Curve counting on the Enriques surface and the Klemm-Marino formula

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October 2, 2023
Intercontinental moduli and algebraic geometry

Motivation

Goal: Compute the Gromov-Witten invariants of a compact Calabi-Yau threefold.

This is very difficult.

Instead it is easier to first consider the local case

$$X = \mathrm{Tot}(\omega_S), \quad S \text{ smooth projective surface.}$$

This is best understood for K3 surfaces and says also something about compact CY3 geometry.

K3 surfaces

The moduli space of stable maps to a K3 surface S has a *reduced* virtual class

$$[\overline{M}_g(S,\beta)]^{\mathrm{red}} \in \mathrm{CH}_g(\overline{M}_g(S,\beta))$$

Let $\mathbb{E} \to \overline{M}_g(S,\beta)$ be the Hodge bundle and let $\lambda_i = c_i(\mathbb{E})$.
We define the invariants

$$N_{g,\beta} := \int_{[\overline{M}_g(S,\beta)]^{\mathrm{red}}} (-1)^g \lambda_g$$

This can be viewed as the definition of GW invariants of the local surface $X = \mathrm{Tot}(\omega_S) = S \times \mathbb{C}$ (as an equivariant residue).

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Theorem (Maulik-Pandharipande, 2007)

Let $\pi : X \to C$ be a K3-fibered Calabi-Yau threefold with at most nodal fibers. Then the Gromov-Witten invariants of X in fiber classes is determined by:

(a) *Noether-Lefschetz numbers of the family π (modular, easily computable)*

(b) *The invariants $N_{g,\beta}$.*

Katz-Klemm-Vafa formula

The invariants $N_{g,\beta}$ have been conjectured by Katz-Klemm-Vafa in 1999 and then proven in a long journey by many authors (Bryan, Leung, Maulik, Pandharipande, Thomas, ...).

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Define the Gopakumar-Vafa invariants $n_{g,\beta}$ by

$$\sum_{\beta}\sum_{g} N_{g,\beta} u^{2g-2} t^{\beta} = \sum_{\beta}\sum_{g} n_{g,\beta} u^{2g-2} \sum_{d\geq 1} \frac{1}{d} \left(\frac{\sin(du/2)}{u/2}\right)^{2g-2} t^{d\beta}$$

This formally subtracts contributions from multiple covers and contracted components.

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This formally subtracts contributions from multiple covers and contracted components.

Theorem (Pandharipande-Thomas, 2015)

(a) *The invariants $n_{g,\beta}$ only depend upon β through $\beta \cdot \beta$.*

(b) *If $\beta^2 = 2h-2$ write $n_{g,h} := n_{g,\beta}$. Then*

$$\sum_{g,h}(-1)^g n_{g,h}(p^{1/2} - p^{-1/2})^{2g} q^h = \prod_{m\geq 1} \frac{1}{(1-p^{-1}q^m)^2(1-q^m)^{20}(1-pq^m)^2}$$

Proof: Prove GW/PT correspondence for $S \times \mathbb{C}$ by reducing it via Noether-Lefschetz theory to GW/PT for compact CY3 where it is known [P-Pixton]; prove (a) by localization of PT on $S \times \mathbb{C}$ and constructing additional cosections.

Enriques surfaces

An Enriques surface is a smooth projective surface Y with non-trivial canonical bundle such that

$$\omega_Y^{\otimes 2} \cong \mathcal{O}_Y, \quad H^1(Y, \mathcal{O}_Y) = 0.$$

The double cover associated to ω_Y is a K3 surface X, so $Y = X/\mathbb{Z}_2$.
Hodge diamond of an Enriques:

$$\begin{array}{ccccc} & & 1 & & \\ & 0 & & 0 & \\ 0 & & 10 & & 0 \\ & 0 & & 0 & \\ & & 1 & & \end{array}$$

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No reduction necessary!

$$[\overline{M}_g(Y, \beta)]^{\mathrm{vir}} \in \mathrm{CH}_{g-1}(\overline{M}_g(Y, \beta))$$

The Gromov-Witten invariants of the local Enriques are defined by

$$N_{g,\beta} = \int_{[\overline{M}_g(Y,\beta)]^{\mathrm{vir}}} (-1)^{g-1} \lambda_{g-1}$$

Comments / History

Some previous conjectures and results:

- 1896: Enriques discovered the Enriques surface
- 1998: Harvey-Moore made proposal for $N_{1,\beta}$
- 2005: Klemm-Marino conjectured formula for $N_{g,\beta}$ (and corrected H-M proposal)
- 2006: Maulik-Pandharipande: Two results:

  (a) If Virasoro constraints hold for $\mathrm{GW}(Y)$ in genus 2, then Klemm-Marino conjecture true in genus 1.

  (b) $N_{g,\beta}$ related to a certain compact K3-fibered CY3 (the Enriques Calabi-Yau).

The Klemm-Marino formula

Define *twisted* Gupakumar-Vafa invariants $n_{g,\beta}$ of the local Enriques surface by

$$\sum_{\beta} \sum_{g} N_{g,\beta} u^{2g-2} t^{\beta} = \sum_{\beta} \sum_{g} n_{g,\beta} u^{2g-2} \sum_{\substack{d \geq 1 \\ d \text{ odd}}} \frac{1}{d} \left(\frac{\sin(du/2)}{u/2} \right)^{2g-2} t^{d\beta}$$

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Conjecture (Klemm-Marino formula)

(a) *The invariants* $n_{g,\beta}$ *only depend upon* β *through* $\beta\cdot\beta$.

(b) *If* $\beta^2 = 2h-2$ *write* $n_{g,h} := n_{g,\beta}$. *Then*

$$\sum_{g,h}(-1)^{g-1} n_{g,h}(p^{1/2}-p^{-1/2})^{2g-2} q^{h-1}$$
$$= \prod_{\substack{m\geq 1 \\ m \text{ odd}}} \frac{1}{(1-p^{-1}q^m)^2(1-q^m)^4(1-pq^m)^2} \prod_{m\geq 1} \frac{1}{(1-q^m)^8}$$

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Theorem (O. 2023)

Conjecture holds.

Reformulation in genus 1 (for fun)

In genus 1 we can restate the Klemm-Marino formula as

$$\exp\left(\sum_{\beta\neq 0} N_{1,\beta} t^{\beta}\right) = \prod_{\beta>0}\left(\frac{1+t^{\beta}}{1-t^{\beta}}\right)^{a(\beta^2/2)}$$

where the coefficients $a(n)$ are defined by

$$\sum_{n\geq 0} a(n)q^n = \prod_{n\geq 1}\frac{(1+q^n)^8}{(1-q^n)^8} = 1 + 16q + 144q^2 + 960q^3 + 5264q^4 + \dots .$$

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$$\mathcal{D}_M = \{x \in \mathbb{P}(M\otimes\mathbb{C}) | x\cdot x = 0, x\cdot\overline{x} > 0\}.$$

The moduli space of Enriques surfaces is the arithmetic quotient $\mathcal{D}_M/O(M) - \mathcal{H}_2$.
Thm(Borcherds) $\exists$ automorphic form $\Phi(t)$ of weight 4 on $\mathcal{D}_M$ for $O(M)$.

Corollary

The series $\exp(\sum_{\beta\neq 0} N_{1,\beta} t^\beta)$ *is the Fourier expansion of the automorphism form* $\Phi(t)^{-1/8}$ *around the level* 1 *cusp.*

Strategy of the proof

Consider splitting (we ignore all torsion)

$$H^2(Y, \mathbb{Z}) \cong U \oplus E_8(-1).$$

U spanned by half-fiber f and 2-section s of an elliptic fibration $Y \to \mathbb{P}^1$

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To prove the Klemm-Marino formula we prove three properties:

(a) $N_{g,\beta}$ depends on β only through the square $\beta \cdot \beta$ and the divisibility of β.

(b) The partial series

$$F_{g,r} = \sum_{d \geq 0} \sum_{\alpha \in E_8(-1)} q^d \zeta^\alpha N_{g, rs+df+\alpha}$$

are quasi-Jacobi forms for $\Gamma_0(2)$ of weight $2g - 2$.

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(c) Explicit computation of $F_{g,1}$ and $F_{g,2}$.

Then some modular form magic determines the rest. Let me explain this 'magic'.

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Lemma We know $N_{g,rs+df}$ if $r \nmid d$.
Proof: Indeed, let $\ell = \gcd(r,d) < r$ and let $(r_0, d_0) = (r,d)/\ell$. Then (r,d) and $\ell(1, r_0 d_0)$ have the same square and divisibility. By Property (a) thus $N_{g,rs+df} = N_{g,\ell(s+r_0 d_0)}$ which is known by induction. $\square$

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Proposition

Let $r \geq 3$. Let $f(q)$ be a modular form for $\Gamma_0(2)$ with Fourier expansion

$$f = a_0 + a_1 q^r + a_2 q^{2r} + a_3 q^{3r} + \dots$$

(so all Fourier coefficients vanish if r does not divide the exponent). Then $f = 0$.

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Proof of Main Theorem.

- Property (b): $F_{g,r} = \sum_d N_{g,rs+df} q^d$ is a quasi-modular form for $\Gamma_0(2)$.
- Conjectural answer $F_{g,r}^{KM}$ is quasi-modular for $\Gamma_0(2)$
- Lemma: $F_{g,r} - F_{g,r}^{KM}$ has Fourier coefficients only if r divides the exponent
- Proposition: $F_{g,r} - F_{g,r}^{KM} = 0$.

□

Proof of Proposition

A modular form for a congruence subgroup $\Gamma \subset \mathrm{SL}_2(\mathbb{Z})$ is a holomorphic function on the upper half plane $f : \mathbb{H} \to \mathbb{C}$ satisfying

$$f\left(\frac{a\tau + b}{c\tau + d}\right) = (c\tau + d)^k f(\tau) \tag{1}$$

for all $\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \Gamma$ (and a boundedness condition).

- If $\left(\begin{smallmatrix} 1 & 1 \\ 0 & 1 \end{smallmatrix}\right) \in \Gamma$ this says that $f(\tau + 1) = f(\tau)$, so f can be expanded in $q = e^{2\pi i \tau}$.
- If $f = a_0 + a_1 q^r + a_2 q^{2r} + \ldots$ then we also have $f(\tau + 1/r) = f(\tau)$.
- $\Rightarrow f$ satisfies (1) for all elements in $\widetilde{\Gamma} := \langle \Gamma_0(2), \left(\begin{smallmatrix} 1 & 1/m \\ 0 & 1 \end{smallmatrix}\right) \rangle$.
- Small check: $\widetilde{\Gamma}$ is dense in $\mathrm{SL}_2(\mathbb{R})$.
- $\Rightarrow f$ satisfies (1) for all elements in $\mathrm{SL}_2(\mathbb{R})$ by continuity.
- $\Rightarrow f = 0$.

$\square$

Rest of proof

It remains to prove the three properties we used:

Property (b): The partial series $F_{g,r} = \sum_{d \geq 0} \sum_{\alpha \in E_8(-1)} q^d \zeta^\alpha N_{g,rs+df+\alpha}$ are quasi-Jacobi forms for $\Gamma_0(2)$ of weight $2g-2$.

Proof: Use degeneration of the elliptic fibration $Y \to \mathbb{P}^1$ to a rational elliptic surface glued with $(\mathbb{P}^1 \times E)/\langle \mathrm{inv}_{\mathbb{P}^1}, -1 \rangle$. Then use work of Pixton and myself.

Property (a): $N_{g,\beta}$ depends on β only through the square $\beta \cdot \beta$ and the divisibility of β.

Proof: Use compact CY3 geometry and sheaves. This is next topic. □

Compact CY3 geometry

- Let $X \to Y$ be the K3 cover, let $\tau : X \to X$ be the covering involution.
- The Enriques Calabi-Yau threefold is

$$Q = (X \times E)/\mathbb{Z}_2$$

  where $\mathbb{Z}_2$ acts by $(x, e) \to (\tau(x), -e)$.
- The projection to the second factor

$$\pi : Q \to E/\mathbb{Z}_2 = \mathbb{P}^1$$

  is an isotrivial K3 fibration (with generic fiber X) and 4 double Enriques fibers (isomorphic to Y).
- The projection to the first factor

$$p : Q \to X/\mathbb{Z}_2 = Y$$

  is an isotrivial elliptic fibration with section.
- $H_2(Q, \mathbb{Z}) = H_2(Y, \mathbb{Z}) \oplus \mathbb{Z}[E]$

Finishing the proof

- Maulik-Pandharipande: For $\beta \in H_2(Y,\mathbb{Z})$, we have $N^Q_{g,\beta} = 4N^{K_Y}_{g,\beta}$
- Pandharipande-Pixton: The GW/PT correspondence holds for Q
- Let $\mathrm{DT}(v)$ be the generalized DT invariants counting semistable sheaves supported on fibers of $\pi : Q \to \mathbb{P}^1$ in class $v \in H^*(Y,\mathbb{Z})$.
- Theorem(O., based on work of Toda):

$$\sum_{\beta \in H_2(Y,\mathbb{Z})} \sum_{n \in \mathbb{Z}} \mathrm{PT}_{n,\beta}(-p)^n q^\beta = \prod_{\substack{r \geq 0 \\ \beta > 0 \\ n \geq 0}} \exp\left((n+r)\mathrm{DT}(r,\beta,n) q^\beta p^n\right)$$

$$\times \prod_{\substack{r > 0 \\ \beta > 0 \\ n > 0}} \exp\left((n+r)\mathrm{DT}(r,\beta,n) q^\beta p^{-n}\right)$$

- Prop(O., based on Toda) $\mathrm{DT}(gv) = \mathrm{DT}(v)$ for all autoequivalences of $D^b(Y)$
- Prop(O., based on Macri-Mehrotra-Stellari) $\mathrm{DMon}(Y) = O^+(H^*(X,\mathbb{Z})^\tau)$.
- $H^*(X,\mathbb{Z})^\tau \cong U \oplus U(2) \oplus E_8(-2)$
- Cor: $\mathrm{DT}(v)$ only depends on square $v \cdot v$, $\mathrm{div}(v)$ and $\mathrm{type}(v) \in \{\mathrm{odd}, \mathrm{even}\}$
- This implies $N^Q_{g,\beta}$ only depends on the square β^2 and the divisibility of β.

□

Further conjectures

What can we say about the Gromov-Witten theory of the Enriques Calabi-Yau?

> *The Enriques Calabi-Yau 3-fold may be the most tractable compact Calabi-Yau with nontrivial Gromov-Witten theory. Certainly the higher genus study of the quintic 3-fold in P4 appears more difficult.*
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The results so far determine the $N^{Q}_{g,(\beta,d)}$ in case $d = 0$.

Another viewpoint and a conjecture

Let $\pi : Y \to \mathbb{P}^1$ be an elliptic fibration on the Enriques surface.
We have the induced abelian surface fibration:

$$\rho : Q \xrightarrow{p} Y \xrightarrow{\pi} \mathbb{P}^1.$$

The generic fiber is the product of two elliptic curves $E \times F$, 12 fibers are of the form $E \times C$ for a nodal genus 1 curve C, and 2 double bielliptic fibers.

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Define the generating series of Pandharipande-invariants counting curves of degree ℓ over the base:

$$Z_\ell^Q(p,t,q,\zeta) = \sum_{d,e,\alpha} \sum_{n \in \mathbb{Z}} (-p)^n t^{e-\frac{1}{2}} q^d \zeta^\alpha \mathrm{PT}_{n,\ell s + df + \alpha + e[E]}.$$

In degree zero this is not difficult to evaluate (e.g. method of Bryan):

$$Z_0^Q(p,t,q,\zeta) = t^{-1/2} \prod_{k \geq 1} (1-t^k)^{-12} \cdot \prod_{d \geq 1} \left(\frac{1+q^d}{1-q^d} \right)^4.$$

$$= \eta(t)^{-12} \cdot \frac{\eta(2\tau)^4}{\eta(\tau)^8}.$$

In degree 1 we get the analoge of the Igusa cusp form formula for $K3 \times E$.

Define two Borcherds lifts

$$\chi_{10}(p,q,t) = pqt \prod_{(\ell,n,r)>0} (1 - p^r q^\ell t^n)^{c_1(4n\ell - r^2)}$$

$$\Phi_4(p,q,t) = pqt^{1/2} \prod_{(\ell,n,r)>0} (1 - p^r q^{2\ell} t^n)^{c_2(n\ell, r)}$$

where $(\ell, n, r) > 0$ stands for $n > 0$ or $\ell > 0$ or $(n = \ell = 0$ and $r < 0)$.

- χ_{10} is the well-known Igusa cusp form (a Siegel modular form for $\mathrm{Sp}_2(\mathbb{Z})$).
- Φ_4 is a cusp form of weight 4 for a level 2 paramodular group

Both product expansions were first found by Gritsenko-Nikulin (1995).

Let also $\vartheta_{E_8}(\zeta, q) = \sum_{\alpha \in E_8} \zeta^\alpha q^{\alpha^2/2}$ the theta function of the E_8-lattice.

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Conjecture (O. 2023)

$$Z_1^Q(p,t,q,\zeta) = 8 \frac{\Phi_4(p,q,t)}{\chi_{10}(p,q,t)} \vartheta_{E_8}(\zeta, q)$$

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Unfortunately, a formula for Z_ℓ^Q for $\ell \geq 2$ seems difficult to guess at this point.
Problem: $N_{g,\beta+d[E]}$ for $\beta \in H_2(Y,\mathbb{Z})$ does not only depend on β^2 and $\mathrm{div}(\beta)$.

The coefficients c_i in the last slide were defined by

$$\sum_{n,r} c_1(4n - r^2) p^r q^n = 24\Theta^2 \wp$$

$$\sum_{n,r} c_2(n, r) p^r q^n = \Theta^4 (12\wp^2 - 20 G_4)$$

where

$$\Theta(p, q) = (p^{1/2} - p^{-1/2}) \prod_{m \geq 1} \frac{(1 - pq^m)(1 - p^{-1} q^m)}{(1 - q^m)^2}$$

$$\wp(p, q) = \frac{1}{12} + \frac{p}{(1 - p)^2} + \sum_{d \geq 1} \sum_{k|d} k(p^k - 2 + p^{-k}) q^d$$

$$G_k(q) = -\frac{B_k}{2 \cdot k} + \sum_{n \geq 1} \sum_{d|n} d^{k-1} q^n$$

Thank you for the attention.