available constraints. □

Proof of Theorem 7.4: Genus 2 case. Using Lemma 7.5 below, the standard intersections

$$c_2(X)\cdot \mathfrak{q}_1(\mathsf{p})\mathfrak{q}_1(1)1 = 27, \qquad c_2(X)\cdot \mathfrak{q}_1(W)\mathfrak{q}_1(F)1 = 3$$

and the genus 1 part of Theorem 7.4, the holomorphic anomaly equation of [O22b] reads:

$$\frac{d}{dG_2}\mathsf{F}_2 = \mathsf{F}_1(U) = 3\left(2q\frac{d}{dq} - \frac{1}{2}\left(y\frac{d}{dy}\right)^2\right)\mathcal{A} + 60\mathcal{B}.$$

Integration with respect to G_2 yields

$$\mathsf{F}_2 = \frac{\Theta^2}{\Delta}\left(\frac{5}{384}\wp E_2^3 + \frac{25}{6144}E_2^4 + \frac{5}{384}\wp E_2 E_4 + \frac{7}{3072}E_2^2 E_4 + \frac{1}{1152}E_2 E_6 + aE_4^2 + b\wp E_6\right)$$

for some $a, b \in \mathbb{C}$. Here we used that F_2 is determined up to the functions $\wp^4, \wp(y\frac{d}{dy}\wp)^2, \wp^2 E_4, E_4$ and that the poles in the first of these functions have to cancel which replaces them with a $E_6\wp$ term and then that $\wp^2 E_4$ can also not appear because of holomorphicity. Finally, using the following evaluations (ref. Proposition 6.1, Lemma 3.3):

$$\langle \varnothing \rangle^{\mathrm{GW}}_{2,B-A} = \frac{1}{128}, \quad \langle \varnothing \rangle^{\mathrm{GW}}_{2,B+F+A} = 0$$

yields that $a = -13/18432$ and $b = -1/96$ and thus

$$\mathsf{F}_2(\varnothing) = \frac{\Theta^2}{\Delta}\left(\frac{5\wp E_2^3}{384} + \frac{25E_2^4}{6144} + \frac{5\wp E_2 E_4}{384} + \frac{7E_2^2E_4}{3072} - \frac{13E_4^2}{18432} - \frac{\wp E_6}{96} + \frac{E_2E_6}{1152}\right).$$

This implies the result by monodromy invariance (we even have one more condition to spare, namely the vanishing of $\langle \varnothing \rangle^{\mathrm{GW}}_{2,B}$). ☐

Lemma 7.5. *Let $\widetilde{\beta}_{d,k} = W + dF + kA$ and let $h_{d,k} = \widetilde{\beta}^{\vee}_{d,k}$ be the dual. Then*

$$h_{d,k}^2\cdot \mathfrak{q}_1(F)^2 1 = 2, \qquad h_{d,k}^2 \cdot \mathfrak{q}_1(\mathsf{p})\mathfrak{q}_1(1)1 = 2d - k^2/4,$$
$$h_{d,k}^2\cdot \mathfrak{q}_2(F)1 = -2k, \qquad h_{d,k}^2 \cdot \mathfrak{q}_1(W)\mathfrak{q}_1(F)1 = 2d - k^2/4,$$
$$h_{d,k}^2\cdot c_2(X) = 30(2d - k^2/2).$$

Proof. Let $\delta = c_1(\mathcal{O}_S^{[2]}) = -\frac{1}{2}\Delta_{S^{[2]}}$ and $D(\alpha) = \mathfrak{q}_1(\alpha)\mathfrak{q}_1(1)1$. We have

$$h_{d,k} = \widetilde{\beta}^{\vee}_{d,k} = D(W) + dD(F) - \frac{k}{2}\delta.$$

This yields, for example

$$\int h_{d,k}^2\cdot \mathfrak{q}_2(F)1 = -k\int \mathfrak{q}_2(F)1\cdot \delta \cdot D(W) = -2k.$$

The other cases are similar (use that $\mathfrak{q}_1(W)\mathfrak{q}_1(F)\cdot \delta^2 = \mathfrak{q}_1(\mathsf{p})\mathfrak{q}_1(1)\cdot\delta^2 = -1$). For the last expression we use the Fujiki constant $C(c_2) = 30$. ☐

7.5. **Genus 1 Gopakumar-Vafa invariants.** A hyperkähler variety X is of $K3^{[2]}$*-type* if it is deformation equivalent to the Hilbert scheme $S^{[2]}$ for a $K3$ surface S. For any primitive curve class $\beta \in H_2(X,\mathbb{Z})$, we define the *genus 1 Gopakumar-Vafa class*

$$n_{1,\beta} \in H^4(X,\mathbb{Q})$$

by

$$\int_X n_{1,\beta}\cup\gamma = n_{1,\beta}(\gamma), \quad \forall\, \gamma \in H^4(X,\mathbb{Q}),$$

where $n_{1,\beta}(\gamma)$ is given in Definition 1.6. In an ideal geometry (ref. §1.5), $n_{1,\beta}$ is the class of the surface swept out by the elliptic curves in class β.

Our discussion above leads to the following formula. Define

$$\mathcal{A}' = \frac{\Theta^2}{\Delta}\left(-\frac{1}{4}\wp - \frac{5}{48}E_2\right) = -\frac{(y+y^{-1})}{8}q^{-1} + 6 + O(q),$$
$$\mathcal{B}' = \frac{\Theta^2}{\Delta}\left(-\frac{1}{96}\wp E_2 - \frac{1}{256}E_2^2 - \frac{1}{2304}E_4\right) = -\frac{(y+y^{-1})}{192}q^{-1} + O(q),$$

and recall the series $\mathcal{A}, \mathcal{B}$ from Theorem 7.4.

Theorem 7.6. *Assume Conjectures A and C of* [O22b]*. For any hyperkähler variety X of $K3^{[2]}$ type and for any primitive curve class $\beta \in H_2(X,\mathbb{Z})$, we have*

$$n_{1,\beta} = \frac{1}{2} a_\beta h_\beta^2 + b_\beta c_2(T_X),$$

where $a_\beta = \mathcal{A}_\beta + \mathcal{A}'_\beta$ and $b_\beta = \mathcal{B}_\beta + \mathcal{B}'_\beta$.

Proof. Since the Chern class $c_2(X)$ is monodromy invariant, we can write

$$\frac{1}{24} \mathrm{ev}_{1*}(\mathrm{ev}_2^*(c_2(X))[\overline{M}_{0,2}(X,\beta)]^{\mathrm{vir}}) = \frac{1}{2}\mathcal{A}'_\beta h_\beta^2 + \mathcal{B}'_\beta c_2(T_X).$$

for some $\mathcal{A}'_\beta, \mathcal{B}'_\beta$. Using Theorem 7.3, one computes that these are precisely the β-coefficients of the functions $\mathcal{A}', \mathcal{B}'$ defined above. The claim now follows from Theorem 7.4 and the definition of genus 1 Gopakumar-Vafa invariants. ☐

The integrality conjecture for Gopakumar-Vafa invariants (Conjecture 1.9) would imply that $n_{1,\beta} \in H^4(X,\mathbb{Q})$ is an integral class. We give the following criterion:

Lemma 7.7. *$n_{1,\beta}$ is integral, i.e. lies in $H^4(X,\mathbb{Z})$ if and only if the following holds:*

(i) *If $(\beta,\beta) \in 2\mathbb{Z}$, then a_β is an even integer and $3b_\beta \in \mathbb{Z}$.*
(ii) *If $(\beta,\beta) = 2d - \frac{1}{2}$, then a_β, $24b_\beta$, $\frac{1}{8}a_\beta - 3b_\beta$ all lie in $\mathbb{Z}$.*

Proof. Using deformation invariance (e.g. [O21a, Cor. 2]), we may work with $X = S^{[2]}$ for an elliptic $K3$ surface S with $\mathrm{Pic}(S)$ generated by the class of a section B and the fiber class F, Moreover, we can use the curve class

$$\beta := \widetilde{\beta}_{d,k} = W + dF + kA,$$

for $d \geqslant -1$ and $k \in \{0,1\}$. With the notation of Lemma 7.5, we then have:

$$n_{1,\beta} = \frac{1}{2}a_\beta D(W)^2 + d a_\beta D(W)D(F) + \frac{1}{2}a_\beta d^2 D(F)^2 + -k a_\beta D(W)\delta - kdD(F)\delta + \frac{k^2}{4}\delta^2.$$

By the main result of [Nova], a basis for the Hodge classes

$$H^{2,2}(S^{[2]},\mathbb{Z}) = H^4(S^{[2]},\mathbb{Z}) \cap H^{2,2}(S^{[2]},\mathbb{C})$$

is given by the 7 classes

$$D(W)^2,\ D(W)D(F),\ D(F)^2,\ \delta^2$$
$$e_x := \frac{1}{2}\left(D(x)^2 + D(x)\delta\right) \text{ for } x \in \{W,F\}$$
$$V := \frac{1}{24}c_2(T_{S^{[2]}}) + \frac{1}{8}\delta^2.$$

The class $n_{1,\beta}$ has the following expansion in this integral basis:

$$\begin{aligned} n_{1,\beta} = &\frac{k+1}{2}a_\beta D(W)^2 - k a_\beta e_W \\ &+ \frac{d(d+k)}{2}a_\beta F^2 - kda_\beta e_F \\ &+ d a_\beta D(W)D(F) \\ &+ 24 b_\beta V + \left(\frac{k^2}{8}a_\beta - 3b_\beta\right)\delta^2. \end{aligned}$$

If $(\beta,\beta) \in \mathbb{Z}$ then $k=0$, so integrality of $n_{1,\beta}$ impies (by the first summand) that $a_\beta \in 2\mathbb{Z}$ and by the last summand that $3b_\beta \in \mathbb{Z}$, and this is clearly sufficient. If $(\beta,\beta) = 2d - \frac{1}{2}$, we have $k=1$, which gives $a_\beta, 24b_\beta, \frac{1}{8}a_\beta - 3b_\beta \in \mathbb{Z}$ and this is clearly sufficient. ☐

The criterion of the lemma can be easily checked using a computer program. We obtain:

Corollary 7.8. *Under the assumptions of Theorem 7.6, $n_{1,\beta}$ is integral for all $(\beta,\beta) \leqslant 100$.*

Example 7.9 (A real life example)**.** *Let $F(Y) \subset \mathrm{Gr}(2,6)$ be the Fano variety of lines on a very general cubic 4-fold $Y \subset \mathbb{P}^5$. Let $\mathcal{U} \subset \mathcal{O}_{\mathrm{Gr}}^{\otimes 6}$ be the universal subbundle on $\mathrm{Gr}(2,6)$ and set*

$$g = c_1(\mathcal{U}^\vee), \quad c = c_2(\mathcal{U}^\vee).$$

The unique primitive curve class is $\beta = \frac{1}{2}g^\vee$ and is of square $(\beta,\beta) = 3/2$ since $(g,g) = 6$. The basic geometry of these classes is discussed in [Ot]*, in particular we have*

$$c_2(X) = 5g^2 - 8c.$$

Theorem 7.6 implies that the surface in $F(Y)$ swept out by elliptic curves in class β has class:

$$n_{1,\beta} = 35(g^2 - c).$$

This is indeed integral and effective (the surface of lines meeting a given line is $\frac{1}{3}(g^2 - c)$).

7.6. **Genus 2 Gopakumar-Vafa invariants.** Since we can control now all Gromov-Witten invariants for $S^{[2]}$ in arbitrary genus (for primitive classes), it is also straightforward to compute genus 2 Gopakumar-Vafa invariants (see [NO] for the computation of the nodal invariants):

Theorem 7.10. *Assume Conjectures A and C of* [O22b]. *For any hyperkähler variety X of $K3^{[2]}$ type and for any primitive curve class $\beta \in H_2(X,\mathbb{Z})$, we have*

$$n_{2,\beta} = \widetilde{I}_\beta,$$

where

$$\widetilde{I}(y,q) = \frac{\Theta^2}{\Delta}\Bigg[\frac{5}{384}\wp E_2^3 + \frac{25}{6144}E_2^4 + \frac{35}{384}\wp E_2^2 - \frac{5}{512}E_2^3 + \frac{5}{384}\wp E_2E_4 + \frac{7}{3072}E_2^2E_4 - \frac{71}{64}\wp E_2 + \frac{27}{512}E_2^2 - \frac{47}{384}\wp E_4 + \frac{5}{4608}E_2E_4 - \frac{13}{18432}E_4^2 - \frac{1}{96}\wp E_6 + \frac{1}{1152}E_2E_6 + \frac{9}{8}\wp - \frac{5}{32}E_2 - \frac{23}{1536}E_4 - \frac{5}{1152}E_6 + \frac{1}{8}\Bigg].$$

Using a computer program, we immediately obtain:

Corollary 7.11. *Under the assumptions of Theorem 7.10, $n_{2,\beta}$ is integral for all $(\beta,\beta) \leqslant 138$.*

7.7. **Genus 0 Gopakumar-Vafa invariants.** For completeness, we also give a proof of the integrality of genus 0 Gopakumar-Vafa invariants discussed in the introduction.

Proof of Theorem 0.11. Inverting the definition of genus 0 Gopakumar-Vafa invariants, we have

$$n_{0,\beta}(\gamma_1,\ldots,\gamma_n) = \sum_{k|\beta}\mu(k)k^{-3+n}\big\langle\tau_0(\gamma_1)\cdots\tau_0(\gamma_n)\big\rangle^{\mathrm{GW}}_{0,\beta},$$

where $\mu(k)$ is the Möbius function. Consider also the "BPS invariants" introduced in [O21a]:

$$\widetilde{n}_{0,\beta}(\gamma_1,\ldots,\gamma_n) = \sum_{k|\beta}\mu(k)k^{-3+n}(-1)^{[\beta]+[\beta/k]}\big\langle\tau_0(\gamma_1)\cdots\tau_0(\gamma_n)\big\rangle^{\mathrm{GW}}_{0,\beta}.$$

Then it is straightforward to show that (see [O21a, Def. 1] for the notation $[-]$):

(i) If $\left[\frac{\beta}{\mathrm{div}(\beta)}\right] = 0$, then $n_{0,\beta}(\gamma_1,\ldots,\gamma_n) = \widetilde{n}_{0,\beta}(\gamma_1,\ldots,\gamma_n)$,
(ii) If $\left[\frac{\beta}{\mathrm{div}(\beta)}\right] = 1$, then

$$n_{0,\beta}(\gamma_1,\ldots,\gamma_n) = \begin{cases}\widetilde{n}_{0,\beta}(\gamma_1,\ldots,\gamma_n) & \text{if } \mathrm{div}(\beta) \text{ is odd or } 4|\,\mathrm{div}(\beta),\\ \widetilde{n}_{0,\beta}(\gamma_1,\ldots,\gamma_n) - \widetilde{n}_{0,\beta/2}(\gamma_1,\ldots,\gamma_n) & \text{if } \mathrm{div}(\beta) \text{ is even but } \mathrm{div}(\beta/2) \text{ is odd.}\end{cases}$$

Hence it suffices to show that $\widetilde{n}_{0,\beta}(\gamma)$ is integral for any effective curve class $\beta \in H_2(X,\mathbb{Z})$. As conjectured in [O21a] and proven in [O21c], the invariant $\widetilde{n}_{0,\beta}(\gamma)$ only depends on

$$\mathsf{q}(\beta), \quad [\beta/\mathrm{div}(\beta)], \quad \text{and} \quad (\beta,\gamma).$$

Hence we may assume that β is primitive. But here the result follows since for a very general pair (X,β), where X is a hyperkähler variety of $K3^{[2]}$-type, it is well-known that $\overline{M}_{0,1}(X,\beta)$ is an algebraic space (there are no non-trivial automorphisms) of expected dimension (e.g. [OSY, §1.1]), therefore

$$\mathrm{ev}_*[\overline{M}_{0,1}(X,\beta)]^{\mathrm{vir}} = \mathrm{ev}_*[\overline{M}_{0,1}(X,\beta)]$$

is integral (the same argument also shows the integrality of $n_{0,\beta}(\gamma_1,\ldots,\gamma_n)$ for arbitrary number of markings if β is primitive). ☐

Appendix A. Imprimitive curve classes

Let X be a holomorphic symplectic 4-fold. We consider here the Gromov-Witten, Gopakumar-Vafa and DT_4 invariants of X in a (possibly imprimitive) curve class $\beta \in H_2(X,\mathbb{Z})$. As discussed in Section 1.4, the ideal geometry of curves in this case is very difficult to control. Moreover, there are only very few geometries where both the GW and DT_4 invariants can be completely computed, and these geometries do not reflect the general structure of the GW/GV/DT_4 invariants. As such, the general definition of Gopakumar-Vafa invariants $n_{g,\beta}$ for $g>0$ and imprimitive β is not clear at this point. Nevertheless, in this section we define genus 1 Gopakumar-Vafa invariants for imprimtive curve classes in the two geometries where all the invariants can be controlled, and then prove a GV/DT_4 relation.

A.1. **Genus 1 Gopakumar-Vafa invariants.** There are two geometries where we know all GW and DT_4 invariants:

(i) The Embedded Rational Curve family of Section 3
(ii) The product of two $K3$ surfaces $S \times T$ for all curve classes which lie in $H_2(S,\mathbb{Z})$.

These geometries are special because all primary GW and DT_4 invariants with insertion $c_2(T_X)$ vanish. This implies that the genus 1 Gopakumar-Vafa invariants do not have any contributions from genus 0 curves. Based on a computation in the ideal geometry following Section 1.6.1, one expects that

$$\big\langle \tau_0(\gamma) \big\rangle^{\mathrm{GW}}_{1,\beta} \text{ `` }=\text{ '' } \sum_{k|\beta} \sigma(k)\, n_{1,\beta/k}(\gamma) + (\cdots), \tag{A.1}$$

where $\sigma(k) := \sum_{l|k} l$ and $(\cdots)$ stands for contributions from genus 0 curves. This suggests that for the geometries (i) and (ii), there should be no contributions in genus 0. Hence we make the following adhoc definition in this case:

Definition A.1. *Let X be a holomorphic symplectic 4-fold and $\beta \in H_2(X,\mathbb{Z})$ be an effective curve class of type (i) or (ii) above. For any $\gamma \in H^4(X,\mathbb{Z})$, we define $n_{1,\beta}(\gamma)$ by:*

$$\big\langle \tau_0(\gamma) \big\rangle^{\mathrm{GW}}_{1,\beta} = \sum_{k\geqslant 1, k|\beta} \sigma(k)\, n_{1,\beta/k}(\gamma). \tag{A.2}$$

We also introduce the following:

Definition A.2. *For any $\gamma \in H^4(X,\mathbb{Z})$, we define $n_{0,\beta}(\gamma;\psi) \in \mathbb{Q}$ by the multiple cover formula:*

$$\big\langle \tau_1(\gamma) \big\rangle^{\mathrm{GW}}_{0,\beta} = \sum_{k\geqslant 1, k|\beta} \frac{1}{k^3}\, n_{0,\beta/k}(\gamma;\psi).$$

We then can prove the following generalization of the genus one part of Conjecture 2.2:

Proposition A.3. *Let X be a holomorphic symplectic 4-fold, and $\beta \in H_2(X,\mathbb{Z})$ be an effective curve class of type (i) or (ii) above. For certain choice of orientation, we have*

$$\big\langle \tau_1(\gamma) \big\rangle^{\mathrm{DT}_4}_{\beta} = -\frac{1}{2} n_{0,\beta}(\gamma;\psi) - \sum_{k\geqslant 1, k|\beta} n_{1,\beta/k}(\gamma), \quad \forall\, \gamma \in H^4(X,\mathbb{Z}). \tag{A.3}$$

Remark A.4. *The second part in the RHS of the above equality is also consistent with the ideal geometry computation (see Remark 2.6).*

A.2. **Proof of Proposition A.3: Embedded rational curve family.** Let X be a holomorphic symplectic 4-fold and let $\beta \in H_2(X,\mathbb{Z})$ be a curve class which satisfy conditions (i-iii) of Section 3. Then by Lemma 3.3, all genus 1 GV invariants $n_{1,\beta}(\gamma)$ vanish. Moreover, by Lemma 3.5 and with the notation of that section, we have

$$n_{0,d\beta}(\gamma;\psi) = \begin{cases} \int_C j^*(\gamma)\, c_1(\omega_p) & \text{if } d=1, \\ 0 & \text{if } d>1. \end{cases}$$

Similarly, by Lemma 3.7, for certain choice of orientation we have

$$\big\langle \tau_1(\gamma) \big\rangle^{\mathrm{DT}_4}_{\beta} = \begin{cases} -\frac{1}{2}\int_C j^*(\gamma)\cdot c_1(\omega_p) & \text{if } d=1, \\ 0 & \text{if } d>1. \end{cases}$$

This implies the claim. ∎

A.3. **Proof of Proposition A.3: $K3 \times K3$.** Let $X = S \times T$ and $\beta \in H_2(S, \mathbb{Z})$ be an effective curve class. Consider a cohomology class $\gamma \in H^4(X, \mathbb{Z})$ with Künneth decomposition

$$\gamma = A_1 \cdot 1 \otimes \mathsf{p} + D_1 \otimes D_2 + A_2 \cdot \mathsf{p} \otimes 1.$$

The claim follows from the following two lemmata:

Lemma A.5. $n_{0,\beta}(\gamma; \psi) = -2A_1 \cdot N_0\left(\frac{\beta^2}{2}\right)$.

Proof. This follows from Lemma 5.2 and the definition. □

Lemma A.6. *We have*

$$\sum_{k \geqslant 1, k|\beta} n_{1,\beta/k}(\gamma) = A_2\, e(T) N_1(\beta^2/2). \tag{A.4}$$

Proof. Recall that by (5.7) we have

$$\left\langle \tau_0(\gamma) \right\rangle_{1,\beta}^{\mathrm{GW}} = A_2\, e(T) \sum_{k \geqslant 1, k|\beta} k \cdot N_1\left(\frac{\beta^2}{2k^2}\right).$$

Hence by Eqn. (A.2), $n_{1,\beta}(\gamma)$'s are the unique (recursively defined) integers which satisfy the relation

$$\sum_{k|\beta} \sum_{l|k} l\, n_{1,\beta/k}(\gamma) = A_2\, e(T) \sum_{k|\beta} k \cdot N_1\left(\frac{\beta^2}{2k^2}\right). \tag{A.5}$$

We show that the integers $n_{1,\beta}(\gamma)$ defined by Eqn. (A.4) satisfy the relation (A.5). This then completes the proof. Indeed, using Eqn. (A.4), the right hand side of Eqn. (A.5) becomes

$$\begin{aligned} A_2\, e(T) \sum_{k|\beta} k \cdot N_1\left(\frac{\beta^2}{2k^2}\right) &= \sum_{k|\beta} k \left(\sum_{a|\beta/k} n_{1,\beta/ka}(\gamma) \right) \\ (\text{set } m := ka) \quad &= \sum_{m|\beta} \sum_{k|m} k n_{1,\beta/m}, \end{aligned}$$

which is precisely the left hand side of Eqn. (A.5). Hence Eqn. (A.4) holds. □

We conclude the proof of Proposition A.3: By Theorem 5.8, we have

$$\left\langle \tau_1(\gamma) \right\rangle_{\beta}^{\mathrm{DT}_4} = A_1\, N_0\left(\frac{\beta^2}{2}\right) - A_2\, e(T)\, N_1\left(\frac{\beta^2}{2}\right),$$

which is precisely the right hand side of (A.3) by the two lemmata above. □

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