DONALDSON–THOMAS INVARIANTS OF ABELIAN THREEFOLDS AND BRIDGELAND STABILITY CONDITIONS

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Abstract. We study the reduced Donaldson–Thomas theory of abelian threefolds using Bridgeland stability conditions. The main result is the invariance of the reduced Donaldson–Thomas invariants under all derived autoequivalences, up to explicitly given wall-crossing terms. We also present a numerical criterion for the absence of walls in terms of a discriminant function. For principally polarized abelian threefolds of Picard rank one, the wall-crossing contributions are discussed in detail. The discussion yield evidence for a conjectural formula for curve counting invariants by Bryan, Pandharipande, Yin, and the first author.

For the proof we strengthen several known results on Bridgeland stability conditions of abelian threefolds. We show that certain previously constructed stability conditions satisfy the full support property. In particular, the stability manifold is non-empty. We also prove the existence of a Gieseker chamber and determine all wall-crossing contributions. A definition of reduced generalized Donaldson–Thomas invariants for arbitrary Calabi–Yau threefolds with abelian actions is given.

Contents

1. Introduction

1.1. **Overview.** Let X be a smooth projective Calabi–Yau threefold with an ample divisor H, and let Γ be the image of the Chern character map

$$\mathrm{ch}\colon K(X) \twoheadrightarrow \Gamma \subset H^{2*}(X, \mathbb{Q}).$$

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For every $v \in \Gamma$ consider the Donaldson–Thomas invariant

$$\mathrm{DT}_H(v) \in \mathbb{Q}.$$

If the moduli space $M_H(v)$ of H-Gieseker semistable sheaves of Chern character v consists of stable sheaves, then $\mathrm{DT}_H(v)$ is defined by

$$\mathrm{DT}_H(v) := \int_{M_H(v)} \nu \, \mathrm{de} := \sum_{k \in \mathbb{Z}} k \cdot e\left(\nu^{-1}(k)\right), \tag{1}$$

where $\nu : M_H(v) \to \mathbb{Z}$ is the Behrend function [Beh09] and $e(-)$ is the topological Euler characteristic. In general, $\mathrm{DT}_H(v)$ is defined via the motivic Hall algebra [JS12]. The invariants $\mathrm{DT}_H(v)$ enumerate (with weights) Gieseker semistable sheaves on the threefold.

An interesting question is the following: *Given a derived autoequivalence* $g \in \operatorname{Aut} D^b(X)$*, how are the Donaldson–Thomas invariants* $\mathrm{DT}_H(v)$ *and* $\mathrm{DT}_H(g_* v)$ *related?* For the dualizing functor and curve counting Donaldson–Thomas invariants such a relation was established in [Tod10, Bri11, Tod] and proved the rationality and functional equation part of the GW/DT correspondence conjecture [MNOP06]. Another instance is [OS2] where an autoequivalence on elliptically fibered Calabi–Yau threefolds yielded modular properties of generating series of Donaldson–Thomas invariants.

In this paper we answer the above question in full generality for the reduced[1] Donaldson–Thomas invariants of abelian threefolds. The results are strong constraints on these invariants, and may be leveraged later for their explicit computation. Our approach is based on Bridgeland stability conditions [Bri07] and wall-crossing techniques. In particular, this paper is the first instance that Bridgeland stability conditions of a compact Calabi–Yau threefold have been applied to Donaldson–Thomas theory in this context (earlier work either used weak/limit stability conditions, e.g. [Tod10, Bri11, Tod, OS2] mentioned above, or considered Bridgeland stability conditions for local surfaces, e.g. [Tod12, MT] for local K3 surfaces).

Abelian threefolds are 'simple' enough among all Calabi–Yau threefolds such that the technical difficulties regarding Bridgeland stability conditions can be overcome. Yet they are also 'complicated' enough for interesting phenomena to appear. We hope this case provides insights into the application of Bridgeland stability conditions to the Donaldson–Thomas theory of compact Calabi–Yau threefolds in general.

1.2. **Reduced Donaldson–Thomas invariants.** Let A be a non-singular abelian threefold over $\mathbb{C}$. With H and Γ as before, let $M_H(v)$ be the moduli space of H-Gieseker semistable sheaves on A of Chern character $v \in \Gamma$. The

[1]For an abelian threefold A with dual $\widehat{A} = \mathrm{Pic}^0(A)$, the group $A \times \widehat{A}$ acts on the moduli spaces $M_H(v)$ and forces the Donaldson–Thomas invariants (1) to vanish. The theory is only interesting after reduction, see Section 1.2.

product $A \times \widehat{A}$ acts on $M_H(v)$ by

$$(a, L) \cdot E = T_a^* E \otimes L$$

where $T_a \colon A \to A$ is the translation $x \mapsto x + a$.

We define *reduced* Donaldson–Thomas invariants $\mathrm{DT}_H(v) \in \mathbb{Q}$ which count $A \times \widehat{A}$-orbits of Gieseker semistable sheaves as follows.[2]

If the $A \times \widehat{A}$ action has finite stabilizers and $M_H(v)$ consists of H-Gieseker stable sheaves, following Gulbrandsen [Gul13] we define reduced Donaldson–Thomas invariants by integrating over the stack quotient:

$$\mathrm{DT}_H(v) := \int_{[\overline{M}_H(v)/(A \times \widehat{A})]} \nu \ \mathrm{d}e$$

where $\nu \colon [\overline{M}_H(v)/(A \times \widehat{A})] \to \mathbb{Z}$ is the Behrend function of the stack and the topological Euler characteristic is taken in the orbifold sense. For arbitrary $v \in \Gamma$ the reduced invariant $\mathrm{DT}_H(v)$ is defined via the $A \times \widehat{A}$-equivariant motivic Hall algebra, see Section 2.

1.3. **Autoequivalences.** A sheaf $E \in \mathrm{Coh}(A)$ is called *semihomogeneous* if its stabilizer group under the $A \times \widehat{A}$ action

$$\Xi(E) = \{(a, L) \in A \times \widehat{A} \ : \ T_a^* E \otimes L \cong E\} \tag{2}$$

is of dimension 3. Consider the subset of semihomogeneous classes

$$\mathcal{C} := \{\pm \mathrm{ch}(E) : E \text{ is a semihomogeneous sheaf } \} \subset \Gamma. \tag{3}$$

Let also $\chi : \Gamma \times \Gamma \to \mathbb{Z}$ be the Euler pairing on Γ.

We prove the following invariance property in Section 4.4.

Theorem 1.1. *Suppose $v \in \Gamma$ can not be written as $\gamma_1 + \gamma_2$ for any $\gamma_i \in \mathcal{C}$ with $\chi(\gamma_1, \gamma_2) \neq 0$. Then $\mathrm{DT}_H(v)$ is independent of H and*

$$\mathrm{DT}_H(g_* v) = \mathrm{DT}_H(v).$$

for every autoequivalence $g \in \mathrm{Aut}(D^b(A))$.

If $v \in \Gamma$ does not satisfy the assumption of Theorem 1.1, then $\mathrm{DT}_H(v)$ and $\mathrm{DT}_H(g_* v)$ are related by a wall-crossing formula. The wall-crossing formula depends only on the derived equivalence g and the possible ways in which v can be written as a sum of two semihomogeneous classes. The wall-crossing contributions are determined in Lemma 4.13. In particular, the precise wall-crossing formula can be worked out explicitly in any concrete case. An example of non-trivial wall-crossing is discussed in Theorem 1.3.

[2] We have chosen here the same notation for the reduced invariants as for the (standard) Donaldson–Thomas invariants defined in (1). However, from now on all our invariants are reduced, so this choice should not create confusion.

The assumption of Theorem 1.1 is often cumbersome to check in practice. We state a numerical criterion in its place. Consider the discriminant

$$\Delta : H^{2*}(A, \mathbb{Q}) \to \mathbb{Q},$$

that is the unique homogeneous degree 4 polynomial function which is invariant under the spin group and is normalized by $\Delta(1 + \mathbf{p}) = -1$. Here $\mathbf{p} \in H^6(A, \mathbb{Z})$ is the class of a point. We refer to Appendix A for details and an explicit formula in case $A = E_1 \times E_2 \times E_3$. We have the following.

Proposition 1.2. *Let $v \in \Gamma$. If $\Delta(v) \geq 0$, then v satisfies the assumption of Theorem 1.1.*

Proposition 1.2 is in perfect agreement with physical arguments by Sen on the behaviour of the partition function of 1/8 BPS dyones under change of stability: wall-crossing contributions can appear only for classes with negative discriminant, see [Sen08, Section 4].

1.4. **Principally polarized abelian threefolds of Picard rank one.** Let (A, H) be a principally polarized abelian threefold with $\rho(A) = 1$. By Mukai [Muk81] the group $\mathrm{SL}_2(\mathbb{Z})$ acts on $D^b(A)$ (modulo shifts) by

$$T = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \mapsto (-) \otimes \mathcal{O}_X(H), \quad S = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \mapsto \Phi_{\mathcal{P}}, \tag{4}$$

where $\Phi_{\mathcal{P}}$ is the Fourier-Mukai transform with kernel the normalized Poincaré line bundle on $A \times A$. Moreover, any autoequivalence acts by an element in $\mathrm{SL}_2(\mathbb{Z})$ (moduli shifts, translation and twisting by degree 0 line bundles).

The image of the Chern character map is

$$\Gamma = \mathbb{Z} \oplus \mathbb{Z}[H] \oplus \mathbb{Z}[H^2/2] \oplus \mathbb{Z}[H^3/6]. \tag{5}$$

Since the only semihomogeneous sheaves on A are vector bundles[3] or have 0-dimensional support the subset of semihomogeneous classes is

$$\mathcal{C} = \{ r(p^3, p^2 q, pq^2, q^3) : (p, q, r) \in \mathbb{Z}^3, r \neq 0, \gcd(p, q) = 1 \}.$$

For any $v = r(p^3, p^2 q, pq^2, q^3) \in \mathcal{C}$ define its slope by

$$\Theta(v) = \frac{q}{p} \in \mathbb{Q} \cup \{\infty\} \tag{6}$$

with the convention $\Theta(v) = \infty$ if $p = 0$. If $\gamma_1, \gamma_2 \in \Gamma$, then $\chi(\gamma_1, \gamma_2) \neq 0$ if and only if $\Theta(\gamma_1) \neq \Theta(\gamma_2)$. We have the following result.

Theorem 1.3. *Suppose $v = \gamma_1 + \gamma_2$ for some $\gamma_i \in \mathcal{C}$ with $\Theta(\gamma_1) < \Theta(\gamma_2)$, and let*

$$g = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}_2(\mathbb{Z}).$$

[3] If E is a semihomogeneous vector bundle, then $\mathrm{ch}(E) = r(E) \exp(c_1(E)/r(E))$ where $r(E)$ is the rank of E, see [Muk].

(i) *If* $-\frac{d}{c} \notin [\Theta(\gamma_1), \Theta(\gamma_2))$ *or* $c = 0$ *then*

$$\mathrm{DT}_H(g_* v) = \mathrm{DT}_H(v).$$

(ii) *If* $-\frac{d}{c} \in [\Theta(\gamma_1), \Theta(\gamma_2))$ *then*

$$\mathrm{DT}_H(v) - \mathrm{DT}_H(g_* v) = (-1)^{r_1 r_2 \alpha} r_1 r_2 \alpha^9 \Bigg(\sum_{\substack{k_1 \geq 1 \\ k_1 | r_1}} \frac{1}{k_1^2} \Bigg) \cdot \Bigg(\sum_{\substack{k_2 \geq 1 \\ k_2 | r_2}} \frac{1}{k_2^2} \Bigg)$$

where $\gamma_i = r_i(p_i^3, p_i^2 q_i, p_i q_i^2, q_i^3)$ *and* $\alpha = p_1 q_2 - p_2 q_1$.

1.5. **Curve counting.** As before let (A, H) be a principally polarized abelian threefold of Picard rank $\rho(A) = 1$. For any non-zero $(\beta, n) \in \mathbb{Z}^2$ define

$$\mathrm{DT}_{\beta,n} = \mathrm{DT}_H(1, 0, -\beta, -n).$$

The invariant $\mathrm{DT}_{\beta,n}$ enumerates algebraic curves $C \subset A$ with $[C] = \beta H^2/2$ and $\chi(\mathcal{O}_C) = n$ up to translation.

A conjecture for $\mathrm{DT}_{\beta,n}$ was proposed in [BOPY18, Section 7.6] as follows. Define the theta functions

$$\theta_2(q) = \sum_{n \in \mathbb{Z}} q^{(n+\frac{1}{2})^2}, \quad \theta_3(q) = \sum_{n \in \mathbb{Z}} q^{n^2}.$$

Let $\mathsf{a}(n) \in \mathbb{Z}$ be defined by the Fourier expansion

$$\sum_n \mathsf{a}(n) q^n = \frac{-16}{\theta_2(q)^4 \theta_3(q)} = -q^{-1} + 2 - 8q^3 + 12q^4 - 39q^7 + 56q^8 + \dots .$$

Let also $\mathsf{n}(\beta, k) = \sum_\delta \delta^2$ where δ runs over all positive divisors of $k, \beta, \beta^2/k$ and β^3/k^2 if these numbers are integers, and let $\mathsf{n}(\beta, k) = 0$ otherwise.

If $\beta < 0$, or $\beta = 0$ and $n < 0$ the invariant $\mathrm{DT}_{\beta,n}$ vanishes since the moduli space is empty. In all other cases we have the following.

Conjecture 1.4 ([BOPY18]). *Assume* $\beta > 0$, *or* $\beta = 0$ *and* $n > 0$. *Then*

$$\mathrm{DT}_{\beta,n} = (-1)^n \sum_{\substack{k \geq 1 \\ k | n}} \frac{1}{k} \mathsf{n}(\beta, k) \mathsf{a}\left(\frac{4\beta^3 - n^2}{k^2} \right) \tag{7}$$

We have the following corollary of Theorem 1.3.

Corollary 1.5. *Let* $(\beta, n) \in \mathbb{Z}^2$ *be non-zero, and suppose* (c, d) *is an integer solution of the equation* $d^3 - 3\beta c^2 d - nc^3 = 1$. *Define*

$$(\beta', n') = (d^2\beta + ncd + \beta^2 c^2, 6\beta^2 d^2 c + 6c^2 d\beta n + n + 2c^3 n^2 - 2c^3 \beta^3).$$

If $4\beta^3 - n^2 \geq 0$, *then* $\mathrm{DT}_{\beta,n} = \mathrm{DT}_{\beta',n'}$, *and moreover* $\mathrm{DT}_{\beta,n}$ *satisfies Conjecture 1.4 if and only if* $\mathrm{DT}_{\beta',n'}$ *does.*

In Corollary 1.5 the pairs (β, n) and (β', n') are related by a derived autoequivalence. The discriminant specializes to $\Delta = 4\beta^3 - n^2$.

Corollary 1.5 yields evidence for Conjecture 1.4. In particular, calculations for primitive curve classes (which are easier) yield informations for imprimitive curve classes. For example, for $(\beta, n) = (1,1)$ and $(c,d) = (1,2)$ we obtain the non-trivial relation

$$\mathrm{DT}_{7,37} = \mathrm{DT}_{1,1} = 8$$

where the last equality follows by a direct computation.

If Δ is negative, then $\mathrm{DT}_{\beta,n}$ and $\mathrm{DT}_{\beta',n'}$ differ by the wall-crossing contributions of Theorem 1.3. We have checked in many cases (using a computer program) that the right hand side of Conjecture 1.4 satisfies the same wall-crossing behaviour. This yields non-trivial evidence for Conjecture 1.4 also in the critical range where the discriminant is negative. We refer to Section 5.6 for further discussions and a proof of Corollary 1.5.

The constraints obtained from Theorem 1.1 are strongest for abelian threefolds with higher Picard number, since these have a large group of derived autoequivalences. The conjecture in [BOPY18, Section 7.6] applies to curve counting invariants of arbitrary abelian threefolds. It would be interesting to show the compatibility of the [BOPY18] conjecture with Theorem 1.1 in general. Another interesting direction is to use Theorem 1.1 to extend the [BOPY18] conjecture to arbitrary primitive vectors $v \in \Gamma$.

1.6. **Idea of the proof of Theorem 1.1.** Reduced Donaldson–Thomas invariants are defined by making the motivic Hall algebra and the integration map equivariant with respect to the action of $\mathbf{A} := A \times \hat{A}$.[4] The equivariant integration map (defined in Section 2.9) takes values in the ring

$$\mathbb{Q}[\mathbf{A}] = \bigoplus_{\substack{B \subset \mathbf{A} \text{ connected} \\ \text{abelian subvarieties}}} \mathbb{Q}\epsilon_B,$$

where the ring structure is defined in terms of the intersection of the subvarieties B. For example, if Z is a variety with $\mathbf{A}$-action and $Z_B \subset Z$ is the stratum of points whose stabilizers contain B with finite index, then its equivariant integral is the polynomial

$$\mathbf{e}(Z) = \sum_{B \subset \mathbf{A}} e([Z_B/(\mathbf{A}/B)])\epsilon_B.$$

Applying the integration map to moduli spaces of semistable sheaves (or certain linear combinations thereof) yields the Donaldson–Thomas polynomial $\mathbf{DT}_H(v) \in \mathbb{Q}[\mathbf{A}]$. Its coefficient of ϵ_0 is the reduced invariant $\mathrm{DT}_H(v)$. Similarly for every Bridgeland stability condition $\sigma \in \mathrm{Stab}(A)$ there is an invariant $\mathbf{DT}_\sigma(v) \in \mathbb{Q}[\mathbf{A}]$ counting σ-semistable objects of Chern character v.

[4] See also [OS] for equivariant Hall algebras and a definition in a simpler case.

For every autoequivalence g we have formally

$$\mathbf{DT}_{\sigma}(v) = \mathbf{DT}_{g_*\sigma}(g_* v). \tag{8}$$

In this paper we prove the following steps:

(i) Stability conditions on A constructed by Maciocia–Piyaratne [MP15, MP16] and Bayer–Macrí–Stellari [BMS16] satisfy the full support property. Hence they define a family in the stability manifold $\mathrm{Stab}(A)$. In particular $\mathrm{Stab}(A) \neq \varnothing$. The connected component $\mathrm{Stab}^{\circ}(A) \subset \mathrm{Stab}(A)$ which contains this family is called the main component.

(ii) $\mathrm{Stab}^{\circ}(A)$ is preserved by all autoequivalences.

(iii) (Gieseker chamber) For every H and v, there exist a $\sigma \in \mathrm{Stab}^{\circ}(A)$ such that $\mathbf{DT}_{\sigma}(v) = \mathbf{DT}_H(v)$.

(iv) If v can not be written as a sum of two semihomogeneous classes, then all wall-crossing contributions vanish. In particular, $\mathrm{DT}_{\sigma}(v)$ is independent of $\sigma \in \mathrm{Stab}^{\circ}(A)$.

We conclude

$$\mathbf{DT}_H(v) \overset{(iii)}{=} \mathbf{DT}_{\sigma}(v) \overset{(8)}{=} \mathbf{DT}_{g_*\sigma}(g_* v) \overset{(ii)+(iv)+(iii)}{=} \mathbf{DT}_H(g_* v) \qquad \square$$

1.7. **Plan of the paper.** In Section 2 we define the integration map for equivariant motivic hall algebras and reduced Donaldson–Thomas invariants. In Section 3 we prove the full support property for certain Bridgeland stability conditions on abelian threefolds and show the existence of a Gieseker chamber. In Section 4 we define reduced Donaldson–Thomas invariants for Bridgeland semistable objects, and discuss their wall-crossing behaviour. This leads to a proof of Theorem 1.1. In Section 5 we specialize to principally polarized abelian threefolds and prove Theorem 1.3. In Appendix A we discuss the discriminant function and spin representations.

1.8. **Conventions.** We always work over $\mathbb{C}$ and all schemes are assumed to be of finite type. Given an algebraic group G we let G° denote the connected component of G which contains the origin. For a derived auto-equivence $g \in \mathrm{Aut}\, D^b(X)$ we let g_* denote its induced action on cohomology.

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2. Reduced Donaldson–Thomas invariants

2.1. **Overview.** Let X be a smooth projective Calabi–Yau threefold equipped with an action of an abelian variety A. The product

$$\mathbf{A} = A \times \mathrm{Pic}^0(X)$$

acts on the moduli spaces of Gieseker semistable sheaves on X by translation by elements in A and tensor product with elements in $\mathrm{Pic}^0(X)$. The goal of Section 2 is to define reduced (generalized) Donaldson–Thomas invariants of X which count $\mathbf{A}$-orbits of Gieseker semistable sheaves. For abelian threefolds our definition generalizes work of Gulbrandsen [Gul13] and [OS]. However the definition is not special to abelian threefolds. A list of examples to keep in mind is the following:

- $X = A$ is an abelian threefold and $\mathbf{A} = A \times \widehat{A}$.
- $X = S \times E$ with S a K3 surface, E an elliptic curve, and $\mathbf{A} = E \times \widehat{E}$.
- $X = (S \times E)/G$ where S is a symplectic surface, E is an elliptic curve, and G is a finite group acting on S by symplectic automorphisms, on E by translation by torsion points, and such that the induced diagonal action on $S \times E$ is free. The E-action on $S \times E$ descends to an action on the quotient, and we can take $\mathbf{A} = E \times \widehat{E/G}$.
- X is a Calabi–Yau threefold with $h^{1,0}(X) > 0$, and $\mathbf{A} = \mathrm{Pic}^0(X)$.

In Section 2.2 and Section 2.4 we discuss equivariant Grothendieck groups of varieties and stacks respectively. This leads to the definition of the equivariant Hall algebra in Section 2.6. In Section 2.9 we begin the construction of the equivariant integration map.

2.2. **Equivariant Grothendieck group of varieties.** Let A be an abelian variety. Following [OS, Section 3] the A-equivariant Grothendieck group of varieties $K_0^A(\mathrm{Var})$ is the free abelian group generated by the classes

$$[X, a_X]$$

of a variety X with an A-action $a_X \colon A \times X \to X$, modulo the equivariant scissor relations

$$[X, a_X] = [Z, a_X|_Z] + [U, a_X|_U]$$

for every A-invariant closed subvariety $Z \subset X$ with $U = X \setminus Z$. Taking products of varieties with the induced diagonal A-action endows $K_0^A(\mathrm{Var})$ with the structure of a commutative ring with unit.

Consider the $\mathbb{Q}$-vector space

$$\mathbb{Q}[A] = \bigoplus_{B \subset A} \mathbb{Q}\epsilon_B$$

where B runs over all *connected* abelian subvarieties of A. We define a $\mathbb{Q}$-linear ring structure on $\mathbb{Q}[A]$ as follows. If connected abelian subvarieties $B_1, B_2 \subset A$ intersect transversely, i.e.

$$\operatorname{codim}(B_1 \cap B_2) = \operatorname{codim} B_1 + \operatorname{codim} B_2,$$

we set

$$\epsilon_{B_1} \cdot \epsilon_{B_2} = \left| \frac{B_1 \cap B_2}{(B_1 \cap B_2)^\circ} \right| \epsilon_{(B_1 \cap B_2)^\circ}.$$

If B_1, B_2 are not transverse we set

$$\epsilon_{B_1} \cdot \epsilon_{B_2} = 0.$$

Lemma 2.1. *$(\mathbb{Q}[A], \cdot)$ is an associative commutative algebra with unit ϵ_A.*

Proof. The key step is to prove associativity: Let $B_1, B_2, B_3 \subset A$ be connected. Then $(\epsilon_{B_1} \cdot \epsilon_{B_2}) \cdot \epsilon_{B_3}$ is non-zero if and only if

$$\operatorname{codim}(B_1 \cap B_2 \cap B_3) = \operatorname{codim}(B_1) + \operatorname{codim}(B_2) + \operatorname{codim}(B_3)$$

in which case we get

$$(\epsilon_{B_1} \cdot \epsilon_{B_2}) \cdot \epsilon_{B_3} = \left| \frac{B_1 \cap B_2 \cap B_3}{(B_1 \cap B_2 \cap B_3)^\circ} \right| \epsilon_{(B_1 \cap B_2 \cap B_3)^\circ}.$$

In particular, the right hand side is invariant under permutation. $\square$

Let X be a variety with A-action a_X. For any abelian subvariety $B \subset A$ let $X_B \subset X$ denote the (reduced) locally closed subscheme of points whose stabilizer contain B with finite index,

$$X_B = \{ x \in X : \operatorname{Stab}(x) \supset B,\ |\operatorname{Stab}(x)/B| < \infty \}.$$

The subscheme $X_B \subset X$ is A-invariant and the induced A-action on X_B descends to an A/B-action with finite stabilizers. The quotient stack

$$[X_B/(A/B)]$$

is hence Deligne–Mumford and its (topological) Euler characteristic is well-defined as a rational number.

We define the A-reduced Euler characteristic of the class $[X, a_X]$ by

$$\mathbf{e}([X, a_X]) := \sum_{B \subset A} e\big([X_B/(A/B)]\big)\epsilon_B \ \in \mathbb{Q}[A]$$

where the sum runs over all connected abelian subvarieties of A.

Lemma 2.2. *The $\mathbb{Q}$-linear map*

$$\mathbf{e} \colon K_0^A(\mathrm{Var}) \to \mathbb{Q}[A],\ \ [X, a_X] \mapsto \mathbf{e}([X, a_X])$$

is a ring homomorphism.

Proof. Since the A-reduced Euler characteristic respects the A-equivariant scissor relation, the map $\mathbf{e}$ is well-defined. We need to show it is a ring homomorphism. Let X_1, X_2 be varieties with A-actions. By a stratification argument we may assume $X_i = (X_i)_{B_i}$ for some connected abelian subvarieties $B_i \subset A$. With respect to the diagonal A-action we have

$$\mathrm{Stab}\big((x_1, x_2)\big) = \mathrm{Stab}(x_1) \cap \mathrm{Stab}(x_2)$$

for all $(x_1, x_2) \in X_1 \times X_2$, and hence

$$\mathbf{e}([X_1 \times X_2, a_{X_1 \times X_2}]) = c\, \epsilon_{(B_1 \cap B_2)^\circ}$$

for some $c \in \mathbb{Q}$. We need to show

$$c = \left| \frac{B_1 \cap B_2}{(B_1 \cap B_2)^\circ} \right| e\big([X_1/(A/B_1)]\big) e\big([X_2/(A/B_2)]\big)$$

if B_1 and B_2 are transverse, and $c = 0$ otherwise.

Consider the commutative diagram of rows of exact sequences of abelian groups,

$$\begin{tikzcd}
0 \arrow[r] & B_1 \cap B_2 \arrow[r] \arrow[d] & A \arrow[r] \arrow[d, "\Delta"] & A/(B_1 \cap B_2) \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & B_1 \times B_2 \arrow[r] & A \times A \arrow[r] & A/B_1 \times A/B_2 \arrow[r] & 0.
\end{tikzcd}$$

Since the left hand square is fibered the induced morphism

$$\alpha : (B_1 \times B_2)/(B_1 \cap B_2) \to A$$

is injective, and we obtain the exact sequence

$$0 \to A/(B_1 \cap B_2) \to A/B_1 \times A/B_2 \to \mathrm{Coker}(\alpha) \to 0.$$

The subvarieties B_1 and B_2 are transverse if and only if the addition map

$$B_1 \times B_2 \to A, (b_1, b_2) \mapsto b_1 + b_2$$

is surjective, hence if and only if $\mathrm{Coker}(\alpha) = 0$. If B_1 and B_2 are not transverse the quotient

$$\Big[(X_1 \times X_2) \Big/ (A/(B_1 \cap B_2)^\circ) \Big]$$

hence carries an action by the positive-dimensional abelian variety $\mathrm{Coker}(\alpha)$ and therefore its Euler characteristic is zero; this implies $c = 0$. If B_1 and

B_2 are transverse, we get $A/(B_1 \cap B_2) = A/B_1 \times A/B_2$ and so

$$\begin{aligned} c &= e\left(\left[(X_1 \times X_2)\Big/(A/(B_1 \cap B_2)^\circ)\right]\right) \\ &= \left|\frac{B_1 \cap B_2}{(B_1 \cap B_2)^\circ}\right| e\left(\left[(X_1 \times X_2)\Big/(A/(B_1 \cap B_2))\right]\right) \\ &= \left|\frac{B_1 \cap B_2}{(B_1 \cap B_2)^\circ}\right| e([X_1/(A/B_1)])e([X_2/(A/B_2)]). \end{aligned}$$ □

2.3. **Preliminaries on stacks.** We will follow Bridgeland [Bri11] for conventions on stacks. In particular, all stacks are assumed to be algebraic and locally of finite type with affine geometric stabilizers. Geometric bijections and Zariski fibrations of stacks are defined in [Bri11, Definition 3.1] and [Bri11, Definition 3.3]. Group actions on stacks are discussed in [Rom05].

2.4. **Equivariant Grothendieck group of stacks.** Let A be an abelian variety, and let $\mathcal{S}$ be an algebraic stack equipped with an A-action $a_{\mathcal{S}}$.

Definition 2.3. *The A-equivariant relative Grothendieck group of stacks $K_0^A(\mathrm{St}/\mathcal{S})$ is defined to be the $\mathbb{Q}$-vector space generated by the classes*

$$[\mathcal{X} \xrightarrow{f} \mathcal{S}, a_{\mathcal{X}}],$$

where $\mathcal{X}$ is an algebraic stack of finite type, $a_{\mathcal{X}}$ is an A-action on $\mathcal{X}$, and f is an A-equivariant morphism, modulo the following relations:

(a) *For every pair of stacks $\mathcal{X}_1$ and $\mathcal{X}_2$ with A-actions a_1 and a_2 respectively a relation*

$$[\mathcal{X}_1 \sqcup \mathcal{X}_2 \xrightarrow{f_1 \sqcup f_2} \mathcal{S}, a_1 \sqcup a_2] = [\mathcal{X}_1 \xrightarrow{f_1} \mathcal{S}, a_1] + [\mathcal{X}_2 \xrightarrow{f_2} \mathcal{S}, a_2]$$

where f_i $(i = 1, 2)$ are A-equivariant.

(b) *For every commutative diagram*

$$\begin{tikzcd} \mathcal{X}_1 \arrow[rr, "g"] \arrow[dr, "f_1"'] & & \mathcal{X}_2 \arrow[dl, "f_2"] \\ & \mathcal{S} & \end{tikzcd}$$

with all morphisms A-equivariant and g a geometric bijection a relation

$$[\mathcal{X}_1 \xrightarrow{f_1} \mathcal{S}, a_1] = [\mathcal{X}_2 \xrightarrow{f_2} \mathcal{S}, a_2].$$

(c) *Let $\mathcal{X}_1, \mathcal{X}_2, \mathcal{Y}$ be stacks equipped with A-actions a_1, a_2, a_Y respectively such that the stabilizer groups of a_1, a_2, a_Y at all $\mathbb{C}$ points have the same connected component, i.e.*

$$\mathrm{Stab}_{a_1}(x_1)^\circ = \mathrm{Stab}_{a_2}(x_2)^\circ = \mathrm{Stab}_{a_Y}(y)^\circ$$

for all $x_1 \in \mathcal{X}_1(\mathbb{C}), x_2 \in \mathcal{X}_2(\mathbb{C}), y \in \mathcal{Y}(\mathbb{C})$.

Then for every pair of A-equivariant Zariski fibrations

$$h_1 : \mathcal{X}_1 \to \mathcal{Y}, \quad h_2 : \mathcal{X}_2 \to \mathcal{Y}$$

with the same fibers and for every A-equivariant morphism $\mathcal{Y} \xrightarrow{g} \mathcal{S}$, a relation

$$[\mathcal{X}_1 \xrightarrow{g \circ h_1} \mathcal{S}, a_1] = [\mathcal{X}_2 \xrightarrow{g \circ h_2} \mathcal{S}, a_2]. \qquad \square$$

Remark 2.4. If A is the the trivial group Definition 2.3 specializes to the relative Grothendieck group of stacks defined by Bridgeland [Bri11, Definition 3.10]. In this case we will usually omit A from the notation, and will write $K_0(\mathrm{St}/\mathcal{S})$. We will follow the same convention throughout the section: the trivial abelian variety is omitted from the notation.

Remark 2.5. For any connected abelian subvariety $B \subset A$ the restriction of A-actions to B-actions induces a morphism

$$K_0^A(\mathrm{St}/\mathcal{S}) \to K_0^B(\mathrm{St}/\mathcal{S}).$$

In particular, if B is the trivial abelian variety,

$$\mathrm{Forg} : K_0^A(\mathrm{St}/\mathcal{S}) \to K_0(\mathrm{St}/\mathcal{S}),$$

is the map that forgets the equivariant structure.

2.5. **Non-equivariant Hall algebras.** Let X be a Calabi–Yau threefold, i.e. a non-singular projective threefold with $K_X = 0$. Let $\mathcal{M}$ be the stack of coherent sheaves on X. By [Bri11, 4.2] the Hall algebra of X is the group

$$H(X) := K_0(\mathrm{St}/\mathcal{M})$$

together with the associative product $*$ defined by extension of sheaves.

Consider the polynomial ring

$$\Lambda = K_0(\mathrm{Var})[\mathbb{L}^{-1}, (1 + \mathbb{L} + \cdots + \mathbb{L}^n)^{-1}, n \geq 1]$$

where $\mathbb{L} = [\mathbb{A}^1] \in K_0(\mathrm{Var})$ is the class of the affine line. The subalgebra of *regular classes* is the Λ-submodule

$$H_{\mathrm{reg}}(X) \subset H(X)$$

generated by all classes $[Z \to \mathcal{M}]$ where Z is a variety. In particular, $H_{\mathrm{reg}}(X)$ is closed under $*$-product. The quotient

$$H_{\mathrm{sc}}(X) = H_{\mathrm{reg}}(X)/(\mathbb{L} - 1)H_{\mathrm{reg}}(X)$$

is called the semi-classical limit and is commutative with respect to $*$. The Poisson bracket defined by

$$\{f, g\} := \frac{f * g - g * f}{\mathbb{L} - 1}, \quad f, g \in H_{\mathrm{sc}}(X) \tag{9}$$

makes $H_{\mathrm{sc}}(X)$ a Poisson algebra with respect to $(*, \{-, -\})$.

2.6. **Equivariant Hall algebras.** Let X be a Calabi–Yau threefold equipped with the action of an abelian variety A. The group

$$\mathbf{A} := A \times \mathrm{Pic}^0(X)$$

acts on the stack of coherent sheaves $\mathcal{M}$ on X by

$$(a, \mathcal{L}) \cdot E = T_a^* E \otimes \mathcal{L} \quad \text{for all } a \in A, \mathcal{L} \in \mathrm{Pic}^0(X), E \in \mathrm{Coh}(X).$$

The $\mathbf{A}$-equivariant motivic Hall algebra is the group

$$H^{\mathbf{A}}(X) := K_0^{\mathbf{A}}(\mathrm{St}/\mathcal{M}).$$

The product $*$ lifts canonically to an associative product on $H^{\mathbf{A}}(X)$ via the diagonal action, see [OS, Section 4.6]. The forgetful morphism of Remark 2.5,

$$\mathrm{Forg} : H^{\mathbf{A}}(X) \to H(X),$$

is a ring homomorphism with respect to this product.

Define the subalgebra of regular classes by

$$H^{\mathbf{A}}_{\mathrm{reg}}(X) := \mathrm{Forg}^{-1}(H_{\mathrm{reg}}(X)). \tag{10}$$

The semi-classical limit is the quotient

$$H^{\mathbf{A}}_{\mathrm{sc}}(X) = H^{\mathbf{A}}_{\mathrm{reg}}(X)/(\mathbb{L} - 1)H^{\mathbf{A}}_{\mathrm{reg}}(X).$$

By an argument parallel to [OS, Proposition 2] the algebra $H^{\mathbf{A}}_{\mathrm{sc}}(X)$ is commutative and the bracket $\{-,-\}$ defined in (9) lifts to a Poisson bracket on $H^{\mathbf{A}}_{\mathrm{sc}}(X)$.[5] Therefore $H^{\mathbf{A}}_{\mathrm{sc}}(X)$ is a Poisson algebra with respect to $(*, \{-,-\})$.

2.7. **Gieseker stability.** Let H be a fixed polarization on X. For a sheaf $E \in \mathrm{Coh}(X)$, its *Hilbert polynomial* is

$$\chi(E \otimes \mathcal{O}_X(mH)) = a_d m^d + a_{d-1} m^{d-1} + \cdots$$

where $a_i \in \mathbb{Q}$, $d = \dim \mathrm{Supp}(E)$ and a_d is a positive rational number. The *reduced Hilbert polynomial* is defined by

$$\overline{\chi}_H(E) := \frac{\chi(E \otimes \mathcal{O}_X(mH))}{a_d} \in \mathbb{Q}[m].$$

Let Γ be the image of the Chern character map

$$\Gamma := \mathrm{Im}\left(\mathrm{ch} \colon K(X) \to H^{2*}(X, \mathbb{Q})\right).$$

Since $\overline{\chi}_H(E)$ only depends on the Chern character of E, there is a map $\overline{\chi}_H \colon \Gamma \to \mathbb{Q}[m]$ such that $\overline{\chi}_H(E) = \overline{\chi}_H(\mathrm{ch}(E))$.

The reduced Hilbert polynomial is used in the definition of Gieseker stability as follows.

[5] The condition (c) in Definition 2.3 is used crucially here.

Definition 2.6. *An object $E \in \mathrm{Coh}(X)$ is H-Gieseker (semi)stable if it is pure and for any non-zero subsheaf $F \subsetneq E$, we have*

$$\overline{\chi}_H(F)(m) < (\leq)\ \overline{\chi}_H(E)(m)$$

for $m \gg 0$.

Let $\Gamma_+ \subset \Gamma$ be the set of Chern characters of coherent sheaves,

$$\Gamma_+ := \mathrm{Im}(\mathrm{ch}|_{\mathrm{Coh}(X)} \colon \mathrm{Coh}(X) \to \Gamma).$$

For any $v \in \Gamma_+$ let

$$\mathcal{M}_H(v) \subset \mathcal{M} \tag{11}$$

be the open substack of finite type parametrizing H-Gieseker semistable sheaves with Chern charecter v. For any fixed $\overline{\chi} \in \mathbb{Q}[m]$ consider the union

$$\mathcal{M}_H(\overline{\chi}) = \coprod_{\overline{\chi}_H(v) = \overline{\chi}} \mathcal{M}_H(v).$$

The Hall algebra of semistable sheaves with reduced Hilbert polynomial $\overline{\chi}$ is defined by

$$H(X, \overline{\chi}) := K_0(\mathrm{St}/\mathcal{M}_H(\overline{\chi})). \tag{12}$$

Since the category of H-Gieseker semistable sheaves with fixed reduced Hilbert polynomial is closed under extension, the natural inclusion map

$$H(X, \overline{\chi}) \hookrightarrow H(X) \tag{13}$$

is a ring homomorphism. As before the Hall algebra $H(X, \overline{\chi})$ has a subalgebra of regular classes (the Λ-module generated by all $[Z \to \mathcal{M}_H(\overline{\chi})]$ where Z is a variety) and a semi-classical limit. We have the natural inclusions[6]

$$H_{\mathrm{reg}}(X, \overline{\chi}) \subset H_{\mathrm{reg}}(X), \quad H_{\mathrm{sc}}(X, \overline{\chi}) \subset H_{\mathrm{sc}}(X). \tag{14}$$

Since (11) is $\mathbf{A}$-equivariant, there exists an $\mathbf{A}$-equivariant version of (12),

$$H^{\mathbf{A}}(X, \overline{\chi}) \subset H^{\mathbf{A}}(X).$$

Similarly one has $\mathbf{A}$-equivariant versions of (14),

$$H^{\mathbf{A}}_{\mathrm{reg}}(X, \overline{\chi}) \subset H^{\mathbf{A}}_{\mathrm{reg}}(X), \quad H^{\mathbf{A}}_{\mathrm{sc}}(X, \overline{\chi}) \subset H^{\mathbf{A}}_{\mathrm{sc}}(X).$$

2.8. **Poisson torus.** By the Riemann-Roch theorem, the Euler paring

$$\chi(E, F) := \sum_{i \in \mathbb{Z}} (-1)^i \dim \mathrm{Ext}^i(E, F), \quad E, F \in D^b(X)$$

descends to a unique bilinear form

$$\chi \colon \Gamma \times \Gamma \to \Gamma$$

[6]The subalgebra of regular classes could also be defined as the preimage of $H_{\mathrm{reg}}(X)$ under the inclusion (13), and similarly for the semi-classical limit.

which satisfies $\chi(E,F) = \chi(\mathrm{ch}(E), \mathrm{ch}(F))$. Consider the group

$$C^{\mathbf{A}}(X) := \bigoplus_{v \in \Gamma} \mathbb{Q}[\mathbf{A}] \cdot c_v.$$

An associative product $*$ and a Poisson bracket on $C^{\mathbf{A}}(X)$ are defined by

$$c_{v_1} * c_{v_2} := (-1)^{\chi(v_1,v_2)} c_{v_1+v_2}$$
$$\{c_{v_1}, c_{v_2}\} := (-1)^{\chi(v_1,v_2)} \chi(v_1,v_2) c_{v_1+v_2}.$$

Then $C^{\mathbf{A}}(X)$ is a Poisson algebra with respect to the above $(*, \{-,-\})$.

2.9. **Equivariant integration map: Overview.** Recall from [Bri11] the integration map

$$\mathcal{I} : H_{\mathrm{sc}}(X) \to C(X).$$

The map $\mathcal{I}$ is a Poisson algebra homomorphism with respect to $(*, \{-,-\})$ such that for every $Z \to \mathcal{M}(v)$ with Z a variety we have

$$\mathcal{I}([Z \xrightarrow{f} \mathcal{M}(v)]) = e(Z, f^*\nu) c_v = \left(\int_Z f^*\nu \ \mathrm{de} \right) c_v.$$

Here $\nu : \mathcal{M} \to \mathbb{Z}$ is the Behrend function [Beh09] and $\mathcal{M}(v) \subset \mathcal{M}$ is the substack of sheaves with Chern character v.

For each $\overline{\chi} \in \mathbb{Q}[m]$, let $H_{\mathrm{sc}}(X, \overline{\chi})$, $H^{\mathbf{A}}_{\mathrm{sc}}(X, \overline{\chi})$ be the semi-classical limits of Hall algebras of semistable sheaves with reduced Hilbert polynomial $\overline{\chi}$ as defined in Section 2.7. The integration map $\mathcal{I}$ restricts to the Poisson algebra homomorphism

$$\mathcal{I}\colon H_{\mathrm{sc}}(X, \overline{\chi}) \to C(X).$$

The goal of the next section is to define an equivariant integration map

$$\mathcal{I}^{\mathbf{A}} : H^{\mathbf{A}}_{\mathrm{sc}}(X, \overline{\chi}) \to C^{\mathbf{A}}(X)$$

which is a Poisson algebra homomorphism with respect to $(*, \{-,-\})$ such that

$$\mathcal{I}^{\mathbf{A}}([Z \xrightarrow{f} \mathcal{M}_H(v), a]) = \left(\sum_{B \subset \mathbf{A}} (-1)^{\dim \mathbf{A}/B} \epsilon_B \int_{[Z_B/(\mathbf{A}/B)]} f^*\nu \ \mathrm{d}e \right) c_v, \tag{15}$$

for every $\mathbf{A}$-equivariant map $Z \xrightarrow{f} \mathcal{M}_H(v)$, where Z is a variety.

If an equivariant regular class α can be written ($\mathbf{A}$-equivariantly) as a Λ-linear combination of classes $[Z_i \to \mathcal{M}, a_i]$ with Z_i varieties, then we may define $\mathcal{I}^{\mathbf{A}}(\alpha)$ directly using (15). However, by our definition of regular classes this only holds after forgetting the equivariant structure. Hence we need to proceed with more caution. We take the following four steps:

1. Integrate regular elements non-equivariantly over the fibers of the map $p : \mathcal{M}_H(\overline{\chi}) \to M_H(\overline{\chi})$, where $M_H(\overline{\chi})$ is the *good moduli space* of $\mathcal{M}_H(\overline{\chi})$.

2. Show the constructible function obtained from (1.) is $\mathbf{A}$-equivariant.
3. Integrate the constructible function of (1.) $\mathbf{A}$-equivariantly over $M_H(\overline{\chi})$ to get an element of $C^{\mathbf{A}}(X)$.
4. Check the integration maps of (1.) and (3.) preserve the Poisson structures. It follows that $\mathcal{I}^{\mathbf{A}}$ is a Poisson algebra homomorphism.

2.10. **Equivariant integration map: Construction.**

Step 1. Let

$$p\colon \mathcal{M}_H(v) \to M_H(v)$$

be the good moduli space of $\mathcal{M}_H(v)$, i.e. $M_H(v)$ is an algebraic space satisfying that p_* on coherent sheaves is exact and the induced morphism $\mathcal{O}_{M_H(v)} \to p_*\mathcal{O}_{\mathcal{M}_H(v)}$ is an isomorphism. The good moduli space $M_H(v)$ parametrizes S-equivalence classes of H-Gieseker semistable sheaves with Chern character v. The existence of $M_H(v)$ as a projective scheme is well-known from the GIT construction of $\mathcal{M}_H(v)$, see [Alp13, Example 8.7]. We set

$$p\colon \mathcal{M}_H(\overline{\chi}) \to M_H(\overline{\chi}) = \coprod_{\overline{\chi}_H(v)=\overline{\chi}} M_H(v). \tag{16}$$

Until the end of this section, we fix $\overline{\chi}$ and only consider classes $v \in \Gamma$ satisfying $\overline{\chi}_H(v) = \overline{\chi}$.

Let $\mathrm{Constr}(M_H(\overline{\chi}))$ be the space of $\mathbb{Q}$-valued constructible[7] functions on $M_H(\overline{\chi})$. Consider the map

$$p_* : H_{\mathrm{reg}}(X, \overline{\chi}) \to \mathrm{Constr}(M_H(\overline{\chi}))$$

defined by integration over fibers as follows: If

$$\alpha = \sum_i a_i [Z_i \xrightarrow{f} \mathcal{M}_H(v)] \in H_{\mathrm{reg}}(X, \overline{\chi})$$

for varieties Z_i and $a_i \in \mathbb{Q}$, then for every $x \in M_H(\overline{\chi})$ we let

$$p_*(\alpha)(x) := \mathrm{Coeff}_{c_v}\Big(\mathcal{I}(\iota_{x*}\iota_x^*\alpha)\Big) = \sum_i a_i \int_{Z_i|_{\mathcal{M}_x}} f^*\nu \, \mathrm{de},$$

where $\mathrm{Coeff}_{c_v}(-)$ denotes the coefficient of c_v, the map $\iota_x : \mathcal{M}_x \to \mathcal{M}_H(\overline{\chi})$ is the inclusion of the fiber of the map (16) over $x \in M_H(\overline{\chi})$, and we used the induced maps[8]

$$\iota_x^* : H_{\mathrm{reg}}(X, \overline{\chi}) \to K_0(\mathrm{St}/\mathcal{M}_x), \quad \iota_{x*} : K_0(\mathrm{St}/\mathcal{M}_x) \to H(X, \overline{\chi}).$$

---

[7]A function $f : \mathcal{X} \to \mathbb{Q}$ is constructible, if $f(\mathcal{X})$ is finite and for every $c \in f(\mathcal{X})$ the preimage $f^{-1}(c)$ is the union of a finite collection of finite type stacks. In particular, $f : M \to \mathbb{Q}$ constructible implies that $f|_{M_v}$ is non-zero only for finitely many $v \in \Gamma$, where $M_v \subset M$ is the component of sheaves with Chern character v.

[8]Since ι_x is representable, the composition $\iota_{x*}\iota_x^*$ preserves the subalgebra of regular classes.

Step 2. The $\mathbf{A}$-action on the stack $\mathcal{M}_H(\overline{\chi})$ descends to an $\mathbf{A}$-action on its good moduli space $M_H(\overline{\chi})$. Consider the subgroup of $\mathbf{A}$-invariant functions

$$\mathrm{Constr}^{\mathbf{A}}(M_H(\overline{\chi})) \subset \mathrm{Constr}(M_H(\overline{\chi})).$$

Lemma 2.7. *The image of the composition*

$$H^{\mathbf{A}}_{\mathrm{reg}}(X,\overline{\chi}) \xrightarrow{\mathrm{Forg}} H_{\mathrm{reg}}(X,\overline{\chi}) \xrightarrow{p_*} \mathrm{Constr}(M_H(\overline{\chi}))$$

lies in $\mathrm{Constr}^{\mathbf{A}}(M_H(\overline{\chi}))$. *Hence we have the commatative diagram*

$$\begin{tikzcd}
H^{\mathbf{A}}_{\mathrm{reg}}(X,\overline{\chi}) \arrow[r, "\mathrm{Forg}"] \arrow[d, "p_*^{\mathbf{A}}"] & H_{\mathrm{reg}}(X,\overline{\chi}) \arrow[d, "p_*"] \\
\mathrm{Constr}^{\mathbf{A}}(M_H(\overline{\chi})) \arrow[r, hook] & \mathrm{Constr}(M_H(\overline{\chi})).
\end{tikzcd}$$

with $p_*^{\mathbf{A}} = p_* \circ \mathrm{Forg}$.

Proof. Consider a regular equivariant class

$$[\mathcal{X} \xrightarrow{f} \mathcal{M}_H(v), a] \in H^{\mathbf{A}}_{\mathrm{reg}}(X,\overline{\chi})$$

where $\mathcal{X}$ is a stack, and let

$$\phi = p_*\mathrm{Forg}([\mathcal{X} \xrightarrow{f} \mathcal{M}_H(v), a]).$$

We need to show $\phi(a \cdot x) = \phi(x)$ for every $x \in M_H(\overline{\chi})$ and $a \in A$.

Since the Behrend function is invariant under the $\mathbf{A}$ action, by stratifying $\mathcal{X}$ we may assume $f^*\nu$ is constant on $\mathcal{X}$. We let $\mathcal{X}_x$ denote be the fiber of $p \circ f : \mathcal{X} \to M_H(\overline{\chi})$ over the point $x \in M_H(\overline{\chi})$. We need to compare the value of the integration map $\mathcal{I}$ applied to

$$[\mathcal{X}_x \to \mathcal{M}_H(v)],\, [\mathcal{X}_{a\cdot x} \to \mathcal{M}_H(v)] \in H_{\mathrm{reg}}(X,\overline{\chi}).$$

Since $\mathcal{X}$ carries an $\mathbf{A}$-action and $p \circ f$ is equivariant, translation by $a \in A$ yields an isomorphism of stacks

$$t_a : \mathcal{X}_x \xrightarrow{\cong} \mathcal{X}_{a\cdot x}.$$

The claim now follows directly from the following Lemma. □

Lemma 2.8. *Let* $[\mathcal{Y} \xrightarrow{f} \mathcal{M}_H(v)] \in H_{\mathrm{reg}}(X,\overline{\chi})$ *such that* $f^*\nu$ *is equal to a constant* $k \in \mathbb{Z}$. *Then the integral*

$$\mathcal{I}([\mathcal{Y} \xrightarrow{f} \mathcal{M}_H(v)])$$

only depends on k, *the class* v *and the isomorphism class of the stack* $\mathcal{Y}$.

Proof of Lemma 2.8. For a variety Y, let $P(Y)(u)$ be its virtual Poincaré polynomial. The stack $\mathcal{Y}$ admits a stratification whose strata is of the form

$[Y_i/\operatorname{GL}_{n_i}(\mathbb{C})]$. Then

$$P(\mathcal{Y})(u) = \sum_i \frac{P(Y_i)(u)}{P(\operatorname{GL}_{n_i}(\mathbb{C}))(u)} \in \mathbb{Q}(u)$$

is independent of a stratification (see [Joy07b, Theorem 4.10]), and we have

$$\mathcal{I}([\mathcal{Y} \xrightarrow{f} \mathcal{M}_H(v)]) = k \lim_{u \to -1} P(\mathcal{Y})(u) c_v$$

where the limit on the right hand side exist since $\mathcal{Y} \to \mathcal{M}_H(v)$ is regular. The right hand side only depends on $\mathcal{Y}$ and k and v, and not on f. $\square$

Step 3. Let $\phi : M_H(v) \to \mathbb{Q}$ be a constructible $\mathbf{A}$-invariant function. Then there exists a stratification

$$M_H(v) = \coprod_i Z_i$$

into $\mathbf{A}$-invariant subspaces Z_i such that

- Z_i is a variety,
- the restriction $\phi|_{Z_i}$ is constant of value $a_i \in \mathbb{Q}$,
- there exists a connected subgroup $B_i \subset \mathbf{A}$ such that $(Z_i)_{B_i} = Z_i$.

Such a stratification can be constructed along the lines of [Bri11, 2.4] and [OS, 3.3]. We define an integration map

$$J : \operatorname{Constr}^{\mathbf{A}}(M_H(\overline{\chi})) \to C^{\mathbf{A}}(X)$$

by sending the constructible function ϕ to

$$J(\phi) = \left(\sum_i (-1)^{\dim(\mathbf{A}/B_i)} a_i e([Z_i/(A/B_i)]\epsilon_{B_i}) \right) c_v. \tag{17}$$

Since any two such stratifications have a common refinement, the map J is well-defined.

Step 4. The direct sum map $\oplus \colon \mathcal{M}_H(\overline{\chi}) \times \mathcal{M}_H(\overline{\chi}) \to \mathcal{M}_H(\overline{\chi})$ descends to a map

$$\oplus \colon M_H(\overline{\chi}) \times M_H(\overline{\chi}) \to M_H(\overline{\chi}).$$

Define an associative product and a Poisson bracket on $\operatorname{Constr}(M_H(\overline{\chi}))$ by

$$f * g := \sum_{v_1, v_2} (-1)^{\chi(v_1,v_2)} \oplus_* (f_{v_1} \times g_{v_2}),$$
$$\{f, g\} := \sum_{v_1, v_2} \chi(v_1, v_2)(-1)^{\chi(v_1,v_2)} \oplus_* (f_{v_1} \times g_{v_2}),$$

for all $f, g \in \operatorname{Constr}(M_H(\overline{\chi}))$, where $f_v = f|_{M_H(v)}$ and similar for g, and we let

$$(f_{v_1} \times g_{v_2})(x_1, x_2) = f_{v_1}(x_1) g_{v_2}(x_2)$$

for all $x_1 \in M_H(v_1), x_2 \in M_H(v_2)$. By a direct check $\mathrm{Constr}(M_H(\overline{\chi}))$ is a Poisson algebra with respect to $(*, \{-,-\})$.

Since taking direct sums is $\mathbf{A}$-equivariant, the operations $*$ and $\{-,-\}$ preserve the space of $\mathbf{A}$-invariant functions and define a Poisson algebra structure on $\mathrm{Constr}^{\mathbf{A}}(M_H(\overline{\chi}))$.

Lemma 2.9. *The map of integration along fibers*

$$p_* : H_{\mathrm{sc}}(X, \overline{\chi}) \to \mathrm{Constr}(M_H(\overline{\chi}))$$

is a Poisson algebra homomorphism with respect to $(*, \{-,-\})$. *The same holds for the equivariant map*

$$p_*^{\mathbf{A}} : H_{\mathrm{sc}}^{\mathbf{A}}(X, \overline{\chi}) \to \mathrm{Constr}^{\mathbf{A}}(M_H(\overline{\chi})).$$

Proof. We first consider the non-equivariant case. We need to show that for all $\alpha_1, \alpha_2 \in H_{\mathrm{sc}}(X, \overline{\chi})$ we have

$$p_*(\alpha_1 * \alpha_2) = p_*(\alpha_1) * p_*(\alpha_2), \tag{18}$$

$$p_*(\{\alpha_1, \alpha_2\}) = \{p_*(\alpha_1), p_*(\alpha_2)\}, \tag{19}$$

Assume first that each α_i is supported over a point $x_i \in M_H(v_i)$, so in particular

$$p_*(\alpha_i) = a_i \delta_{x_i}, \quad i = 1, 2,$$

where $a_i \in \mathbb{Q}$ and we let δ_x is the characteristic function at the point x. Then $\alpha_1 * \alpha_2$ is supported over the point $x = x_1 \oplus x_2$ and hence

$$\begin{aligned} p_*(\alpha_1 * \alpha_2) &= \mathrm{Coeff}_{c_v}\left(\mathcal{I}(\alpha_1 * \alpha_2)\right) \delta_x \\ &= \mathrm{Coeff}_{c_v}\left(\mathcal{I}(\alpha_1) * \mathcal{I}(\alpha_2)\right) \delta_x \\ &= \left(a_1 a_2 (-1)^{\chi(v_1, v_2)}\right) \delta_x \\ &= p_*(\alpha_1) * p_*(\alpha_2), \end{aligned}$$

where we have set $v = v_1 + v_2$. Similarly,

$$\begin{aligned} p_*(\{\alpha_1, \alpha_2\}) &= \mathrm{Coeff}_{c_v}\left(\mathcal{I}(\{\alpha_1, \alpha_2\})\right) \delta_x \\ &= \mathrm{Coeff}_{c_v}\left(\{\mathcal{I}(\alpha_1), \mathcal{I}(\alpha_2)\}\right) \delta_x \\ &= \left(a_1 a_2 (-1)^{\chi(v_1, v_2)} \chi(v_1, v_2)\right) \delta_x \\ &= \{p_*(\alpha_1), p_*(\alpha_2)\}. \end{aligned}$$

For the general case let $\alpha_i = [X_i \to \mathcal{M}_H(v_i)]$ where X_i is a variety. Let $x \in M_H(v_1 + v_2)$ be a fixed point, and consider all possible decompositions

$$x = x_{1j} \oplus x_{2j}, \quad j = 1, \ldots, \ell$$

with $x_{ij} \in M_H(v_i)$ for $i = 1, 2$. Then, to compute the value of $p_*(\alpha_1 * \alpha_2)$ at x we may replace X_i by

$$\bigsqcup_{j=1}^{\ell} X_i|_{\mathcal{M}_{x_{ij}}}$$

By bilinearity of both sides of (18) we may further assume that $\ell = 1$, or equivalently, that there is only a unique decomposition $x = x_1 \oplus x_2$. But then we are in the case considered before and the claim follows. The argument for $\{-,-\}$ is parallel. This completes the non-equivariant case.

The equivariance case follows immediately: we have $p_*^{\mathbf{A}} = p_* \circ \mathrm{Forg}$, and Forg and p_* are both ring and Poisson algebra homomorphisms. □

Lemma 2.10. *The map*

$$J : \mathrm{Constr}^{\mathbf{A}}(M_H(\overline{\chi})) \to C^{\mathbf{A}}(X)$$

is a Poisson algebra homomorphism.

Proof. For every $i \in \{1, 2\}$, let

$$X_i \subset M_H(v_i)$$

be an $\mathbf{A}$-invariant subspace such that $(X_i)_{B_i} = X_i$ for some connected subgroup $B_i \subset \mathbf{A}$. We prove the claim for the A-invariant functions

$$\delta_{X_i} \in \mathrm{Constr}^{\mathbf{A}}(M_H(\overline{\chi})).$$

The general case follows by a stratification argument.

By definition we have

$$\delta_{X_1} * \delta_{X_2} = (-1)^{\chi(v_1,v_2)} \oplus_* (\delta_{X_1 \times X_2})$$

Applying J yields

$$\begin{aligned} J(\delta_{X_1} * \delta_{X_2}) &= (-1)^{\chi(v_1,v_2)+\dim(\mathbf{A}/B)} e\big([X_1 \times X_2/(\mathbf{A}/B)]\big)\epsilon_B c_{v_1+v_2} \\ &= (-1)^{\chi(v_1,v_2)+\dim(\mathbf{A}/B)} \mathbf{e}([X_1 \times X_2]) c_{v_1+v_2}, \end{aligned} \tag{20}$$

where $B = (B_1 \cap B_2)^\circ$ and $\mathbf{e}$ denotes the equivariant Euler characteristic.

On the other hand,

$$\begin{aligned} J(\delta_{X_i}) &= (-1)^{\dim(\mathbf{A}/B_i)} e([X_i/(\mathbf{A}/B_i)])\epsilon_{B_i} c_{v_i} \\ &= (-1)^{\dim(\mathbf{A}/B_i)} \mathbf{e}(X_i) c_{v_i}. \end{aligned}$$

By Section 2.2 we have

$$\mathbf{e}([X_1 \times X_2]) = \mathbf{e}(X_1)\mathbf{e}(X_2).$$

Hence if B_1 and B_2 are not transverse, then (20) and $J(\delta_{X_1}) * J(\delta_{X_2})$ both vanish. If B_1 and B_2 are transverse, then

$$\dim(\mathbf{A}/B) = \mathrm{codim}(B) = \mathrm{codim}(B_1) + \mathrm{codim}(B_2) = \dim(\mathbf{A}/B_1) + \dim(\mathbf{A}/B_2)$$

which gives the desired equality:

$$J(\delta_{X_1} * \delta_{X_2}) = J(\delta_{X_1}) * J(\delta_{X_2}).$$

The check that J preserves the Poisson bracket is parallel. $\square$

Definition 2.11. *The equivariant integration map is defined by*

$$\mathcal{I}^{\mathbf{A}} := J \circ p_*^{\mathbf{A}} : H_{\mathrm{sc}}^{\mathbf{A}}(X, \overline{\chi}) \to \mathrm{Constr}^{\mathcal{A}}(M_H(\overline{\chi})) \to C^{\mathbf{A}}(X).$$

We have the following result.

Theorem 2.12. *The equivariant integration map $\mathcal{I}^{\mathbf{A}}$ is a Poisson algebra homomorphism. Moreover, for every $\mathbf{A}$-equivariant map $f : Z \to \mathcal{M}_H(v)$, where Z is a variety, we have*

$$\mathcal{I}^{\mathbf{A}}([Z \xrightarrow{f} \mathcal{M}_H(v), a]) = \left(\sum_{B \subset \mathbf{A}} (-1)^{\dim(\mathbf{A}/B)} \epsilon_B \int_{[Z_B/(\mathbf{A}/B)]} f^* \nu \ \mathrm{d}e \right) c_v,$$

Proof. The first claim follows from Lemma 2.9 and 2.10. For the second we may assume Z is a $\mathbf{A}$-invariant subvariety of $\mathcal{M}_H(v)$, that $Z_B = Z$ for some connected abelian subvariety $B \subset \mathbf{A}$ and that the Behrend function is constant of value k on Z. Let $p_Z : Z \to Z' \subset M_H(v)$ be the restriction of $p : \mathcal{M}_H(v) \to M_H(v)$ to Z. Then

$$\begin{aligned} \mathcal{I}^{\mathbf{A}}([Z \to \mathcal{M}_H(v), a]) &= k(-1)^{\dim(\mathbf{A}/B)} \epsilon_B c_v \int_{[Z'/(\mathbf{A}/B)]} p_{Z*}(1) \ \mathrm{d}e \\ &= k(-1)^{\dim(\mathbf{A}/B)} \epsilon_B c_v \int_{[Z/(\mathbf{A}/B)]} 1 \ \mathrm{d}e. \end{aligned}$$ $\square$

2.11. **Definition of Donaldson-Thomas invariants.** As above let A be an abelian variety acting on a Calabi–Yau threefold X, and set $\mathbf{A} = A \times \mathrm{Pic}^0(X)$. The stack of semistable sheaves (11) defines an element

$$\delta_H(v) := [\mathcal{M}_H(v) \subset \mathcal{M}_H(\overline{\chi})] \in H^{\mathbf{A}}(X, \overline{\chi}).$$

Applying a formal logarithm defines the element

$$\epsilon_H(v) := \sum_{\substack{l \geq 1, v_1 + \cdots + v_l = v \\ \overline{\chi}_H(v_i) = \overline{\chi}}} \frac{(-1)^{l-1}}{l} \delta_H(v_1) * \cdots * \delta_H(v_l). \tag{21}$$

The following is the equivariant analog of a result of Joyce.

Proposition 2.13. $(\mathbb{L} - 1)\epsilon_H(v) \in H_{\mathrm{reg}}^{\mathbf{A}}(X, \overline{\chi})$.

Proof. By Joyce [Joy07a, Theorem 8.7] the element is regular after forgetting the equivariant structure. Hence it is regular by definition (10). $\square$

Define the class

$$\overline{\epsilon}_H(v) := [(\mathbb{L} - 1)\epsilon_H(v)] \in H_{\mathrm{sc}}^{\mathbf{A}}(X, \overline{\chi}).$$

Definition 2.14. *The* $\mathbf{A}$*-reduced Donaldson–Thomas invariant of* X *in class* $v \in \Gamma_+$ *is the unique element* $\mathbf{DT}_H(v) \in \mathbb{Q}[\mathbf{A}]$ *such that*

$$\mathcal{I}^{\mathbf{A}}(\overline{\epsilon}_H(v)) = \mathbf{DT}_H(v) \cdot c_v.$$

Remark 2.15. We expect $\mathbf{DT}_H(v) \in \mathbb{Q}[\mathbf{A}]$ to be invariant under deformations of X under which $v \in H^*(X, \mathbb{Q})$ remains algebraic. If v is primitive the deformation invariance property can be proven by constructing a slice of the $\mathbf{A}$-action, see [Gul13] for abelian threefolds and [Obe18] for K3 $\times E$.

It is convenient to define Donaldson-Thomas invariants for every $v \in \Gamma$ by the following convention:

- If $v \in -\Gamma_+$ define $\mathbf{DT}_H(v) := \mathbf{DT}_H(-v)$.
- If $v \notin \pm\Gamma_+$ define $\mathbf{DT}_H(v) := 0$.

For any $v \in \Gamma$ and connected abelian subvariety $B \subset \mathbf{A}$, we further define $\mathrm{DT}_H(v)_B \in \mathbb{Q}$ by the expansion

$$\mathbf{DT}_H(v) = \sum_{B \subset \mathbf{A}} \mathrm{DT}_H(v)_B \cdot \epsilon_B.$$

Moreover we write $\mathrm{DT}_H(v) := \mathrm{DT}_H(v)_{B=(0,0)}$.

Remark 2.16. Let $v \in \Gamma$. We expect that the stabilizer of an element $E \in \mathcal{M}_H(v)$ only depends on its Chern character and not on its moduli. In particular, for every $v \in \Gamma$ we expect to have $\mathbf{DT}_H(v) = \mathrm{DT}_H(v)_B \epsilon_B$ for a B determined by v. Partial results in this direction were obtained by Gulbrandsen, see [Gul13, Proposition 3.5].

3. Bridgeland stability conditions on abelian threefolds

3.1. **Review of stability conditions.** Let X be a smooth projective variety, and $D^b(X)$ its bounded derived category of coherent sheaves. Here we review Bridgeland stability conditions on $D^b(X)$. We fix a finitely generated free abelian group Λ, and a group homomorphism $\mathrm{cl} \colon K(X) \to \Lambda$.

Definition 3.1. ([Bri07]) *A stability condition on* $D^b(X)$ *with respect to* (Λ, cl) *is a pair*

$$\sigma = (Z, \mathcal{A}), \quad \mathcal{A} \subset D^b(X)$$

where $Z \colon \Lambda \to \mathbb{C}$ *is a group homomorphism and* $\mathcal{A}$ *is the heart of a bounded t-structure such that the following conditions hold:*

(i) *For any non-zero* $E \in \mathcal{A}$*, we have*

$$Z(E) := Z(\mathrm{cl}(E)) \in \{re^{\pi i \phi} : r > 0, \phi \in (0, 1]\}.$$

(ii) (Harder-Narasimhan property) *For any* $E \in \mathcal{A}$*, there is a filtration*

$$0 = E_0 \subset E_1 \subset \cdots \subset E_N$$

in $\mathcal{A}$ such that each subquotient $F_i = E_i/E_{i-1}$ is Z-semistable with $\arg Z(F_i) > \arg Z(F_{i+1})$.

(iii) (Support property) *There is a quadractic form Q on Λ such that* $Q(\mathrm{cl}(E)) \geq 0$ *for any Z-semistable object E and Q is negative definite on* $\mathrm{Ker}(Z)$.

Here an object $E \in \mathcal{A}$ is Z-*(semi)stable* if we have

$$\arg Z(F) < (\leq) \arg Z(E)$$

in $(0, \pi]$ for any subobject $0 \neq F \subsetneq E$.

For group homomorphisms $Z, Z' \colon \Lambda \to \mathbb{C}$, we write $Z \sim Z'$ if we have

$$\operatorname{Re} Z' = \lambda_1 \operatorname{Re} Z + \lambda_2 \operatorname{Im} Z,\ \operatorname{Im} Z' = \lambda_3 \operatorname{Im} Z$$

for some $\lambda_i \in \mathbb{R}$ with λ_1, λ_3 positive. Then $(Z, \mathcal{A})$ is a stability condition if and only if $(Z', \mathcal{A})$ is a stability condition, and Z-(semi)stable objects coincide with Z'-(semi)stable objects. In this case, we say that $(Z, \mathcal{A})$, $(Z', \mathcal{A})$ are equivalent and write $(Z, \mathcal{A}) \sim (Z', \mathcal{A})$.

Given a Bridgeland stability condition $\sigma = (Z, \mathcal{A})$ the category of σ-semistable objects with phase $\phi \in \mathbb{R}$ is defined in case $\phi \in (0, 1]$ by

$$\mathcal{P}(\phi) := \{E \in \mathcal{A} : E \text{ is } Z\text{-semistable with } Z(E) \in \mathbb{R}_{>0} e^{\pi i \phi}\} \cup \{0\}.$$

and for general $\phi \in \mathbb{R}$ by the condition

$$\mathcal{P}(\phi + 1) = \mathcal{P}(\phi)[1].$$

The data of a stability condition σ is equivalent to the data

$$(Z, \{\mathcal{P}(\phi)\}_{\phi \in \mathbb{R}}),\ Z \colon \Lambda \to \mathbb{C},\ \mathcal{P}(\phi) \subset D^b(X) \tag{22}$$

satisfying some properties, see [Bri07, Section 5] for details.

Let $\mathrm{Stab}_\Lambda(X)$ be the set of stability conditions on $D^b(X)$ with respect to (Λ, cl). By [Bri07] there is a natural topology on $\mathrm{Stab}_\Lambda(X)$ such that the forgetful map

$$\Pi \colon \mathrm{Stab}_\Lambda(X) \to \Lambda_{\mathbb{C}}^{\vee},\ (Z, \mathcal{A}) \mapsto Z$$

is a local homeomorphism.

Let $\Gamma \subset H^{2*}(X, \mathbb{Q})$ be the image of the Chern character map. We call the support property with respect to (Γ, ch) the *full support property.*[9] The space of stability conditions with respect to (Γ, ch) is denoted by

$$\mathrm{Stab}(X) := \mathrm{Stab}_\Gamma(X).$$

[9]This will be used in the following way. Suppose that $\sigma = (Z, \mathcal{A})$ is a stability condition with respect to (Λ, cl) and that $\mathrm{cl} : K(X) \to \Lambda$ factors through the Chern character map, i.e. $\mathrm{cl} = \mathrm{cl}' \circ \mathrm{ch}$ for some $\mathrm{cl}' : \Gamma \to \Lambda$. Then the pair $\sigma' = (Z \circ \mathrm{cl}', \mathcal{A})$ automatically satisfies conditions (i,ii) of Definition 3.1, but not necessarily the full support property (iii). Hence the stability condition σ induces a stability condition with respect to (Γ, ch) if and only if σ (or more precisely σ') satisfies the full support property.

Every (co-variant) autoequivalence $g \in \mathrm{Aut}(D^b(X))$ induces an action $g_* \in \mathrm{Aut}(\Gamma)$ which commutes with Chern character maps, see also Section 3.5 for further details. Therefore g also acts on $\mathrm{Stab}(X)$ by

$$g_*(Z, \mathcal{A}) := \big(g_* Z, g(\mathcal{A})\big), \tag{23}$$

where $g_* Z(-) := Z \circ g_*^{-1}(-)$. The induced action on the manifold $\mathrm{Stab}(X)$ is a homeomorphism, and the assignment $g \mapsto g_*$ defines a left $\mathrm{Aut}(D^b(X))$-action on $\mathrm{Stab}(X)$.

3.2. **Double tilting constructions.** Let X be a smooth projective 3-fold, and let $B + i\omega \in \mathrm{NS}(X)_{\mathbb{C}}$ with ω ample. The B-twisted Chern character of an object $E \in D^b(X)$ is defined by

$$\mathrm{ch}^B(E) := e^{-B}\,\mathrm{ch}(E) \in H^{2*}(X, \mathbb{R}).$$

For any $E \in K(X)$ let

$$\begin{aligned} Z_{\omega,B}(E) &:= -\int_X e^{-i\omega}\,\mathrm{ch}^B(E) \\ &= \left(-\mathrm{ch}_3^B(E) + \frac{1}{2}\mathrm{ch}_1^B(E)\omega^2\right) + i\left(\mathrm{ch}_2^B(E)\omega - \frac{1}{6}\mathrm{ch}_0^B(E)\omega^3\right). \end{aligned}$$

If X is an abelian 3-fold, we have

$$Z_{\omega,B}(E) = -\chi(e^{B+i\omega}, \mathrm{ch}(E)). \tag{24}$$

The homomorphism $Z_{\omega,B} \colon K(X) \to \mathbb{C}$ descends to a homomorphism

$$Z_{\omega,B} \colon \Gamma \to \mathbb{C}.$$

In [BMT14] a heart of a t-structure $\mathcal{A}_{\omega,B}$ was constructed as a candidate for a Bridgeland stability condition $(Z_{\omega,B}, \mathcal{A}_{\omega,B})$. We review the construction. Consider the B-twisted ω-slope function on $\mathrm{Coh}(X)$,

$$\mu_{\omega,B}(E) = \frac{\mathrm{ch}_1^B(E)\cdot\omega^2}{\mathrm{rank}(E)} \in \mathbb{R} \cup \{\infty\}.$$

It defines the usual slope stability on $\mathrm{Coh}(X)$. Define a torsion pair $(\mathcal{T}_{\omega,B}, \mathcal{F}_{\omega,B})$ on $\mathrm{Coh}(X)$ by

$$\begin{aligned} \mathcal{T}_{\omega,B} &= \langle E \in \mathrm{Coh}(X) : E \text{ is } \mu_{\omega,B}\text{-semistable with } \mu_{\omega,B}(E) > 0\rangle, \\ \mathcal{F}_{\omega,B} &= \langle E \in \mathrm{Coh}(X) : E \text{ is } \mu_{\omega,B}\text{-semistable with } \mu_{\omega,B}(E) \le 0\rangle, \end{aligned}$$

where we let $\langle * \rangle$ denote the extension closure. Its tilt is the heart

$$\mathcal{B}_{\omega,B} = \langle \mathcal{F}_{\omega,B}[1], \mathcal{T}_{\omega,B}\rangle \subset D^b(X).$$

The slope function $\nu_{\omega,B}$ on $\mathcal{B}_{\omega,B}$ is defined by

$$\nu_{\omega,B}(E) = \frac{\mathrm{Im} Z_{\omega,B}(E)}{\mathrm{ch}_1^B(E)\cdot\omega^2} \in \mathbb{R} \cup \{\infty\}.$$

It also defines the $\nu_{\omega,B}$-stability on $\mathcal{B}_{\omega,B}$. Similarly to above, the torsion pair $(\mathcal{T}'_{\omega,B}, \mathcal{F}'_{\omega,B})$ of $\mathcal{B}_{\omega,B}$ is defined by

$$\mathcal{T}'_{\omega,B} = \langle E \in \mathcal{B}_{\omega,B} : E \text{ is } \nu_{\omega,B}\text{-semistable with } \nu_{\omega,B}(E) > 0\rangle,$$
$$\mathcal{F}'_{\omega,B} = \langle E \in \mathcal{B}_{\omega,B} : E \text{ is } \nu_{\omega,B}\text{-semistable with } \nu_{\omega,B}(E) \leq 0\rangle.$$

By tilting a second time we obtain the heart

$$\mathcal{A}_{\omega,B} = \langle \mathcal{F}'_{\omega,B}[1], \mathcal{T}'_{\omega,B}\rangle \subset D^b(X).$$

In [BMT14] it was conjectured that the pairs

$$\sigma_{\omega,B} := (Z_{\omega,B}, \mathcal{A}_{\omega,B})$$

are Bridgeland stability conditions.

3.3. **Bogomolov-Gieseker inequalities.** In order to show that pairs $\sigma_{\omega,B}$ are stability conditions, and in particular satisfy the support property, we need to investigate quadractic inequalities for semistable objects. First we recall quadractic inequalities for $\nu_{\omega,B}$-semistable objects in $\mathcal{B}_{\omega,B}$.

Let H be a fixed ample divisor on X and consider the case $\omega = \alpha H$ for some $\alpha \in \mathbb{R}_{>0}$. By [BMT14], there is a constant $C_H \geq 0$ such that for every effective divisor D on X, we have

$$C_H(H^2D)^2 + (H^3)(HD^2) \geq 0.$$

If X is an abelian 3-fold, we can take $C_H = 0$. Let us also take $B \in \mathrm{NS}(X)_{\mathbb{R}}$ and for any $E \in D^b(X)$ define

$$\Delta(E) := (\mathrm{ch}_1(E))^2 - 2\,\mathrm{ch}_0(E)\,\mathrm{ch}_2(E),$$
$$\overline{\Delta}_{H,B}(E) := (H^2\,\mathrm{ch}_1^B(E))^2 - 2(H^3\,\mathrm{ch}_0^B(E))(H\,\mathrm{ch}_2^B(E)).$$

By the Hodge index theorem we have $\overline{\Delta}_{H,B}(E) \geq H^3 \cdot H\Delta(E)$ which is an equality when the Picard rank of X is one.

Proposition 3.2. ([BMT14]) *For any $\nu_{\omega,B}$-semistable object $E \in \mathcal{B}_{\omega,B}$, where $\omega = \alpha H$ for an ample divisor H and $\alpha \in \mathbb{R}_{>0}$, we have the inequlaities*

$$\overline{\Delta}_{H,B}(E) \geq 0, \qquad \textit{and} \qquad H^3 \cdot H\Delta(E) + C_H(H^2\,\mathrm{ch}_1^B(E))^2 \geq 0.$$

For any $E \in D^b(X)$ define

$$\overline{\nabla}_{H,B}(E) = 12(H^2\,\mathrm{ch}_1^B(E))^2 - 18(H^3\,\mathrm{ch}_0^B(E))(H\,\mathrm{ch}_2^B(E)).$$

The following conjecture is proposed in [BMT14, BMS16]:

Conjecture 3.3. ([BMT14, BMS16], [PT, Theorem 1.4]) *For any $\nu_{\omega,B}$-semistable object $E \in \mathcal{B}_{\omega,B}$, where $\omega = \alpha H$ for an ample divisor H and $\alpha \in \mathbb{R}_{>0}$, we have*

$$\alpha^2\overline{\Delta}_{H,B}(E) + \overline{\nabla}_{H,B}(E) \geq 0.$$

For fixed (H, B), let $\Lambda_{H,B} \subset \mathbb{R}^4$ be the free abelian group of rank 4 given by the image of the map

$$\mathrm{cl}\colon K(X) \to \mathbb{R}^4,\ E \mapsto (H^3 \operatorname{ch}_0^B(E), H^2 \operatorname{ch}_1^B(E), H \operatorname{ch}_2^B(E), \operatorname{ch}_3^B(E)).$$

The following result is proven in [BMS16, Theorem 8.6] when B is proportional to H, and the general case follows by a parallel argument.

Proposition 3.4. ([BMS16, Theorem 8.6]) *If Conjecture 3.3 holds for X and some $\alpha \in \mathbb{R}_{>0}$, then we have*

$$(Z^{a,b}_{\alpha H,B}, \mathcal{A}_{\alpha H,B}) \in \operatorname{Stab}_{\Lambda_{H,B}}(X), \tag{25}$$

where $Z^{a,b}_{\alpha H,B}$ is defined by

$$Z^{a,b}_{\alpha H,B} = \left(-\operatorname{ch}_3^B + bH \operatorname{ch}_2^B + aH^2 \operatorname{ch}_1^B\right) + i\left(\alpha H \operatorname{ch}_2^B - \frac{\alpha^3}{6} H^3 \operatorname{ch}_0^B\right) \tag{26}$$

with $a, b \in \mathbb{R}$ satisfying

$$a > \frac{\alpha^2}{18} + \frac{\sqrt{3}}{6}|b|\alpha. \tag{27}$$

Moreover, there is an interval $I^{a,b}_{\alpha} \subset (\alpha^2, 18a)$ such that for all $K \in I^{a,b}_{\alpha}$, the quadratic form defined by

$$Q_K(-) = K\overline{\Delta}_{H,B}(-) + \overline{\nabla}_{H,B}(-)$$

establishes the support property for the stability condition (25).

Conjecture 3.3 is known to hold for abelian threefolds A by [MP15, MP16, BMS16, PT]. Hence by Proposition 3.4 the pairs

$$\sigma_{\omega,B} = (Z^{a=\alpha^2/2, b=0}_{\alpha H,B}, \mathcal{A}_{\alpha H,B}),\ \omega = \alpha H$$

define Bridgeland stability conditions on A with respect to $(\Lambda_{H,B}, \mathrm{cl})$ and define points in $\operatorname{Stab}_{\Lambda_{H,B}}(A)$. In the following subsections we show that the pairs (25) are stability conditions also with respect to (Γ, ch). In particular, they form a family in $\operatorname{Stab}(A)$.

3.4. **Projection maps in cohomologies.** Let X be an n-dimensional smooth projective variety, and $H \in \mathrm{NS}_{\mathbb{Q}}(X)$ an ample class. Let

$$H^{2*}_{\mathrm{alg}}(X, \mathbb{Q}) \subset H^{2*}(X, \mathbb{Q})$$

be the subspace spanned by algebraic classes. We fix some notation on the projection maps on $H^{2*}_{\mathrm{alg}}(X, \mathbb{Q})$. For any i, we define

$$p_{H,i} : H^{2i}_{\mathrm{alg}}(X, \mathbb{Q}) \to H^{2i}_{\mathrm{alg}}(X, \mathbb{Q}), \quad \gamma_i \mapsto \frac{\gamma_i \cdot H^{n-i}}{H^n} H^i.$$

This gives us the map

$$p_H : H^{2*}_{\mathrm{alg}}(X, \mathbb{Q}) \to H^{2*}_{\mathrm{alg}}(X, \mathbb{Q}), \quad (\gamma_0, \dots, \gamma_n) \mapsto (p_{H,0}(\gamma_0), \dots, p_{H,n}(\gamma_n)).$$

Also we define

$$p_{H,i}^{\perp} : H^{2i}_{\mathrm{alg}}(X, \mathbb{Q}) \to H^{2i}_{\mathrm{alg}}(X, \mathbb{Q}), \quad \gamma_i \mapsto \gamma_i - p_{H,i}(\gamma_i),$$

and

$$p_H^{\perp} : H^{2*}_{\mathrm{alg}}(X, \mathbb{Q}) \to H^{2*}_{\mathrm{alg}}(X, \mathbb{Q}), \quad (\gamma_0, \dots, \gamma_n) \mapsto \left(p_{H,0}^{\perp}(\gamma_0), \dots, p_{H,n}^{\perp}(\gamma_n)\right).$$

We define

$$H^{2i}_{\mathrm{alg}}(X, \mathbb{Q})^{H,\|} = \mathrm{im}\left(p_{H,i}\right), \quad H^{2i}_{\mathrm{alg}}(X, \mathbb{Q})^{H,\perp} = \mathrm{im}\left(p_{H,i}^{\perp}\right),$$
$$H^{2*}_{\mathrm{alg}}(X, \mathbb{Q})^{H,\|} = \mathrm{im}\left(p_{H}\right), \quad H^{2*}_{\mathrm{alg}}(X, \mathbb{Q})^{H,\perp} = \mathrm{im}\left(p_{H}^{\perp}\right).$$

So we have

$$H^{2i}_{\mathrm{alg}}(X, \mathbb{Q}) = H^{2i}_{\mathrm{alg}}(X, \mathbb{Q})^{H,\|} \oplus H^{2i}_{\mathrm{alg}}(X, \mathbb{Q})^{H,\perp},$$
$$H^{2*}_{\mathrm{alg}}(X, \mathbb{Q}) = H^{2*}_{\mathrm{alg}}(X, \mathbb{Q})^{H,\|} \oplus H^{2*}_{\mathrm{alg}}(X, \mathbb{Q})^{H,\perp}.$$

By abuse of notation we will write p_H for $p_{H,i}$, and $p_H^{\perp}$ for $p_{H,i}^{\perp}$. We have

$$\mathrm{id} = p_H + p_H^{\perp}. \tag{28}$$

We write

$$\mathrm{ch}_i^{H,\|}(E) = p_H\left(\mathrm{ch}_i(E)\right), \ \mathrm{ch}_i^{H,\perp}(E) = p_H^{\perp}\left(\mathrm{ch}_i(E)\right).$$

Then we have $H^{n-i} \cdot \mathrm{ch}_i^{H,\|}(E) = H^{n-i} \cdot \mathrm{ch}_i(E)$, and $H^{n-i} \cdot \mathrm{ch}_i^{H,\perp}(E) = 0$. From the Hodge Index Theorem, we have

$$H^{n-2} \cdot \left(\mathrm{ch}_1^{H,\perp}(E)\right)^2 \leq 0. \tag{29}$$

Remark 3.5. Let $\Lambda_H^{\|}$ be the image of the composition

$$K(X) \xrightarrow{\mathrm{ch}} H^{2*}_{\mathrm{alg}}(X, \mathbb{Q}) \xrightarrow{p_H^{\|}} H^{2*}_{\mathrm{alg}}(X, \mathbb{Q})^{H,\|}.$$

If B is proportional to H, then the support properties for $(\Lambda_{H,B}, \mathrm{cl})$ and $(\Lambda_H^{\|}, p_H^{\|} \circ \mathrm{ch})$ are equivalent. So in Proposition 3.4, we obtain stability conditions in $\mathrm{Stab}_{\Lambda_H^{\|}}(X)$.

We define $\Lambda_H^{\sharp}$, $\Lambda_H^{\flat}$ to be the images of maps

$$\mathrm{cl}^{\sharp} \colon K(X) \to H^{2*}_{\mathrm{alg}}(X, \mathbb{Q}), \ E \mapsto (\mathrm{ch}_0(E), \mathrm{ch}_1(E), \mathrm{ch}_2^{H,\|}(E), \mathrm{ch}_3(E)),$$
$$\mathrm{cl}^{\flat} \colon K(X) \to H^{2*}_{\mathrm{alg}}(X, \mathbb{Q}), \ E \mapsto (\mathrm{ch}_0(E), \mathrm{ch}_1^{H,\|}(E), \mathrm{ch}_2(E), \mathrm{ch}_3(E)) \tag{30}$$

respectively. In the next lemma, we observe that stability conditions in Proposition 3.4 satisfy the support property with respect to the $(\Lambda_H^{\sharp}, \mathrm{cl}^{\sharp})$.

Lemma 3.6. *In the situation of Proposition 3.4, suppose that B is proportional to H and $C_H = 0$. Then the stability conditions (25) satisfy the support property with respect to $(\Lambda_H^{\sharp}, \mathrm{cl}^{\sharp})$. For an interval $I_{\alpha}^{a,b} \subset (\alpha^2, 18a)$*

and $K \in I_{\alpha}^{a,b}$*, the quadractic form is*

$$Q_K^{\sharp} = K\overline{\Delta}_{H,B}(-) + \overline{\nabla}_{H,B}(-) + (K-\alpha^2)H^3 \cdot H(\mathrm{ch}_1^{H,\perp}(-))^2.$$

Proof. The proof of [BMS16, Lemma 8.8] is applied, by replacing the inequality $\overline{\Delta}_{H,B}(-) \geq 0$ for $\nu_{\omega,B}$-semistable objects with the inequality (see Proposition 3.2)

$$H^3 \cdot H\Delta(-) = \overline{\Delta}_{H,B}(-) + H^3 \cdot H(\mathrm{ch}_1^{H,\perp}(-))^2 \geq 0.$$

Then the quadractic form $\alpha^2\overline{\Delta}_{H,B} + \overline{\nabla}_{H,B} + (K-\alpha^2)H^3 \cdot H\Delta$ gives the desired support property. □

3.5. **Fourier-Mukai transforms and abelian 3-folds.** Our strategy for the proof of the full support property is to use Fourier-Mukai transforms. Let us quickly recall some of the important notions in Fourier-Mukai theory. Further details can be found in [Huy06].

Let X, Y be smooth projective varieties and let p_i, $i = 1, 2$ be the projection maps from $X \times Y$ to X and Y, respectively. The *Fourier-Mukai functor* $\Phi_{\mathcal{E}}^{X \to Y} : D^b(X) \to D^b(Y)$ with kernel $\mathcal{E} \in D^b(X \times Y)$ is defined by

$$\Phi_{\mathcal{E}}^{X \to Y}(-) = \mathbf{R}p_{2*}(\mathcal{E} \overset{\mathbf{L}}{\otimes} p_1^*(-)).$$

When $\Phi_{\mathcal{E}}^{X \to Y}$ is an equivalence of the derived categories, usually it is called a *Fourier-Mukai transform.* Any Fourier-Mukai functor $\Phi_{\mathcal{E}}^{X \to Y} : D^b(X) \to D^b(Y)$ induces a linear map

$$\Phi_{\mathcal{E}}^{\mathrm{H}} \colon H^{2*}_{\mathrm{alg}}(X, \mathbb{Q}) \to H^{2*}_{\mathrm{alg}}(Y, \mathbb{Q}).$$

Here $H^{2*}_{\mathrm{alg}}(X, \mathbb{Q}) \subset H^{2*}(X, \mathbb{Q})$ is the subspace sppaned by algebraic classes. The above linear map is a linear isomorphism when $\Phi_{\mathcal{E}}^{X \to Y}$ is a Fourier-Mukai transform. The induced transform fits into the following commutative diagram, due to the Grothendieck-Riemann-Roch theorem.

$$\begin{tikzcd}
D^b(X) \arrow[r, "\Phi_{\mathcal{E}}^{X \to Y}"] \arrow[d, "v_X(-)"] & D^b(Y) \arrow[d, "v_Y(-)"] \\
H^{2*}_{\mathrm{alg}}(X, \mathbb{Q}) \arrow[r, "\Phi_{\mathcal{E}}^{\mathrm{H}}"] & H^{2*}_{\mathrm{alg}}(Y, \mathbb{Q}).
\end{tikzcd}$$

Here $v_X(-) = \mathrm{ch}(-)\sqrt{\mathrm{td}_X}$ is the Mukai vector map. Note that for an abelian variety X, $\mathrm{td}_X = 1$. Hence the Mukai vector $v(E)$ of $E \in D^b(X)$ is the same as its Chern character $\mathrm{ch}(E)$.

Let $X = A$ be an abelian variety, and $\widehat{A} = \mathrm{Pic}^0(A)$ its dual abelian variety. The *Poincaré line bundle* $\mathcal{P}$ on the product $A \times \widehat{A}$ is the uniquely determined line bundle satisfying (i) $\mathcal{P}_{A \times \{\widehat{x}\}} \in \mathrm{Pic}(A)$ is represented by $\widehat{x} \in \widehat{A}$, and (ii)

$\mathcal{P}_{\{e\}\times\widehat{A}} \cong \mathcal{O}_{\widehat{A}}$. In [Muk81], Mukai proved that the Fourier-Mukai functors

$$\Phi_{\mathcal{P}}^{A\to\widehat{A}} : D^b(A) \to D^b(\widehat{A}),\ \Phi_{\mathcal{P}^\vee}^{\widehat{A}\to A} : D^b(\widehat{A}) \to D^b(A)$$

are equivalences of derived categories, i.e. Fourier-Mukai transforms. Moreover, he proved that

$$\Phi_{\mathcal{P}^\vee}^{\widehat{A}\to A} \circ \Phi_{\mathcal{P}}^{A\to\widehat{A}} \cong \mathrm{id}[-n], \quad \Phi_{\mathcal{P}}^{A\to\widehat{A}} \circ \Phi_{\mathcal{P}^\vee}^{\widehat{A}\to A} \cong \mathrm{id}[-n],$$

where n is the dimension of A and $\widehat{A}$.

Let L be an ample line bundle on A. Its image under $\Phi_{\mathcal{P}}^{A\to\widehat{A}}$ is a semihomogeneous vector bundle[10] $\widehat{L}$ of rank $\chi(L) = c_1(L)^n/n!$,

$$\Phi_{\mathcal{P}}^{A\to\widehat{A}}(L) \cong \widehat{L}.$$

Moreover, $-c_1(\widehat{L})$ is an ample divisor class on $\widehat{A}$. See [BL99] for further details. We have the following:

Lemma 3.7 ([BL99])**.** *Let $H \in \mathrm{NS}_{\mathbb{Q}}(A)$ be an ample class on A. Under the induced cohomological transform $\Phi_{\mathcal{P}}^{\mathrm{H}} : H^{2*}_{\mathrm{alg}}(A,\mathbb{Q}) \to H^{2*}_{\mathrm{alg}}(\widehat{A},\mathbb{Q})$ of $\Phi_{\mathcal{P}}^{A\to\widehat{A}}$ we have*

$$\Phi_{\mathcal{P}}^{\mathrm{H}}(e^H) = (H^n/n!)\, e^{-\widehat{H}}$$

for some ample class $\widehat{H} \in \mathrm{NS}_{\mathbb{Q}}(\widehat{A})$, satisfying

$$(H^n/n!)(\widehat{H}^n/n!) = 1.$$

Moreover, for each $0 \le i \le n$, the induced cohomological transform gives rise to an isomorphism $\Phi_{\mathcal{P}}^{\mathrm{H}} : H^{2i}_{\mathrm{alg}}(A,\mathbb{Q}) \to H^{2(n-i)}_{\mathrm{alg}}(\widehat{A},\mathbb{Q})$, satisfying

$$\Phi_{\mathcal{P}}^{\mathrm{H}}\left(\frac{H^i}{i!}\right) = \frac{(-1)^{n-i}H^n}{n!(n-i)!}\,\widehat{H}^{n-i}.$$

Let H, $\widehat{H}$ be as in Lemma 3.7. We write

$$v_i(-) = i!H^{n-i}\cdot \mathrm{ch}_i(-), \qquad \widehat{v}_i(-) = i!\widehat{H}^{n-i}\cdot \mathrm{ch}_i(-). \tag{31}$$

For $\gamma = (\gamma_0,\dots,\gamma_n) \in H^{2*}(A,\mathbb{Q})$, we also write $v_i(\gamma) = i!H^{n-i}\gamma_i$ and similarly for $\widehat{v}_i(-)$. The following is a particular case of [Piy, Theorem 3.6].

Lemma 3.8. *We have the following equality for the induced cohomological transform $\Phi_{\mathcal{P}}^{\mathrm{H}} : H^{2*}_{\mathrm{alg}}(A,\mathbb{Q}) \to H^{2*}_{\mathrm{alg}}(\widehat{A},\mathbb{Q})$:*

$$\widehat{v}_i\left(\Phi_{\mathcal{P}}^{\mathrm{H}}(\gamma)\right) = \frac{(-1)^i n!}{H^n}\, v_{n-i}(\gamma).$$

We also have the following corollary:

[10] See Section 3.6 for more details on semihomogeneous bundles.

Corollary 3.9. *The induced cohomological transform* $\Phi^{\mathrm{H}}_{\mathcal{P}} : H^{2*}_{\mathrm{alg}}(A,\mathbb{Q}) \to H^{2*}_{\mathrm{alg}}(\widehat{A},\mathbb{Q})$ *of* $\Phi^{A\to\widehat{A}}_{\mathcal{P}}$ *fits into the following diagrams:*

$$\begin{array}{ccc} H^{2i}_{\mathrm{alg}}(A,\mathbb{Q}) & \xrightarrow{\Phi^{\mathrm{H}}_{\mathcal{P}}} & H^{2(n-i)}_{\mathrm{alg}}(\widehat{A},\mathbb{Q}) \\ \downarrow{\scriptstyle p_{H,i}} & & \downarrow{\scriptstyle p_{\widehat{H},n-i}} \\ H^{2i}_{\mathrm{alg}}(A,\mathbb{Q}) & \xrightarrow{\Phi^{\mathrm{H}}_{\mathcal{P}}} & H^{2(n-i)}_{\mathrm{alg}}(\widehat{A},\mathbb{Q}), \end{array} \qquad \begin{array}{ccc} H^{2i}_{\mathrm{alg}}(A,\mathbb{Q}) & \xrightarrow{\Phi^{\mathrm{H}}_{\mathcal{P}}} & H^{2(n-i)}_{\mathrm{alg}}(\widehat{A},\mathbb{Q}) \\ \downarrow{\scriptstyle p^{\perp}_{H,i}} & & \downarrow{\scriptstyle p^{\perp}_{\widehat{H},n-i}} \\ H^{2i}_{\mathrm{alg}}(A,\mathbb{Q}) & \xrightarrow{\Phi^{\mathrm{H}}_{\mathcal{P}}} & H^{2(n-i)}_{\mathrm{alg}}(\widehat{A},\mathbb{Q}). \end{array}$$

Proof. The first diagram is a direct consequence of Lemma 3.7. The second diagram follows from the relation (28). □

In the case of $n=3$, the Fourier-Mukai transform $\Phi^{A\to\widehat{A}}_{\mathcal{P}}$ with the Poincaré bundle as kernel preserves double tilt hearts as follows:

Lemma 3.10. ([Piy, Theorem 5.3]) *Suppose that* A *is an abelian 3-fold. Then for any* $t \in \mathbb{R}_{>0}$, *we have*

$$\Phi^{A\to\widehat{A}}_{\mathcal{P}}[1]\left(\mathcal{A}_{\sqrt{3}tH/2,tH/2}\right) = \mathcal{A}_{\sqrt{3}\widehat{H}/2t,-\widehat{H}/2t}$$

where $\widehat{H} \in \mathrm{NS}_{\mathbb{Q}}(\widehat{A})$ *is the induced ample class as in Lemma 3.7.*

3.6. **(Semi)homogeneous sheaves.** We recall (semi)homogeneous sheaves on abelian varieties, and study the effect of tensoring them to the stability. The arguments here will be also used in the proof of full support property.

A vector bundle E on an abelian variety A is called *homogeneous* if we have $T^*_x E \cong E$ for all $x \in A$.

Proposition 3.11 ([Muk78])**.** *A vector bundle* E *on* A *is homogeneous if and only if* E *can be filtered by line bundles from* $\mathrm{Pic}^0(A)$.

For a coherent sheaf E on A, we define

$$\Xi(E) := \{(x,L) \in A \times \widehat{A} : T^*_x E \otimes L \cong E\}. \tag{32}$$

By [Muk, Proposition 4.5], we have $\dim \Xi(E) \leq n$, where n is the dimension of A. A coherent sheaf E on A is *semihomogeneous* if $\dim \Xi(E) = n$. If E is a vector bundle, this is equivalent to that for every $x \in A$ there exists a flat line bundle $\mathcal{P}_{A\times\{\widehat{x}\}}$ on A such that $T^*_x E \cong E \otimes \mathcal{P}_{A\times\{\widehat{x}\}}$. Also a coherent sheaf E is called *simple* if we have $\mathrm{End}_A(E) \cong \mathbb{C}$.

Lemma 3.12 ([Muk78, Theorem 5.8])**.** *Let* E *be a simple vector bundle on an* n*-dimensional abelian variety* A. *Then the following conditions are equivalent:*

(i) $\dim H^1(A, \mathcal{E}nd(E)) = n$,
(ii) E *is semihomogeneous,*
(iii) $\mathcal{E}nd(E)$ *is a homogeneous vector bundle.*

Lemma 3.13 ([Muk78, Orl02]). *The following holds.*

(i) *A rank r simple semihomogeneous bundle E has the Chern character*

$$\mathrm{ch}(E) = r \cdot e^{c_1(E)/r}.$$

(ii) *For any $D_A \in \mathrm{NS}_{\mathbb{Q}}(A)$, there exists a simple semihomogeneous bundle E on A with $\mathrm{ch}(E) = r \cdot e^{D_A}$ for some $r \in \mathbb{Z}_{>0}$.*

(iii) *Let E be a semihomogeneous bundle on A. Then E is Gieseker semistable with respect to any ample line bundle L, and if E is simple then it is slope stable with respect to $c_1(L)$.*

Below we assume that A is an abelian 3-fold. Let $\omega, B \in \mathrm{NS}_{\mathbb{Q}}(A)$ such that ω is an ample class.

Proposition 3.14. *Let V be a simple semihomogeneous bundle on A and let*

$$D = \frac{c_1(V)}{\mathrm{rk}(V)}.$$

Then we have the following:

(i) *$E \in \mathrm{Coh}(A)$ is $\mu_{\omega,B}$-semistable if and only if $E \otimes V$ is $\mu_{\omega,B+D}$ semistable.*

(ii) *$E \in \mathcal{B}_{\omega,B}$ is $\nu_{\omega,B}$-semistable if and only if $E \otimes V \in \mathcal{B}_{\omega,B+D}$ is $\nu_{\omega,B+D}$-semistable.*

(iii) *$E \in \mathcal{A}_{\omega,B}$ is $\sigma_{\omega,B}$-semistable if and only if $E \otimes V \in \mathcal{A}_{\omega,B+D}$ is $\sigma_{H,B+D}$-semistable.*

Proof. (i) This follows from the fact that slope semistability is preserved under tensoring by semistable vector bundles and from Lemma 3.13 the simple semihomogeneous bundle V is slope stable.

(ii) From part (i), we have $\mathcal{B}_{\omega,B} \otimes V \subset \mathcal{B}_{\omega,B+D}$; so $E \otimes V \in \mathcal{B}_{\omega,B+D}$. From Lemma 3.13,

$$\mathrm{ch}(V) = \mathrm{rk}(V) \cdot e^{D},$$

so $\mathrm{ch}^{B+D}(E \otimes V) = \mathrm{rk}(V)\,\mathrm{ch}^{B}(E)$. Hence

$$\nu_{\omega,B+D}(E \otimes V) = \nu_{\omega,B}(E). \tag{33}$$

Suppose for a contradiction $E \otimes V \in \mathcal{B}_{\omega,B+D}$ is not $\nu_{\omega,B+D}$ semistable; so the following destabilizing short exact sequence exists in $\mathcal{B}_{\omega,B+D}$:

$$0 \to P \to E \otimes V \to Q \to 0.$$

By tensoring with the dual $V^{\vee}$ we get the following short exact sequence exists in $\mathcal{B}_{\omega,B}$:

$$0 \to P \otimes V^{\vee} \to E \otimes \mathcal{E}nd(V) \to Q \otimes V^{\vee} \to 0. \tag{34}$$

From Lemma 3.12, the bundle $\mathcal{E}nd(V) = V \otimes V^{\vee}$ is a homogeneous bundle, and from Proposition 3.11 it can be filtered by line bundles $\{L_j\}$ from

$\mathrm{Pic}^0(A)$. Therefore, $E \otimes \mathcal{E}nd(V) \in \mathcal{B}_{\omega,B}$ is filtered by $\nu_{\omega,B}$-semistable objects $\{E \otimes L_j\}$ in $\mathcal{B}_{\omega,B}$; hence, $E \otimes \mathcal{E}nd(V) \in \mathcal{B}_{\omega,B}$ is $\nu_{\omega,B}$-semistable. However, according to (33), the short exact sequence (34) destabilizes $E \otimes \mathcal{E}nd(V)$. This is the required contradiction.

(iii) From part (ii), we have $\mathcal{A}_{\omega,B} \otimes V \subset \mathcal{A}_{\omega,B+D}$; so $E \otimes V \in \mathcal{A}_{\omega,B+D}$. Then the rest of the proof is similar to part (ii). $\square$

3.7. **Full support property via FM transforms on abelian 3-folds.** Let A be an abelian 3-fold and $H \in \mathrm{NS}_{\mathbb{Q}}(A)$ be an ample class. Let v_i be the vectors as in (31), and consider the following form of central charge functions

$$W_{H,t}^{p,q} = (-v_3 + qv_2 + pv_0) + i\,(v_2 - tv_1)$$

for $t, p, q \in \mathbb{R}$.

Proposition 3.15. *Let $t \neq 0$, and $a, b \in \mathbb{R}$. Then we have the following:*

$$Z^{a,b}_{\sqrt{3}|t|H/2, tH/2} \sim W^{p,q}_{H,t}$$

for some $p, q \in \mathbb{R}$. Here $\alpha = \sqrt{3}|t|/2, a, b$ satisfy (27), that is $a > (t^2/24) + (|tb|/4)$, if and only if t, p, q satisfy $t(t-q) < \frac{p}{t} < 0$.

Proof. From the definition of v_i and $\mathrm{ch}^{tH/2}(-) = e^{-tH/2}\,\mathrm{ch}(-)$, we have

$$H^3 \mathrm{ch}_0^{tH/2} = v_0,\ H^2 \mathrm{ch}_1^{tH/2} = v_1 - tv_0/2,\ 2H \mathrm{ch}_2^{tH/2} = v_2 - tv_1 + t^2 v_0/4,$$
$$6\,\mathrm{ch}_3^{tH/2} = v_3 - 3tv_2/2 + 3t^2 v_1/4 - t^3 v_0/8.$$

Now, by direct substitution one can check that

$$\begin{aligned}
&Z^{a,b}_{\sqrt{3}|t|H/2, tH/2} \\
&= \left(-\mathrm{ch}_3^{tH/2} + bH\,\mathrm{ch}_2^{tH/2} + aH^2\,\mathrm{ch}_1^{tH/2}\right) + i\frac{t}{2}\left(H\,\mathrm{ch}_2^{tH/2} - \frac{t^2}{8} H^3\,\mathrm{ch}_0^{tH/2}\right) \\
&= \frac{1}{6}\left(-v_3 + qv_2 + pv_0 + r(v_2 - tv_1)\right) + i\frac{t}{4}\,(v_2 - tv_1) \\
&\sim W^{p,q}_{H,t},
\end{aligned}$$

where

$$q = \frac{3t}{4} + \frac{6a}{t},\ p = -3at + \frac{3bt^2}{4} + \frac{t^3}{8},\ r = \frac{3t}{8} + 3b - \frac{6a}{t}.$$

By straightforward computation one can check that $|t|, a, b$ satisfy $a > (t^2/24) + (|tb|/4)$, if and only if t, p, q satisfy $t(t-q) < \frac{p}{t} < 0$. $\square$

Consequently, we get the following particular case of Proposition 3.4 and Lemma 3.6 in an alternative form.

Proposition 3.16. *Let the numbers $t, p, q \in \mathbb{R}$ satisfy*

$$t \neq 0,\ t(t-q) < \frac{p}{t} < 0. \tag{35}$$

Then the pair

$$\left(W^{p,q}_{H,t}, \mathcal{A}_{\sqrt{3}|t|H/2, tH/2}\right)$$

defines a Bridgeland stability condition on A with respect to $(\Lambda^{\sharp}_{H}, \mathrm{cl}^{\sharp})$.

Let us write

$$\Psi := \Phi^{A\to\widehat{A}}_{\mathcal{P}}[1] : D^b(A) \to D^b(\widehat{A}),\ \widehat{\Psi} := \Phi^{\widehat{A}\to A}_{\mathcal{P}^{\vee}}[2] : D^b(\widehat{A}) \to D^b(A).$$

Then $\widehat{\Psi}$ is the quasi inverse of Ψ, and Ψ is the quasi inverse of $\widehat{\Psi}$. Recall that $\Psi_* W^{p,q}_{H,t} : K(\widehat{A}) \to \mathbb{C}$ is the function defined by

$$\Psi_* W^{p,q}_{H,t}(-) = W^{p,q}_{H,t}(\widehat{\Psi}(-)).$$

Let $\widehat{H}$ be the induced ample divisor on $\widehat{A}$ as in Lemma 3.7, and $\widehat{v}_i$ be the vectors as in (31).

Proposition 3.17. *Let $t, p, q \in \mathbb{R}$ such that $t > 0$. We have*

$$\Psi_* W^{p,q}_{H,t} \sim W^{p',q'}_{\widehat{H},t'}$$

for some $t', p', q' \in \mathbb{R}$ defined by

$$t' = -\frac{1}{t} < 0,\ p' = -\frac{1}{p},\ q' = \frac{tq}{p}. \tag{36}$$

Moreover, if $\{t > 0, p, q\}$ satisfies (35), then $\{t' < 0, p', q'\}$ also satisfies (35).

Proof. From Lemma 3.7, we have

$$v_i(\widehat{\Psi}(-)) = (-1)^i \frac{H^3}{6}\ \widehat{v}_{3-i}(-).$$

Hence

$$\operatorname{Im}\left(\Psi_* W^{p,q}_{H,t}\right) = \frac{H^3}{6}\left(\widehat{v}_1 + t\widehat{v}_2\right) = \frac{H^3 t}{6} \operatorname{Im} W^{p',q'}_{\widehat{H},t'},$$
$$\operatorname{Re}\left(\Psi_* W^{p,q}_{H,t}\right) = \frac{H^3}{6}\left(\widehat{v}_0 + q\widehat{v}_1 + p\widehat{v}_3\right) = \frac{H^3}{6}\left(-p \cdot \operatorname{Re} W^{p',q'}_{\widehat{H},t'} + tq \cdot \operatorname{Im} W^{p',q'}_{\widehat{H},t'}\right).$$

Therefore the first claim holds. By direct computation one can check that if $\{t > 0, p, q, r\}$ satisfies (35), then we have

$$t'(t'-q') < \frac{p'}{t'} < 0.$$

That is (35) holds for $\{t' < 0, p', q'\}$. ☐

For $t \in \mathbb{R}_{>0}$, by Lemma 3.10, Proposition 3.17 and Proposition 3.15, we have the following:

Lemma 3.18. *Let $t > 0, p, q \in \mathbb{R}$ satisfy* (35)*. Then we have the following equivalence of Bridgeland stability conditions:*

$$\Psi_* \left(W^{p,q}_{H,t}, \mathcal{A}_{\sqrt{3}tH/2, tH/2} \right) \sim \left(W^{p',q'}_{\widehat{H},t'}, \mathcal{A}_{-\sqrt{3}t'\widehat{H}/2, t'H/2} \right)$$

for $t' < 0, p', q' \in \mathbb{R}$ defined as in (36) *satisfying* (35)*.*

Consequently we prove the following:

Lemma 3.19. *If $t > 0, p, q \in \mathbb{R}$ satisfy* (35)*, then the Bridgeland stability condition defined by the pair*

$$\left(W^{p,q}_{H,t}, \mathcal{A}_{\sqrt{3}tH/2, tH/2} \right) \tag{37}$$

satisfies the full support property, i.e. it is an element of $\mathrm{Stab}(A)$*.*

Proof. From Lemma 3.6, there exists a quadratic form, say Q_1, which establishes the support property for the stability condition (37) with respect to $(\Lambda^\sharp_H, \mathrm{cl}^\sharp)$. Choose $t' < 0, p', q' \in \mathbb{R}$ as in Lemma 3.18. Now from Lemma 3.6, there exists a quadratic form, say Q_2, which establishes the support property for the stability condition $\left(W^{p',q'}_{\widehat{H},t'}, \mathcal{A}_{-\sqrt{3}t'\widehat{H}/2, t'H/2} \right)$ with respect to $(\Lambda_{\widehat{H}^\sharp}, \mathrm{cl}^\sharp)$. Hence, from Lemma 3.18 and Corollary 3.9, the quadratic form $Q_2(\Psi(-))$ establishes the support property for the stability condition (37) with respect to $(\Lambda^\flat_H, \mathrm{cl}^\flat)$ defined in (30). Therefore, the quadratic form

$$Q(-) = Q_1(-) + \lambda Q_2(\Psi(-)), \quad \text{for any } \lambda \in \mathbb{R}_{>0} \tag{38}$$

establishes the support property for the stability condition (37) with respect to (Γ, ch), that is the full support property. $\square$

Theorem 3.20. *Let $B \in \mathrm{NS}_{\mathbb{Q}}(A)$, $\alpha = \sqrt{3}t/2$ for some $t \in \mathbb{Q}_{>0}$ and $a, b \in \mathbb{R}$ satisfying (27). Then the stability condition $(Z^{a,b}_{\alpha H,B}, \mathcal{A}_{\alpha H,B})$ in Proposition 3.4 satisfies the full support property.*

Proof. Let us fix a slope semistable semihomogeneous bundle V on A such that

$$\frac{c_1(V)}{\mathrm{rk}(V)} = -B + \frac{t}{2}H. \tag{39}$$

From Lemma 3.13, $\mathrm{ch}(V) = \mathrm{rk}(V) \cdot e^{(-B+tH/2)}$. Let E be a $(Z^{a,b}_{\alpha H,B}, \mathcal{A}_{\alpha H,B})$-semistable object. By Proposition 3.14, $E \otimes V$ is a $(Z^{a,b}_{\sqrt{3}tH/2,tH/2}, \mathcal{A}_{\sqrt{3}tH/2,tH/2})$-semistable object. Let Q be the quadractic form on Γ which establishes the full support property for (37), which exists by Theorem 3.19. Since $\mathrm{ch}(E \otimes V) = \mathrm{rk}(V) \cdot \mathrm{ch}^{B-tH/2}(E)$, the quadractic form $Q(e^{-B+tH/2}(-))$ establish the support property for $(Z^{a,b}_{\alpha H,B}, \mathcal{A}_{\alpha H,B})$. $\square$

Consequently, we arrive at the following, which is the main result of Section 3. It implies in particular the existence of stability conditions on A with respect to (Γ, ch), or equivalently that $\mathrm{Stab}(A) \neq \varnothing$.

Theorem 3.21. *There is a continuous family of Bridgeland stability conditions in* $\mathrm{Stab}(A)$*, parameterized by the set*

$$(\omega, B, a, b) \in \mathrm{Amp}_{\mathbb{R}}(A) \times \mathrm{NS}_{\mathbb{R}}(A) \times \mathbb{R} \times \mathbb{R}, \quad a > \frac{1}{18} + \frac{\sqrt{3}}{6}|b|$$

via

$$(\omega, B, a, b) \mapsto \left(Z^{a,b}_{\omega,B}, \mathcal{A}_{\omega,B}\right).$$

In particular, there is a continuous embedding $\mathrm{Amp}_{\mathbb{C}}(A) \to \mathrm{Stab}(A)$ *given by* $B + i\omega \mapsto \sigma_{B,\omega}$*. The action of* $\mathrm{Aut}(D^b(A))$ *on* $\mathrm{Stab}(A)$ *preserves the connected component* $\mathrm{Stab}^{\circ}(A)$ *which contains the image of the above map.*

Proof. The first statement is similar to the proof of [BMS16, Proposition 8.10], using Theorem 3.20. Below, we give a proof of the second statement. Let F be a derived autoequivalence of A. If the Fourier-Mukai kernel of F is a vector bundle (up to a shift) then the claim is a direct consequence of [Piy, Theorem 1.1]. Suppose that the Fourier-Mukai kernel of F is not a vector bundle up to a shift. By a theorem of Orlov [Huy06, Proposition 9.53], the kernel of an auto-equivlance between two abelian varieties is represented by a sheaf up to shift. Therefore for a derived equivalence defined by $F' = \Phi_{\mathcal{P}}^{A \to \widehat{A}} \circ \otimes \mathcal{O}_A(nH) \circ F$, where H is ample and n is sufficiently large, the Fourier-Mukai kernel of F' is a vector bundle up to a shift. Again from [Piy, Theorem 1.1], F' takes $\mathrm{Stab}^0(A)$ to $\mathrm{Stab}^0(\widehat{A})$. Since $\Phi_{\mathcal{P}}^{A \to \widehat{A}}$ and $\otimes \mathcal{O}_A(nH)$ preserve connected components $\mathrm{Stab}^0(A)$, $\mathrm{Stab}^0(\widehat{A})$, the equivalence F also preserves $\mathrm{Stab}^0(A)$. $\square$

3.8. **Standard slice.** In what follows, we focus on some subspace of $\mathrm{Stab}(A)$ and find stability conditions on it where semistable objects coincide with Gieseker semistable sheaves.

We fix an ample divisor H and consider $B + i\omega$ written as

$$\omega = \alpha H, \ B = \beta H, \ \alpha \in \mathbb{R}_{>0}, \beta \in \mathbb{R}.$$

We write $\sigma_{\alpha H, \beta H} = (Z_{\alpha H, \beta H}, \mathcal{A}_{\alpha H, \beta H})$ as $\sigma_{\alpha,\beta} = (Z_{\alpha,\beta}, \mathcal{A}_{\alpha,\beta})$ and so on. Recall that we considered the surjective map

$$\Gamma_{\mathbb{Q}} \twoheadrightarrow \mathbb{Q}^4, \ \mathrm{ch}_i \mapsto v_i = i! H^{3-i} \mathrm{ch}_i \,. \tag{40}$$

For $\beta \in \mathbb{R}$ let $(v_0^{\beta}, v_1^{\beta}, v_2^{\beta}, v_3^{\beta}) \in \mathbb{R}^4$ be the vector corresponding to $v(\mathrm{ch}^{\beta H})$,

$$v_0^{\beta} = v_0, \ v_1^{\beta} = v_1 - \beta v_0, \ v_2^{\beta} = v_2 - 2\beta v_1 + \beta^2 v_0,$$
$$v_3^{\beta} = v_3 - 3\beta v_2 + 3\beta^2 v_1 - \beta^3 v_0.$$

Consider the subspace

$$\mathrm{Stab}_H(A) \subset \mathrm{Stab}(A)$$

of stability conditions $(Z, \mathcal{A})$ such that Z factors through the map (40). Let

$$\mathrm{Stab}_H^\circ(A) \subset \mathrm{Stab}_H(A)$$

denote the component which contains the elements $\sigma_{\alpha,\beta}$ (the component exists by Theorem 3.21). The space $\mathrm{Stab}_H^\circ(A)$ is completely described in [BMS16] as follows. Let $\mathfrak{B} \subset \mathbb{R}^4$ be the open subset given by

$$\mathfrak{B} = \left\{ (\alpha, \beta, a, b) \in \mathbb{R}^4 : \alpha > 0, a > \frac{\alpha^2}{18} + \frac{\sqrt{3}}{6}|b|\alpha \right\}.$$

For $(\alpha, \beta, a, b) \in \mathfrak{B}$, the central charge $Z_{\alpha,\beta}^{a,b} := Z_{\alpha H, \beta H}^{a,b}$ in (26) is written as

$$Z_{\alpha,\beta}^{a,b} = \frac{1}{6}\left(-v_3^\beta + 3bv_2^\beta + 6av_1^\beta + i\alpha\left(3v_2^\beta - \alpha^2 v_0^\beta\right)\right).$$

Theorem 3.22. ([BMS16]) *We have the continous embedding*

$$\mathfrak{B} \to \mathrm{Stab}_H^\circ(A),\ (\alpha, \beta, a, b) \mapsto \sigma_{\alpha,\beta}^{a,b} := (Z_{\alpha,\beta}^{a,b}, \mathcal{A}_{\alpha,\beta}) \tag{41}$$

whose image gives a slice of the $\widetilde{\mathrm{GL}}_2^+(\mathbb{R})$*-action on* $\mathrm{Stab}_H^\circ(A)$.

The upper-half plane $\mathbb{H} \subset \mathbb{C}$ is embedded into $\mathfrak{B}$ by

$$\beta + i\alpha \mapsto (\alpha, \beta, \alpha^2/2, b = 0)$$

and its image under the embedding (41) is $\sigma_{\alpha,\beta} = \sigma_{\alpha,\beta}^{a=\alpha^2/2, b=0}$.

3.9. **Gieseker chamber.** We keep the notation from the previous subsection. Let $\overline{\Gamma}_+ \subset \Gamma$ be the subset of $v \in \Gamma$ such that either

$$v_0 > 0, \text{ or } v_1 > 0 = v_0, \text{ or } v_2 > 0 = v_1 = v_0, \text{ or } v_3 > 0 = v_2 = v_1 = v_0.$$

The set $\overline{\Gamma}_+$ contains Γ_+, the set of Chern characters of coherent sheaves.

We first consider $\nu_{\alpha,\beta}$-semistable objects in $\mathcal{B}_{\alpha,\beta}$. For $v \in \overline{\Gamma}_+$, by the same arguments as in [Mac14, Theorem 3.1], we can describe the wall and chamber structure for $\nu_{\alpha,\beta}$-semistable objects on $\mathcal{B}_{\alpha,\beta}$ with Chern character v on the (α, β)-plane:

$$\mathcal{H} = \{\beta + i\alpha : \alpha \in \mathbb{R}_{>0}, \beta \in \mathbb{R}\}.$$

The walls are (after rescaling α by $\sqrt{3}\alpha$) finite nested semi-circles: each wall is a semi-circle contained in $\beta < v_1/v_0$ (where $v_1/v_0 = \infty$ for $v_0 = 0$), whose center lies on the β-axis, and for any two walls one of them is contained in the interior of the other.

When (α, β) lies in the outer of every wall, the $\nu_{\alpha,\beta}$-semistable objects are described in terms of stability conditions on sheaves. For this purpose, we introduce the following notion, which lies between slope stability and Gieseker stability:

Definition 3.23. *For a smooth projective 3-fold X and an ample divisor H on it, a coherent sheaf $E \in \mathrm{Coh}(X)$ is called ν_H-semistable if it is pure and for any subsheaf $0 \subsetneq F \subset E$, we have*

$$\overline{\chi}_H^{\dagger}(F)(m) \leq \overline{\chi}_H^{\dagger}(E)(m)$$

for $m \gg 0$. Here for a polynomial $p(m)$ in m we let $p^{\dagger}(m) = p(m) - p(0)$.

In the case of $X = A$ we have the following.

Lemma 3.24. (i) *A torsion free sheaf $E \in \mathrm{Coh}(A)$ is ν_H-semistable if and only if for any subsheaf $F \subset E$, we have*

$$\frac{v_1(F)}{v_0(F)} \leq \frac{v_1(E)}{v_0(E)}, \quad \textit{and} \quad \frac{v_2(F)}{v_0(F)} \leq \frac{v_2(E)}{v_0(E)} \quad \textit{if} \quad \frac{v_1(F)}{v_0(F)} = \frac{v_1(E)}{v_0(E)}.$$

In particular, it is slope semistable.

(ii) *A ν_H-semistable torsion free sheaf $E \in \mathrm{Coh}(A)$ is Gieseker-semistable if and only if for any ν_H-semistable subsheaf $F \subset E$ with the same $(v_1/v_0, v_2/v_0)$, we have*

$$\frac{v_3(F)}{v_0(F)} \leq \frac{v_3(E)}{v_0(E)}.$$

(iii) *The same statements of (i), (ii) hold after replacing v_i with v_i^{β} for any $\beta \in \mathbb{R}$.*

Lemma 3.25. *For $v \in \overline{\Gamma}_+$, let $(\alpha, \beta) \in \mathcal{H}$ lies in the outer of every wall with respect to the $\nu_{\alpha,\beta}$-stability with Chern character v. Then for $E \in D^b(A)$ with $\mathrm{ch}(E) = v$, it is a $\nu_{\alpha,\beta}$-semistable object in $\mathcal{B}_{\alpha,\beta}$ if and only if it is ν_H-semistable coherent sheaf.*

Proof. The proof is similar to the surface case, for example see [LQ14, Theorem 1,2, Lemma 2.6]. □

For any $v \in \overline{\Gamma}_+$ with $(v_0, v_1) \neq (0, 0)$ the curve $\nu_{\alpha,\beta}(v) = 0$, i.e.

$$v_0\beta^2 - \frac{v_0}{3}\alpha^2 - 2v_1\beta + v_2 = 0$$

intersects each wall at the top of the semi-circle. We define

$$\mathcal{S}_v \subset \mathcal{H}$$

to be the intersection of the outer of every wall and the region $\nu_{\alpha,\beta}(v) > 0$. If $(v_0, v_1) = 0$, then there is no wall with respect to the $\nu_{\alpha,\beta}$-stability, and $\nu_{\alpha,\beta}(v) = \infty$, so we set $\mathcal{S}_v = \mathcal{H}$. In any case for fixed $\alpha > 0$, we have $(\alpha, \beta) \in \mathcal{S}_v$ for $\beta \ll 0$.

The following proposition proves the existence of a Gieseker chamber on $\mathrm{Stab}^{\circ}(A)$.

Proposition 3.26. *For any $(\alpha, \beta) \in \mathcal{S}_v$, there exists $s(\alpha, \beta) > 0$ such that for any $s > s(\alpha, \beta)$ the following holds: an object $E \in D^b(A)$ with $\mathrm{ch}(E) = v$ is a $Z_{\alpha,\beta}^{a=s,b=0}$-semistable object in $\mathcal{A}_{\alpha,\beta}$ if and only if it is a H-Gieseker semistable sheaf.*

Proof. For $t \geq 0$, consider the central charge

$$W_t = (1+t)\alpha^2 v_1^\beta - 3tv_3^\beta + i\alpha\left(3v_2^\beta - \alpha^2 v_0^\beta\right).$$

For all $t > 0$ we have

$$W_t \sim Z_{\alpha,\beta}^{a=s,b=0}, \; s = \frac{1+t}{18t}\alpha^2 > \frac{\alpha^2}{18}.$$

Hence by Theorem 3.22 the pair $(W_t, \mathcal{A}_{\alpha,\beta})$ is a Bridgeland stability condition for any $t > 0$. These stability conditions degenerate to the very weak stability condition $(W_0, \mathcal{A}_{\alpha,\beta})$ at $t = 0$, see [PT, Section 3.4].

Let $\mathcal{D}_v \subset D^b(A)$ be the set of objects with Chern character v. By the definition of $\mathcal{S}_v$, we have $\mathrm{Im}\, W_0(E) > 0$ for any $E \in \mathcal{D}_v$. Therefore by [PT, Lemma 2.19], we have

$$\begin{aligned} &\{E \in \mathcal{D}_v : E \text{ is } W_0\text{-semistable in } \mathcal{A}_{\alpha,\beta}\} \\ &= \{E \in \mathcal{D}_v : E \text{ is } \nu_{\alpha,\beta}\text{-semistable in } \mathcal{B}_{\alpha,\beta}\}. \end{aligned} \tag{42}$$

By Lemma 3.25 and the definition of $\mathcal{S}_v$, (42) coincides with

$$\{E \in \mathcal{D}_v : E \text{ is } \nu_H\text{-semistable in } \mathrm{Coh}(A)\}. \tag{43}$$

On the other hand by [PT, Proposition 2.27], for $0 < t \ll 1$ we have

$$\begin{aligned} &\{E \in \mathcal{D}_v : E \text{ is } W_t\text{-semistable in } \mathcal{A}_{\alpha,\beta}\} \\ &= \left\{ E \in \mathcal{D}_v : \begin{array}{c} E \text{ is } \xi\text{-semistable among } W_0\text{-semistable} \\ \text{objects in } \mathcal{A}_{\alpha,\beta} \text{ with } \arg W_0(-) = \arg W_0(v) \end{array} \right\}, \end{aligned}$$

where ξ is the slope function given by

$$\xi = \frac{3v_3^\beta - \alpha^2 v_1^\beta}{3v_2^\beta - \alpha^2 v_0^\beta}.$$

By Lemma 3.24 and (42), (43), for $v_0 > 0$ the last set of objects is the set of H-Gieseker semistable sheaves $E \in \mathrm{Coh}(A)$ with $\mathrm{ch}(E) = v$. Since $s = (1+t)\alpha^2/18t$ goes to ∞ for $t \to +0$, this implies the Lemma in case $v_0 > 0$. The case $v_0 = 0$ is similar. $\square$

4. Wallcrossing on abelian threefolds

Let A be an abelian threefold and let $\widehat{A} = \mathrm{Pic}^0(A)$ be its dual. We set

$$\mathbf{A} = A \times \widehat{A}.$$

Let also $H \in \mathrm{Pic}(A)$ be a fixed ample class.

4.1. **Reduced DT invariants for Bridgeland semistable objects.** In Section 2.11, we defined $\mathbf{A}$-reduced Donaldson-Thomas invariants

$$\mathbf{DT}_H(v) \in \mathbb{Q}[\mathbf{A}]$$

counting H-Gieseker semistable sheaves on A. Here we define reduced Donaldson–Thomas invariants counting Bridgeland semistable objects on A. The construction is completely parallel to above and we will be brief.

Let $\sigma \in \mathrm{Stab}^{\circ}(A)$ be a Bridgeland stability condition which satisfies the full support property, and let $v \in \Gamma$. We consider the moduli stack

$$\mathcal{M}_\sigma(v, \phi) \tag{44}$$

of σ-semistable objects $E \in D^b(A)$ with $\mathrm{ch}(E) = v$ and phase $\phi \in \mathbb{R}$. By [PT], the stack (44) is an algebraic stack of finite type. Moreover it is announced in [AHLH] that the stack (44) admits a good moduli space

$$p \colon \mathcal{M}_\sigma(v, \phi) \to M_\sigma(v, \phi)$$

for a separated algebraic space $M_\sigma(v, \phi)$ of finite type. We set

$$p \colon \mathcal{M}_\sigma(\phi) := \coprod_v \mathcal{M}_\sigma(v, \phi) \to M_\sigma(\phi) := \coprod_v M_\sigma(v, \phi)$$

By the argument in [PT, Proof of Theorem 5.6] we may assume that σ is defined over $\mathbb{Q}$. Let $\phi \in \mathbb{R}$ be fixed, and let $\mathcal{P}(\phi)$ be the category of σ-semistable objects with phase ϕ. Then there exist a noetherian heart

$$\mathcal{A} = \mathcal{P}((\psi - 1, \psi]) \subset D^b(X)$$

for some $\psi \in \mathbb{R}$ with $\phi \in (\psi - 1, \psi]$. The heart $\mathcal{A}$ is closed under the $\mathbf{A}$-action, since the $\mathbf{A}$ action leaves all the Chern characters invariant. Then by [PT, Corollary 4.21] the stack $\mathcal{O}bj(\mathcal{A})$ of objects in $\mathcal{A}$ is an algebraic stack locally of finite type with $\mathbf{A}$-action. As in Section 2.6 consider the $\mathbf{A}$-equivariant motivic Hall algebra with respect to the heart $\mathcal{A}$,

$$H^{\mathbf{A}}(\mathcal{A}) = K_0^{\mathbf{A}}(\mathrm{St}/\mathcal{O}bj(\mathcal{A})).$$

Then similarly to Section 2.7, we have the subalgebra

$$H^{\mathbf{A}}(\mathcal{A}, \phi) := K_0^{\mathbf{A}}(\mathrm{St}/\mathcal{M}_\sigma(\phi)) \subset H^{\mathbf{A}}(\mathcal{A}).$$

We define $H^{\mathbf{A}}_{\mathrm{reg}}(\mathcal{A}, \phi)$, $H^{\mathbf{A}}_{\mathrm{sc}}(\mathcal{A}, \phi)$ and the integration map

$$\mathcal{I}^{\mathbf{A}} \colon H^{\mathbf{A}}_{\mathrm{sc}}(\mathcal{A}, \phi) \xrightarrow{p_*^{\mathbf{A}}} \mathrm{Constr}^{\mathbf{A}}(M_\sigma(\phi)) \xrightarrow{J} C^{\mathbf{A}}(X) \tag{45}$$

as in Section 2.9. The stack (44) defines the element

$$\delta_\sigma(v, \phi) := [\mathcal{M}_\sigma(v, \phi) \subset \mathcal{M}_\sigma(\phi)] \in H^{\mathbf{A}}(\mathcal{A}, \phi).$$

Using the result of Joyce, the logarithm

$$\epsilon_\sigma(v,\phi) := \sum_{l\geq 1, v_1+\cdots+v_l=v} \frac{(-1)^{l-1}}{l} \delta_\sigma(v_1,\phi) * \cdots * \delta_\sigma(v_l,\phi) \tag{46}$$

yields the regular element $(\mathbb{L}-1)\epsilon_\sigma(v,\phi)$ which in turn defines

$$\overline{\epsilon}_\sigma(v,\phi) := [(\mathbb{L}-1)\epsilon_\sigma(v,\phi)] \in H^{\mathbf{A}}_{\mathrm{sc}}(\mathcal{A},\phi).$$

We define the $\mathbf{A}$-reduced Donaldson–Thomas invariant $\mathbf{DT}_\sigma(v,\phi) \in \mathbb{Q}[\mathbf{A}]$ by

$$\mathcal{I}^{\mathbf{A}}(\overline{\epsilon}_\sigma(v,\phi)) = \mathbf{DT}_\sigma(v,\phi)\cdot c_v.$$

Since $\mathbf{DT}_\sigma(v,\phi) = \mathbf{DT}_\sigma(v,\phi+1)$ the following convention makes sense.

Definition 4.1. *For all $\sigma = (Z,\mathcal{A}) \in \mathrm{Stab}(A)$ and $v \in \Gamma$ define*

$$\mathbf{DT}_\sigma(v) := \begin{cases} \mathbf{DT}_\sigma(v,\phi), & \textit{if } Z(v) \in \mathbb{R}_{>0}e^{\pi i\phi} \textit{ for some } \phi \in \mathbb{R} \\ 0, & \textit{if } Z(v) = 0. \end{cases}$$

For any connected abelian subvariety $B \subset \mathbf{A}$, we define $\mathrm{DT}_\sigma(v)_B \in \mathbb{Q}$ by

$$\mathbf{DT}_\sigma(v) = \sum_{B\subset \mathbf{A}} \mathrm{DT}_\sigma(v)_B \cdot \epsilon_B.$$

As before we usually write $\mathrm{DT}_\sigma(v) := \mathrm{DT}_\sigma(v)_{B=(0,0)}$.

We have the following comparision result.

Proposition 4.2. *For any $v \in \Gamma$ and ample divisor H on A, there exists a $\sigma \in \mathrm{Stab}^\circ(A)$ such that $\mathbf{DT}_\sigma(v) = \mathbf{DT}_H(v)$.*

Proof. By Proposition 3.26 and since $\mathbf{DT}_\sigma(v) = \mathbf{DT}_\sigma(-v)$ by convention. $\square$

4.2. **Comparison under change of stability conditions.** The integration map $\mathcal{I}^{\mathbf{A}}$ defined in Section 4.1 depended on a choice of stability condition. We check the definition is well-behaved under change of stability condition.

Consider a pair of stability conditions

$$\sigma = (Z,\mathcal{A}), \sigma' = (Z',\mathcal{A}') \in \mathrm{Stab}^\circ(A).$$

Let $v \in \Gamma$ be fixed and let $\phi, \phi' \in \mathbb{R}$ be phases such that $Z(v) \in \mathbb{R}_{>0}e^{\pi i\phi}$ and $Z'(v) \in \mathbb{R}_{>0}e^{\pi i\phi'}$. We assume that there is an open embedding of stacks

$$\iota\colon \mathcal{M}_{\sigma'}(v,\phi') \subset \mathcal{M}_\sigma(v,\phi). \tag{47}$$

The inclusion ι induces the map

$$\iota_*\colon K_0^{\mathbf{A}}(\mathrm{St}/\mathcal{M}_{\sigma'}(v,\phi')) \to K_0^{\mathbf{A}}(\mathrm{St}/\mathcal{M}_\sigma(v,\phi)).$$

Recall also from Section 4.1 the integration maps

$$\mathcal{I}^{\mathbf{A}} : K^{\mathbf{A}}_{0,\mathrm{reg}}(\mathrm{St}/\mathcal{M}_\sigma(v,\phi)) \to \mathbb{Q}[\mathbf{A}]c_v,\ \ \mathcal{I}'^{\mathbf{A}} : K^{\mathbf{A}}_{0,\mathrm{reg}}(\mathrm{St}/\mathcal{M}_{\sigma'}(v,\phi')) \to \mathbb{Q}[\mathbf{A}]c_v.$$

obtained from the stability conditions σ and σ' respectively. Here reg stands for regular elements.

Proposition 4.3. *We have $\mathcal{I}^{\mathbf{A}} = \mathcal{I}'^{\mathbf{A}} \circ \iota_*$. In particular,*

$$\mathcal{I}^{\mathbf{A}}\left((\mathbb{L}-1)\iota_*\epsilon_{\sigma'}(v,\phi')\right) = \mathrm{DT}_{\sigma'}(v,\phi')\cdot c_v.$$

Proof. By the universal property of good moduli spaces, we have the commutative diagram

$$\begin{tikzcd} \mathcal{M}_{\sigma'}(v,\phi') \arrow[r, "\iota"] \arrow[d, "p'"] & \mathcal{M}_{\sigma}(v,\phi) \arrow[d, "p"] \\ M_{\sigma'}(v,\phi') \arrow[r, "\tau"] & M_{\sigma}(v,\phi), \end{tikzcd}$$

where the left arrow is the good moduli space for $\mathcal{M}_{\sigma'}(v,\phi')$. Then it is enough to show that the following diagram is commutative

$$\begin{tikzcd} K_{0,\mathrm{reg}}^{\mathbf{A}}(\mathrm{St}/\mathcal{M}_{\sigma'}(v,\phi')) \arrow[r, "\iota_*"] \arrow[d, "p_*'^{\mathbf{A}}"] & K_{0,\mathrm{reg}}^{\mathbf{A}}(\mathrm{St}/\mathcal{M}_{\sigma}(v,\phi)) \arrow[d, "p_*^{\mathbf{A}}"] \\ \mathrm{Constr}^{\mathbf{A}}(M_{\sigma'}(v,\phi')) \arrow[r, "\tau_*"] \arrow[d, "J'"] & \mathrm{Constr}^{\mathbf{A}}(M_{\sigma}(v,\phi)) \arrow[d, "J"] \\ C^{\mathbf{A}}(X) \arrow[r, "\mathrm{id}"] & C^{\mathbf{A}}(X). \end{tikzcd}$$

Here J, J' are defined as in (17), and τ_* is defined as follows: for any $\mathbf{A}$-invariant subspace $Z \subset M_{\sigma'}(v,\phi')$ and $x \in M_{\sigma}(v,\phi)$ let $\tau_*(1_Z)(x) = e(\tau^{-1}(x)\cap Z)$.

The upper diagram is commutative since both $p_*^{\mathbf{A}} \circ \iota_*$ and $\tau_* \circ p_*'^{\mathbf{A}}$ compute the Behrend function weighted Euler numbers of fibers to the map to $M_\sigma(v,\phi)$, and the Behrend weights agree since (47) is an open embedding. To show that the lower diagram is commutative, by the definition of equivariant Euler number it is enough to show that the map τ preserves the connected component of the stabilizer groups of $\mathbf{A}$-actions, i.e. for any $x \in M_{\sigma'}(v,\phi')$, the induced map $\mathrm{Stab}(x)^\circ \to \mathrm{Stab}(\tau(x))^\circ$ is an isomorphism. By the diagram on good moduli spaces and since the open immersion ι preserves the connected component of the stabilizer group, it is enough to show the following lemma. □

Lemma 4.4. *For any $x \in M_\sigma(v,\phi)$, the connected component of the stablizer $B = \mathrm{Stab}(x)^\circ$ acts trivially on the geometric points of $p^{-1}(x) \subset \mathcal{M}_\sigma(v,\phi)$.*

Proof. For a fixed $x \in M_\sigma(v,\phi)$, there is a finite number of B-fixed σ-stable objects $E_1,\dots,E_n$ with phase ϕ such that any point in $p^{-1}(x)$ corresponds to iterated extensions of $E_1,\dots,E_n$. By the induction argument, it is enough to prove the following: for any σ-semistable objects P, Q fixed by B and with phase ϕ, and for any extension

$$0 \to P \to R \to Q \to 0$$

we have $g(R) \cong R$ for any $g \in B$.

The last claim is proved as follows. For $g \in B$, let

$$a_g \colon g(P) \stackrel{\cong}{\to} P, \ b_g \colon g(Q) \stackrel{\cong}{\to} Q$$

be isomorphisms. For $u \in \mathrm{Ext}^1(Q, P)$, we set

$$g(u)' = b_g \circ g(u) \circ a_g^{-1} \in \mathrm{Ext}^1(Q, P),$$

where $g(u) \in \mathrm{Ext}^1(g(Q), g(P))$ is the extension induced by the B-action. The assignment $g \mapsto (u \mapsto g(u)')$ is well-defined up to choices of a_g, b_g, so defines a map

$$B \to \mathrm{GL}(\mathrm{Ext}^1(Q, P)) / \mathrm{Aut}(Q) \times \mathrm{Aut}(P).$$

The target is an affine variety and B is an abelian variety, so the image must be an identity. This gives the proof of the above claim. $\square$

4.3. **Reduced DT invariants for semihomogeneous sheaves.** Recall the subset of semihomogeneous sheaves $\mathcal{C} \subset \Gamma$ defined in (3). Since the stabilizer $B \subset \mathbf{A}$ of every non-zero coherent sheaf on A is at most 3-dimensional [Muk, Proposition 4.5], and the sheaf is semihomogeneous if and only if $\dim(B) = 3$, we have the following.

Lemma 4.5. *Let $v \in \Gamma$ and let $B \subset \mathbf{A}$ be a connected abelian subvariety.*

(a) *If $\dim B > 3$, then $\mathrm{DT}_H(v)_B = 0$.*
(b) *If $\dim B = 3$ and $\mathrm{DT}_H(v)_B \neq 0$ then $v \in \mathcal{C}$.*

We have the following generalization of Lemma 4.5.

Lemma 4.6. *Let $\sigma \in \mathrm{Stab}^\circ(A)$. Let $v \in \Gamma$ and let $B \subset \mathbf{A}$ be connected.*

(a) *If $\dim B > 3$, then $\mathrm{DT}_\sigma(v)_B = 0$.*
(b) *If $\dim B = 3$ and $\mathrm{DT}_\sigma(v)_B \neq 0$, then $v \in \mathcal{C}$.*

The above lemma follows immediately from the following:

Lemma 4.7. *For every $E \in D^b(A)$, let $\Xi(E) \subset \mathbf{A}$ be as in (2). Then we have $\dim \Xi(E) \leq 3$. If $\dim \Xi(E) = 3$, we have $\mathrm{ch}(E) \in \mathcal{C}$.*

Proof. For every $E \in D^b(A)$ with $F_i = \mathcal{H}^i(E)$, we have

$$\Xi(E) \subset \bigcap_{i \in \mathbb{Z}} \Xi(F_i)$$

and $\dim \Xi(F_i) \leq 3$ by [Muk, Proposition 4.5]. Suppose that $\dim \Xi(E) = 3$. Then $\dim \Xi(E) = \dim \Xi(F_i) = 3$ for any $i \in \mathbb{Z}$ such that $F_i \neq 0$. In particular each F_i is a semihomogeneous sheaf. It is enough to show that $\mathrm{ch}(F_i)$ is proportional to $\mathrm{ch}(F_j)$ for each pair (i, j).

First suppose that each F_i is a vector bundle. Then $\mathrm{ch}(F_i)$ is written as $r(F_i)e^{c_1(F_i)/r(F_i)}$. Let $\Xi^{\circ}(-) \subset \Xi(-)$ be the connected component which contains $(0,0)$. Then we have $\Xi^{\circ}(E) = \Xi^{\circ}(F_i)$ for any $i \in \mathbb{Z}$ with $F_i \neq 0$. By [Muk, Theorem 4.9 (3)], the subabelian variety $\Xi^{\circ}(F_i) \subset \mathbf{A}$ determines $c_1(F_i)/r(F_i)$. Therefore for each (i,j), we have $c_1(F_i)/r(F_i) = c_1(F_j)/r(F_j)$, and $\mathrm{ch}(F_i)$, $\mathrm{ch}(F_j)$ are proportional.

When F_i is not a vector bundle, we can apply a Fourier-Mukai transform $\Phi_{\mathcal{P}}^{A\to\widehat{A}} \circ \otimes\mathcal{O}_A(mH)$ for $m \gg 0$ and use Theorem 4.8 below to reduce to the case that every F_i is a vector bundle. □

Theorem 4.8. ([Orl02]) *There is a map*

$$\mathrm{Aut}(D^b(A)) \to \mathrm{Aut}(A \times \widehat{A}), \ g \mapsto g_* \tag{48}$$

such that $g_*\Xi(E) = \Xi(g(E))$ *for any* $E \in D^b(A)$.

4.4. **Independence of stability conditions.** We show the absence of walls in good cases.

Theorem 4.9. *Suppose that* $v \in \Gamma$ *is not written as* $\gamma_1 + \gamma_2$ *for some* $\gamma_i \in \mathcal{C}$ *with* $\chi(\gamma_1, \gamma_2) \neq 0$. *Then for any* $\sigma, \sigma' \in \mathrm{Stab}^{\circ}(A)$ *we have*

$$\mathbf{DT}_{\sigma}(v) = \mathbf{DT}_{\sigma'}(v)$$

Proof. We prove that for any $\sigma = (Z, \mathcal{A}) \in \mathrm{Stab}^{\circ}(A)$ there is an open neighborhood $\sigma \in U \subset \mathrm{Stab}^{\circ}(A)$ with $\mathbf{DT}_{\sigma}(v) = \mathbf{DT}_{\sigma'}(v)$ for any $\sigma' \in U$.

Suppose that $Z(v) = 0$. Then there is no σ-semistable object E with $\mathrm{ch}(E) = v$. By the wall and chamber structure on $\mathrm{Stab}^{\circ}(A)$, there is an open neighborhood $\sigma \in U \subset \mathrm{Stab}^{\circ}(A)$ such that for any $\sigma' \in U$ there is no σ'-semistable object E with $\mathrm{ch}(E) = v$. It follows $\mathbf{DT}_{\sigma}(v) = \mathbf{DT}_{\sigma'}(v) = 0$.

Hence we may assume that $Z(v) \neq 0$. Let $\phi \in \mathbb{R}$ such that $Z(v) \in \mathbb{R}_{>0}e^{\pi i \phi}$. For an open neighborhood $\sigma \in U \subset \mathrm{Stab}^{\circ}(A)$, we take $\sigma' = (Z', \mathcal{A}') \in U$. For $\psi \in \mathbb{R}$, let $\mathcal{P}(\psi)$, $\mathcal{P}'(\psi)$ be the σ, σ'-semistable objects with phase ψ. By shrinking U and applying a $\mathbb{C}$-action on $\mathrm{Stab}^{\circ}(A)$ if necessary, we can assume that

$$\mathcal{P}(\phi) \subset \mathcal{P}'((\phi - \varepsilon, \phi + \varepsilon)) \subset \mathcal{A}$$

for some $0 < \varepsilon \ll 1$, where the right hand side is the extension closure of objects in $\mathcal{P}'(\psi)$ with $\psi \in (\phi - \varepsilon, \phi + \varepsilon)$.

We then have the following identity in $H^{\mathbf{A}}(\mathcal{A}, \phi)$,

$$\delta_{\sigma}(v, \phi) = \sum_{\substack{l \geq 1, \gamma_1 + \cdots + \gamma_l = v \\ Z(\gamma_i) \in \mathbb{R}_{>0}e^{\pi i \phi}, \\ Z'(\gamma_i) \in \mathbb{R}_{>0}e^{\pi i \phi_i}, \\ \phi_1 > \cdots > \phi_l, \phi_i \in (\phi - \varepsilon, \phi + \varepsilon).}} \delta_{\sigma'}(\gamma_1, \phi_1) * \cdots * \delta_{\sigma'}(\gamma_l, \phi_l). \tag{49}$$

Here $\mathcal{M}_{\sigma'}(\gamma_i, \phi_i)$ is an open substack of $\mathcal{M}_{\sigma}(\gamma_i, \phi)$, and[11]

$$\delta_{\sigma'}(\gamma_i, \phi_i) = [\mathcal{M}_{\sigma'}(\gamma_i, \phi_i) \subset \mathcal{M}_{\sigma}(\gamma_i, \phi)] \in H^{\mathbf{A}}_{\mathrm{sc}}(\mathcal{A}, \phi).$$

By substituting (46) and multiplying $(\mathbb{L}-1)$, we obtain an identity in $H^{\mathbf{A}}_{\mathrm{sc}}(\mathcal{A}, \phi)$ of the form

$$\begin{aligned} \overline{\epsilon}_{\sigma}(v, \phi) &= \overline{\epsilon}_{\sigma'}(v, \phi') + \sum_{\gamma_1+\gamma_2=v} a_{\gamma_1,\gamma_2}\{\overline{\epsilon}_{\sigma'}(\gamma_1, \phi_1), \overline{\epsilon}_{\sigma'}(\gamma_2, \phi_2)\} \\ &+ \sum_{\gamma_1+\gamma_2+\gamma_3=v} a_{\gamma_1,\gamma_2,\gamma_3}\{\{\overline{\epsilon}_{\sigma'}(\gamma_1, \phi_1), \overline{\epsilon}_{\sigma'}(\gamma_2, \phi_2)\}, \overline{\epsilon}_{\sigma'}(\gamma_3, \phi_3)\} + \cdots \end{aligned} \tag{50}$$

for some $a_{\gamma_1,\cdots,\gamma_l} \in \mathbb{Q}$.

We apply the equivariant integration map $\mathcal{I}^{\mathbf{A}}$ to (50). If we write

$$\mathcal{I}^{\mathbf{A}}(\overline{\epsilon}_{\sigma'}(\gamma_i, \phi_i)) = \sum_k b_k \epsilon_{B_k} c_{\gamma_i}$$

for some $b_k \in \mathbb{Q}$ and $B_k \subset \mathbf{A}$, then by Lemma 4.6 the B_k are of codimension ≥ 3. By the definition of $\mathbb{Q}[\mathbf{A}]$ it follows that only linear or quadratic terms in the $\overline{\epsilon}_{\sigma'}(\gamma_i, \phi_i)$ contribute when applying $\mathcal{I}^{\mathbf{A}}$ to (50). Moreover using Proposition 4.3, the contribution of the quadratic term is

$$\begin{aligned} &\sum_{\gamma_1+\gamma_2=v} \mathcal{I}^{\mathbf{A}}(a_{\gamma_1,\gamma_2}\{\overline{\epsilon}_{\sigma'}(\gamma_1, \phi_1), \overline{\epsilon}_{\sigma'}(\gamma_2, \phi_2)\}) \\ &= \sum_{\gamma_1+\gamma_2=v} a_{\gamma_1,\gamma_2} \sum_{\substack{B_i \subset \mathbf{A}, \\ i=1,2}} (-1)^{\chi(\gamma_1,\gamma_2)} \chi(\gamma_1,\gamma_2)\, \mathrm{DT}_{\sigma'}(\gamma_1)_{B_1}\, \mathrm{DT}_{\sigma'}(\gamma_2)_{B_2}\, |B_1 \cap B_2|\, \epsilon_{(0,0)}. \end{aligned}$$

where the sum is over connected abelian subvarieties $B_i \subset \mathbf{A}$ of dimension 3 such that B_1 and B_2 are transversal. By Lemma 4.6 and its proof, the above is non-zero only if $v = \gamma_1 + \gamma_2$ with $\gamma_i \in \mathcal{C}$ such that $\chi(\gamma_1, \gamma_2) \neq 0$. $\square$

The proof of Theorem 4.9 also shows the following:

Corollary 4.10. *For every $v \in \Gamma$ and every positive-dimensional connected abelian subvariety $B \subset \mathbf{A}$ we have*

$$\mathrm{DT}_{\sigma}(v)_B = \mathrm{DT}_H(v)_B$$

for all $\sigma \in \mathrm{Stab}^{\circ}(A)$ and ample divisors H.

Combining Theorem 4.9 and Proposition 4.2 yields the following.

Corollary 4.11. *Under the assumptions of Proposition 4.9,*

$$\mathbf{DT}_{\sigma}(v) = \mathbf{DT}_H(v). \tag{51}$$

for all $\sigma \in \mathrm{Stab}^{\circ}(A)$ and ample divisors H. In particular, $\mathbf{DT}_{\sigma}(v)$, $\mathbf{DT}_H(v)$ are independent of σ and H.

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[11]More precisely $\delta_{\sigma'}(\gamma_i, \phi_i)$ is the push-forward under the open embedding $\mathcal{M}_{\sigma'}(\gamma_i, \phi_i) \subset \mathcal{M}_{\sigma}(\gamma_i, \phi)$ as in Section 4.2, and we have omitted the notation of the push-forward.

We have now all ingredients for the proof of Theorem 1.1.

Proof of Theorem 1.1. Suppose that $v \in \Gamma$ is not written as $\gamma_1 + \gamma_2$ for some $\gamma_i \in \mathcal{C}$ with $\chi(\gamma_1, \gamma_2) \neq 0$. Let $g \in \operatorname{Aut} D^b(A)$ be a derived autoequivalence and let $\sigma \in \operatorname{Stab}^\circ(A)$ be a stability condition which lies in the Gieseker chamber with respect to v. By Corollary 4.11 and an application of g we have

$$\mathbf{DT}_H(v) = \mathbf{DT}_\sigma(v) = \mathbf{DT}_{g_*\sigma}(g_* v)$$

By Theorem 3.21 the stability condition $g_*\sigma$ lies in the component $\operatorname{Stab}^\circ(A)$. Hence again by Corollary 4.11,

$$\mathbf{DT}_{g_*\sigma}(g_* v) = \mathbf{DT}_H(g_* v). \qquad \square$$

4.5. **The discriminant.** From Appendix A recall the discriminant

$$\Delta : H^{2*}(A, \mathbb{Z}) \to \mathbb{Z}.$$

By construction Δ is invariant under all derived autoequivalences of A. The following lemma directly implies Proposition 1.2.

Lemma 4.12. *Let $v \in \Gamma$.*

(1) *If $\Delta(v) > 0$, then v is not of the form $\gamma_1 + \gamma_2$ with $\gamma_i \in \mathcal{C}$.*
(2) *If $\Delta(v) = 0$ and $v = \gamma_1 + \gamma_2$ with $\gamma_i \in \mathcal{C}$, then $\chi(\gamma_1, \gamma_2) = 0$.*

Hence, if $\Delta(v) \geq 0$ then v satisfies the assumption of Proposition 4.9.

Proof. By Theorem 4.8 the set $\mathcal{C}$ is preserved by derived autoequivalences. Therefore as in the proof of Lemma 4.7 we may assume $\gamma_i = r_i e^{c_1/r_i}$ for some $c_1 \in H^2(A)$ and $r_i \in \mathbb{Z}$. The claims follow now from Theorem A.2. $\square$

4.6. **Reduced DT invariants for semihomogeneous sheaves II.** We calculate the Donaldson-Thommas invariants of semihomogeneous sheaves.

Lemma 4.13. *Let $v \in \mathcal{C}$. Then*

$$\mathbf{DT}_\sigma(v) = \left(\sum_{k \geq 1, k | v} \frac{1}{k^2} \right) \epsilon_B$$

for some three-dimensional $B \subset \mathbf{A}$ determined by v.

Proof. By [Muk, Proposition 4.11], there exists another abelian variety A' and a equivalence $F \colon D^b(A) \xrightarrow{\sim} D^b(A')$ such that

$$F_* v = (0, 0, 0, r)$$

for some $r \geq 1$. Hence $\Delta(v) = \Delta(F_* v) = 0$. Using Lemma 4.12 and Corollary 4.10 we conclude

$$\mathbf{DT}_\sigma(v) = \mathbf{DT}_{F_*\sigma}(0,0,0,r) = \mathbf{DT}_H(0,0,0,r) = \left(\sum_{k \geq 1, k | r} \frac{1}{k^2} \right) \epsilon_{\{0\} \times \widehat{A}}.$$

where the last equality is [OS, Proposition 6]. □

5. Principally polarized abelian threefolds

5.1. **Setup.** Let (A, H) be a principally polarized abelian 3-fold of Picard rank $\rho(A) = 1$. We identify A with its dual $\widehat{A}$ via the isomorphism

$$A \stackrel{\cong}{\to} \widehat{A}, \ x \mapsto T_x^* \mathcal{O}_A(H) \otimes \mathcal{O}_A(-H).$$

We also identify elements in Γ with vectors $(v_0, v_1, v_2, v_3) \in \mathbb{Z}^4$ via the isomorphism

$$\mathbb{Z}^4 \stackrel{\cong}{\to} \Gamma, \quad (v_0, v_1, v_2, v_3) \mapsto (v_0, v_1[H], v_2[H^2/2], v_3[H^3/6]). \tag{52}$$

Under this identification the Euler pairing χ on Γ is

$$\chi\big((v_0, v_1, v_2, v_3), (v_0', v_1', v_2', v_3')\big) = v_0 v_3' - 3v_1 v_2' + 3v_2 v_1' - v_3 v_0'. \tag{53}$$

The discriminant defined in Appendix A takes the form

$$\Delta(v_0, v_1, v_2, v_3) = -4(v_0 v_2^3 + v_1^3 v_3) - v_0^2 v_3^2 + 3v_1^2 v_2^2 + 6 v_0 v_1 v_2 v_3. \tag{54}$$

5.2. **Action of autoequivalences on cohomology.** Recall that the group $\mathrm{SL}_2(\mathbb{Z})$ is generated by the elements

$$T = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}, \ S = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$$

with relations $S^2 = (TS)^3$ and $S^4 = 1$. Let $\widetilde{\mathrm{SL}}_2(\mathbb{Z})$ be the group generated by $\widetilde{S}$, $\widetilde{T}$ with the relation $\widetilde{S}^2 = (\widetilde{T}\widetilde{S})^3$. There is an exact sequence of groups

$$1 \to \mathbb{Z} \xrightarrow{i} \widetilde{\mathrm{SL}}_2(\mathbb{Z}) \xrightarrow{j} \mathrm{SL}_2(\mathbb{Z}) \to 1 \tag{55}$$

where the map i sends 1 to $\widetilde{S}^4$ and j sends $\widetilde{S}$, $\widetilde{T}$ to S, T respectively.

By a result of Mukai [Muk81] there is a group homomorphism

$$\widetilde{\mathrm{SL}}_2(\mathbb{Z}) \to \mathrm{Aut}(D^b(A)) \tag{56}$$

sending $\widetilde{S}$, $\widetilde{T}$ to $\Phi_{\mathcal{P}} := \Phi_{\mathcal{P}}^{A \to A}$ and $\otimes \mathcal{O}_A(H)$ respectively. Because $\Phi_{\mathcal{P}}^4 = [-6]$ acts on Γ trivially, (56) descends to a homomorhism

$$\mathrm{SL}_2(\mathbb{Z}) \to \mathrm{Aut}(\Gamma). \tag{57}$$

In terms of the generators (S, T) this representation is given by

$$T \mapsto \begin{pmatrix} 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 1 & 2 & 1 & 0 \\ 1 & 3 & 3 & 1 \end{pmatrix}, \quad S \mapsto \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \\ 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \end{pmatrix}. \tag{58}$$

For $g \in \mathrm{SL}_2(\mathbb{Z})$, we let $g_* \in \mathrm{Aut}(\Gamma)$ denote the induced isomorphism.

We can interpret the action (57) as a $\mathrm{SL}_2(\mathbb{Z})$-action on two variable homogeneous polynomials as follows. Identify elements in Γ with certain cubic homogeneous polynomials in two variables via the map

$$(v_0, v_1, v_2, v_3) \mapsto v_0 x^3 + 3v_1 x^2 y + 3v_2 x y^2 + v_3 y^3. \tag{59}$$

The group $\mathrm{SL}_2(\mathbb{Z})$ acts on the homogeneous cubic polynomials in (x, y) by the transformation

$$g_* \colon (x, y) \mapsto (dx + by, cx + ay) \tag{60}$$

where $g = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \mathrm{SL}_2(\mathbb{Z})$. This action coincides with $g_* \in \mathrm{Aut}(\Gamma)$ under the identification (59).[12]

5.3. **Action of autoequivalences on stability conditions.** We next describe the action of $\widetilde{\mathrm{SL}}_2(\mathbb{Z})$ on $\mathrm{Stab}^\circ(A)$. Let $\mathbb{H} \subset \mathbb{C}$ be the upper half plane. By Theorem 3.22, we have the embedding

$$\mathbb{H} \to \mathrm{Stab}^\circ(A), \ \tau = \beta + i\alpha \mapsto \sigma_\tau := \sigma_{\alpha,\beta} = \sigma_{\alpha H, \beta H}. \tag{61}$$

The group $\mathrm{SL}_2(\mathbb{Z})$ acts on the upper half plane $\mathbb{H}$ by

$$\tau \mapsto g \cdot \tau = \frac{a\tau + b}{c\tau + d}$$

for all $\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \mathrm{SL}_2(\mathbb{Z})$ and $\tau \in \mathbb{H}$. The following Lemma shows that, modulo the $\widetilde{\mathrm{GL}}_2^+(\mathbb{R})$ action, these two actions coincide.

Lemma 5.1. *For any $g \in \mathrm{SL}_2(\mathbb{Z})$ with lift $\widetilde{g} \in \widetilde{\mathrm{SL}}_2(\mathbb{Z})$ and for any $\tau \in \mathbb{H}$, there exists a unique $\xi \in \mathbb{C} \subset \widetilde{\mathrm{GL}}_2^+(\mathbb{R})$ such that*

$$\widetilde{g}_* \sigma_\tau = \sigma_{g\tau} \cdot \xi.$$

Proof. By Theorem 3.22 and since $\mathrm{Aut}\, D^b(A)$ preserves the main component of the stability manifold, we have

$$\widetilde{g}_* \sigma_\tau = \sigma' \cdot \xi$$

for some $\sigma' \in \mathfrak{B}$ and $\xi \in \widetilde{\mathrm{GL}}_2^+(\mathbb{R})$. Therefore it is enough to show that the central charge of $\widetilde{g}_* \sigma_\tau$ is of the desired form.

By (24) the central charge of σ_τ is written as

$$Z_\tau(v) = -\chi(e^{\tau H}, v)$$

for all $v \in \Gamma$. Hence the central charge of $\widetilde{g}_* \sigma_{\alpha,\beta}$ is

$$Z_\tau(g_*^{-1} v) = -\chi(e^{\tau H}, g_*^{-1} v) = -\chi(g_* e^{\tau H}, v). \tag{62}$$

Under the correspondence (59) we have $e^{\tau H} = (x + \tau y)^3$ which implies

$$g_* e^{\tau H} = (c\tau + d)^3 (x + (g\tau) y)^3 = (c\tau + d)^3 e^{(g\tau)H}.$$

Inserting back into (62) the Lemma follows. ∎

[12] The identification (59) also gives motivation to call $\Delta(v)$ the discriminant, since it coincides with the discriminant of the cubic polynomial on the right hand side of (59).

5.4. **Wall and chamber structure.** We consider classes $v \in \Gamma$ which can be written as $\gamma_1 + \gamma_2$ for some $\gamma_i \in \mathcal{C}$ such that $\chi(\gamma_1, \gamma_2) \neq 0$. Since (A, H) is principally polarized we have

$$\mathcal{C} = \{r(p^3, p^2 q, pq^2, q^3) : (p, q, r) \in \mathbb{Z}^3, r \neq 0, \gcd(p, q) = 1\}.$$

Hence v can be written as

$$v = \gamma_1 + \gamma_2,\ \gamma_i = r_i(p_i^3, p_i^2 q_i, p_i q_i^2, q_i^3) \in \mathcal{C},\ \Theta(\gamma_1) < \Theta(\gamma_2), \tag{63}$$

where $\Theta(\gamma_i) = q_i/p_i$.

Lemma 5.2. *If v is written as in (63), then γ_1, γ_2 are uniquely determined from v.*

Proof. Each $\gamma_i \in \mathcal{C}$ is either written as $u_i(1, \theta_i, \theta_i^2, \theta_i^3)$ for some $u_i \in \mathbb{Z}$ and $\theta_i \in \mathbb{Q}$, or proportional to $(0, 0, 0, 1)$. If γ_2 is proportional to $(0, 0, 0, 1)$, then the lemma holds. Therefore it is enough to show that, for fixed $v = (v_0, v_1, v_2, v_3)$, the equation

$$v_j = u_1 \theta_1^j + u_2 \theta_2^j,\ \theta_1 < \theta_2,\ 0 \le j \le 3 \tag{64}$$

has at most one solution of $(u_1, u_2, \theta_1, \theta_2)$. The equations (64) for $j = 0, 1$ give

$$u_1 = \frac{v_1 - v_0\theta_2}{\theta_1 - \theta_2},\ u_2 = \frac{v_0\theta_1 - v_1}{\theta_1 - \theta_2}. \tag{65}$$

By substituting this into (64) for $j = 3, 4$, we obtain

$$\begin{aligned} &v_1(\theta_1 + \theta_2) - v_0\theta_1\theta_2 = v_2, \\ &v_1(\theta_1^2 + \theta_1\theta_2 + \theta_2^2) - v_0\theta_1\theta_2(\theta_1 + \theta_2) = v_3 \end{aligned} \tag{66}$$

respectively. By substituting $v_0\theta_1\theta_2 = v_1(\theta_1 + \theta_2) - v_2$ into the second, we obtain

$$v_2(\theta_1 + \theta_2) - v_1\theta_1\theta_2 = v_3. \tag{67}$$

On the other hand if (64) has a solution, we have

$$v_1^2 - v_0 v_2 = -u_1 u_2 (\theta_1 - \theta_2)^2 \neq 0.$$

Therefore (66), (67) give

$$\theta_1 + \theta_2 = \frac{v_1 v_2 - v_0 v_3}{v_1^2 - v_0 v_2},\ \theta_1\theta_2 = \frac{v_2^2 - v_1 v_3}{v_1^2 - v_0 v_2}.$$

The number of $(\theta_1, \theta_2) \in \mathbb{Q}^2$ with $\theta_1 < \theta_2$ satisfying the above equation is at most one, and (u_1, u_2) is determined by (θ_1, θ_2). $\square$

If v is written as (63), by Lemma 5.2 and the proof of Proposition 4.9 the only possible wall in $\mathrm{Stab}^\circ(A)$ where $\mathrm{DT}_\sigma(v)$ can jump is

$$\mathcal{W}_v := \{(Z, \mathcal{A}) \in \mathrm{Stab}^\circ(A) : Z(\gamma_2) \in \mathbb{R}_{>0} Z(\gamma_1)\}.$$

Lemma 5.3. *For a fixed $\alpha > 0$, there is $\beta_0 \in \mathbb{R}$ such that if $\beta < \beta_0$, then the image of the map*

$$\mathbb{R}_{\geq 1/2} \to \mathrm{Stab}^{\circ}(A), \ s \mapsto \sigma_{\alpha,\beta}^{a=s\alpha^2, b=0}$$

does not intersect with $\mathcal{W}_v$.

Proof. As in the proof of Lemma 5.2, suppose either $\gamma_i = u_i(1, \theta_i, \theta_i^2, \theta_i^3)$ for some $u_i \in \mathbb{Z}$ and $\theta_i = \Theta(\gamma_i) \in \mathbb{Q}$, or γ_2 is proportional to $(0,0,0,1)$. We have

$$\begin{aligned} &Z_{\alpha,\beta}^{a=s\alpha^2, b=0}(u_i(1, \theta_i, \theta_i^2, \theta_i^3)) \\ &= u_i \left\{ -(\theta_i - \beta)^3 + 6s\alpha^2(\theta_i - \beta) + \sqrt{-1}\left(3\alpha(\theta_i - \beta)^2 - \alpha^3\right)\right\}. \end{aligned}$$

First suppose that γ_2 is proportional to $(0,0,0,1)$. If $\sigma_{\alpha,\beta}^{a=s\alpha^2, b=0}$ lies in $\mathcal{W}_v$, then we have $Z_{\alpha,\beta}^{a=s\alpha^2, b=0}(\gamma_1) \in \mathbb{R}$, hence

$$3(\theta_1 - \beta)^2 - \alpha^2 = 0.$$

Hence the lemma holds by setting $\beta_0 = \theta_1 - \alpha/\sqrt{3}$.

Next suppose that γ_2 is not proportional to $(0,0,0,1)$. If $\sigma_{\alpha,\beta}^{a=s\alpha^2, b=0}$ lies in $\mathcal{W}_v$, we have

$$\frac{(\theta_1 - \beta)^3 - 6s\alpha^2(\theta_1 - \beta)}{3\alpha(\theta_1 - \beta)^2 - \alpha^3} = \frac{(\theta_2 - \beta)^3 - 6s\alpha^2(\theta_2 - \beta)}{3\alpha(\theta_2 - \beta)^2 - \alpha^3}.$$

By setting $\theta = \theta_2 - \theta_1$, $\overline{\beta} = \beta - \theta_1$ and simplifying, we obtain

$$3\overline{\beta}^2(\overline{\beta} - \theta)^2 + 6s\alpha^4 + 3(6s-1)\alpha^2\overline{\beta}^2 + 3\theta(1-6s)\alpha^2\overline{\beta} - \theta^2\alpha^2 = 0.$$

Since $\theta > 0$, the above equation gives

$$3(1-6s)\overline{\beta} - \theta \leq 0.$$

Using $s \geq 1/2$, we obtain $\beta \geq 7\theta_1/6 - \theta_2/6$. Hence the lemma follows by setting $\beta_0 = 7\theta_1/6 - \theta_2/6$. $\square$

Corollary 5.4. *For any fixed $\alpha > 0$, we have*

$$\mathrm{DT}_{\sigma_{\alpha,\beta}}(v) = \mathrm{DT}_H(v), \ \beta \ll 0.$$

Proof. If $(\alpha, \beta) \in \mathcal{S}_v$ and $s \gg 0$, then by Proposition 3.26 we have

$$\mathrm{DT}_{\sigma}(v) = \mathrm{DT}_H(v), \ \sigma = \sigma_{\alpha,\beta}^{a=s\alpha^2, b=0}.$$

By Lemma 5.3, for $\beta < \beta_0$, the wall $\mathcal{W}_v$ does not intersect with a path from $\sigma_{\alpha,\beta} = \sigma_{\alpha,\beta}^{a=\alpha^2/2}$ to $\sigma_{\alpha,\beta}^{a=s\alpha^2, b=0}$, $s \gg 0$. Therefore $\mathrm{DT}_{\sigma_{\alpha,\beta}}(v) = \mathrm{DT}_H(v)$. $\square$

We next describe the wall $\mathcal{W}_v$ on the (α, β)-plane, i.e. $\mathbb{H} \cap \mathcal{W}_v$ where $\mathbb{H} = \{\beta + i\alpha \in \mathbb{C} : \alpha > 0\}$ is embedded into $\mathrm{Stab}^{\circ}(A)$ via the map (61).

Lemma 5.5. *Suppose that $\gamma_i \in \mathcal{C}$ is written as $\gamma_i = u_i(1, \theta_i, \theta_i^2, \theta_i^3)$, $0 \neq u_i \in \mathbb{Z}$, $\theta_i \in \mathbb{Q}$ with $\theta_1 < \theta_2$. Then $\mathbb{H} \cap \mathcal{W}_v$ is*

$$\left(\alpha \pm \frac{\sqrt{3}}{6}(\theta_1 - \theta_2)\right)^2 + \left(\beta - \frac{\theta_1 + \theta_2}{2}\right)^2 = \frac{1}{3}(\theta_1 - \theta_2)^2, \ \mp u_1/u_2 > 0, \tag{68}$$

If $\gamma_1 = u_1(1, \theta_1, \theta_1^2, \theta_1^3)$ and $\gamma_2 = (0, 0, 0, u_2)$, then $\mathbb{H} \cap \mathcal{W}_v$ is

$$\beta = \pm\frac{\sqrt{3}}{3}\alpha + \theta_1, \ \pm u_1/u_2 > 0.$$

Proof. If $\gamma_i = u_i(1, \theta_i, \theta_i^2, \theta_i^3)$, we have

$$Z_{\alpha,\beta}(\gamma_i) = u_i(\beta - \theta_i + i\alpha)^3.$$

Then $\mathbb{H} \cap \mathcal{W}_v$ is

$$\frac{(\beta - \theta_1 + i\alpha)^3}{(\beta - \theta_2 + i\alpha)^3} \in \begin{cases} \mathbb{R}_{>0}, & u_1/u_2 > 0 \\ \mathbb{R}_{<0}, & u_1/u_2 < 0. \end{cases} \tag{69}$$

Since we have

$$\frac{(\beta - \theta_1 + i\alpha)}{(\beta - \theta_2 + i\alpha)} = \frac{1}{\alpha^2 + (\beta - \theta_2)^2}\left\{\alpha^2 + (\beta - \theta_1)(\beta - \theta_2) + i\alpha(\theta_1 - \theta_2)\right\}$$

and its imaginary part is negative, the condition (69) is equivalent to

$$\frac{\alpha(\theta_1 - \theta_2)}{\alpha^2 + (\beta - \theta_1)(\beta - \theta_2)} = \pm\sqrt{3}, \ \pm u_1/u_2 > 0.$$

By simplifying, we obtain the desired equation (68). The latter case is similar. □

The walls (68) are circles which intersects with the β-axis at $\beta = \theta_1, \theta_2$, see Figure 1.

5.5. **Proof of Theorem 1.3.** Suppose $v \in \Gamma$ is written as

$$v = \gamma_1 + \gamma_2, \ \gamma_i = r_i(p_i^3, p_i^2 q_i, p_i q_i^2, q_i^3) \in \mathcal{C} \tag{70}$$

with $\Theta(\gamma_1) < \Theta(\gamma_2)$ and let

$$g = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}_2(\mathbb{Z}).$$

Case 1. $-\frac{d}{c} \notin [\Theta(\gamma_1), \Theta(\gamma_2))$ or $c = 0$.

We take $\sigma_{\alpha,\beta}$ with $\beta \ll 0$. Then we have

$$\mathrm{DT}_H(g_* v) = \mathrm{DT}_{\sigma_{\alpha,\beta}}(g_* v) = \mathrm{DT}_{g_*^{-1}\sigma_{\alpha,\beta}}(v) = \mathrm{DT}_{\sigma_{\alpha',\beta'}}(v),$$

where by Lemma 5.1, we have

$$\beta' + i\alpha' = \frac{d(\beta + i\alpha) - b}{-c(\beta + i\alpha) + a}. \tag{71}$$

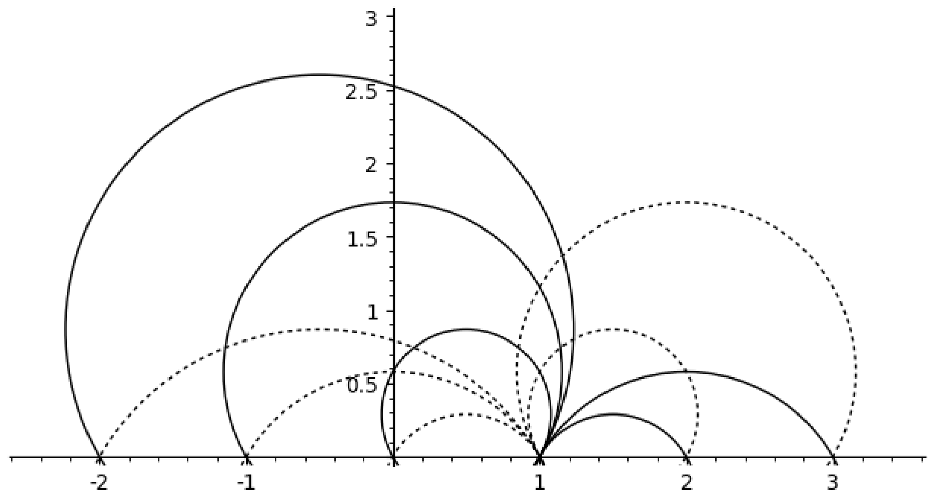

FIGURE 1. The walls $\mathcal{W}_v$ of type (68) for $\theta_1 = 1$ and $\theta_2 \in \{-2, -1, 0, 1/2, 3/2, 2, 3\}$. The circles are drawn dotted/solid depending on $u_1/u_2 \gtrless 0$.

For $\beta \to -\infty$ we get

$$\beta' + i\alpha' \to \begin{cases} -d/c + 0, & c \neq 0, \\ -\infty, & c = 0. \end{cases}$$

Therefore there exists a path in $\mathbb{H}$ which connects (α, β), $\beta \ll 0$ and (α', β'), and does not intersect with $\mathbb{H} \cap \mathcal{W}_v$. We conclude

$$\mathrm{DT}_{\sigma_{\alpha', \beta'}}(v) = \mathrm{DT}_{\sigma_{\alpha, \beta}}(v) = \mathrm{DT}_H(v)$$

as desired.

Case 2. $-\frac{d}{c} \in [\Theta(\gamma_1), \Theta(\gamma_2))$.

With the notation and argument of Step 1 it is enough to compute the right hand side of

$$\mathrm{DT}_H(v) - \mathrm{DT}_H(g_* v) = \mathrm{DT}_{\sigma_{\alpha, \beta}}(v) - \mathrm{DT}_{\sigma_{\alpha', \beta'}}(v)$$

for $\beta \ll 0$. By the asymptotic behavior (71), (α', β') lies inside (resp. RHS) of the wall $\mathbb{H} \cap \mathcal{W}_v$ if $\Theta(\gamma_2) < \infty$ (resp. $\Theta(\gamma_2) = \infty$). Let (α_0, β_0) lies on the wall $\mathbb{H} \cap \mathcal{W}_v$ and take $\sigma_0 = \sigma_{\alpha_0, \beta_0}$. Let $\sigma_\pm$ be small deformations of σ_0 such that their central charges $Z_\pm$ satisfy

$$\arg Z_+(\gamma_1) > \arg Z_+(\gamma_2), \ \arg Z_-(\gamma_1) < \arg Z_-(\gamma_2).$$

From the computations in Lemma 5.5, if $\Theta(\gamma_2) < \infty$ (resp. $\Theta(\gamma_2) = \infty$) then σ_+ lies in the outer (resp. LHS) of the wall $\mathbb{H} \cap \mathcal{W}_v$ and σ_- lies inside (resp. RHS) of it. Therefore we have

$$\mathrm{DT}_{\sigma_{\alpha, \beta}}(v) = \mathrm{DT}_{\sigma_+}(v), \ \mathrm{DT}_{\sigma_{\alpha', \beta'}}(v) = \mathrm{DT}_{\sigma_-}(v).$$

On the other hand, the equation (49) yields

$$\begin{aligned}\delta_{\sigma_0}(v,\phi) &= \delta_{\sigma_+}(v,\phi_+) + \delta_{\sigma_+}(\gamma_1,\phi_1) * \delta_{\sigma_+}(\gamma_2,\phi_2) + \cdots \\ &= \delta_{\sigma_-}(v,\phi_-) + \delta_{\sigma_-}(\gamma_2,\phi_2') * \delta_{\sigma_-}(\gamma_1,\phi_1') + \cdots .\end{aligned}$$

From this we obtain

$$\overline{\epsilon}_{\sigma_+}(v,\phi_+) - \overline{\epsilon}_{\sigma_-}(v,\phi_-) = -\{\overline{\epsilon}_{\sigma_0}(\gamma_1,\phi), \overline{\epsilon}_{\sigma_0}(\gamma_2,\phi)\} + \cdots .$$

By the proof of Proposition 4.9, after applying $\mathcal{I}^{\mathbf{A}}$, only the first term on the right contributes to the difference $\mathrm{DT}_{\sigma_+}(v) - \mathrm{DT}_{\sigma_-}(v)$. Since we have $\chi(\gamma_1,\gamma_2) = r_1 r_2 (p_1 q_2 - p_2 q_1)^3$ and using Corollary 4.13 we find

$$\begin{aligned}\mathrm{DT}_{\sigma_+}(v) - \mathrm{DT}_{\sigma_-}(v) =& (-1)^{r_1 r_2 (p_1 q_2 - p_2 q_1)} r_1 r_2 (p_1 q_2 - p_2 q_1)^3 \\ & \cdot \left(\sum_{k_1 \geq 1, k_1 | r_1} \frac{1}{k_1^2} \right) \left(\sum_{k_2 \geq 1, k_2 | r_2} \frac{1}{k_2^2} \right) |B_1 \cap B_2|,\end{aligned}$$

where $B_i \subset \Xi(E_i)$ is the connected component which contains $(0,0)$ for a semihomogeneous sheaf E_i with Chern character $\pm\gamma_i \in \Gamma_+$.

By [Muk, Theorem 4.9 (i)], we have $B_i = \Xi(F_i)$ for a Jordan-Holder factor of E_i, whose Chern character is $\overline{\gamma}_i = \pm(p_i^3, p_i^2 q_i, p_i q_i^2, q_i^3) \in \Gamma_+$. By [Muk, Theorem 4.9 (ii)], we hence obtain

$$|B_1 \cap B_2| = \chi(\overline{\gamma}_1, \overline{\gamma}_2)^2 = (p_1 q_2 - p_2 q_1)^6.$$

Therefore the result follows. $\square$

5.6. **Curve counting invariants.** For any $(\beta, n) \in \mathbb{Z}^2$ consider the rank one reduced Donaldson–Thomas invariant

$$\mathrm{DT}_{\beta,n} = \mathrm{DT}_H(1,0,-\beta,-n).$$

We want to study the behaviour of $\mathrm{DT}_{\beta,n}$ under Fourier-Mukai transforms.

The following Lemma gives a strong constraint when two such rank 1 classes can be related by a Fourier-Mukai transform.

Lemma 5.6. *Let $(\beta,n) \in \mathbb{Z}^2$ and suppose that*

$$g(1,0,-\beta,-n) = (1,0,-\beta',-n') \tag{72}$$

for some $(\beta',n') \in \mathbb{Z}^2$ and $g \in \mathrm{SL}_2(\mathbb{Z})$. Then there is $(c,d) \in \mathbb{Z}^2$ satisfying

$$d^3 - 3\beta c^2 d - n c^3 = 1$$

such that we have

$$(\beta', n') = (d^2\beta + ncd + \beta^2 c^2, 6\beta^2 d^2 c + 6c^2 d\beta n + n + 2c^3 n^2 - 2c^3\beta^3). \tag{73}$$

Proof. Let $g = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \mathrm{SL}_2(\mathbb{Z})$. The condition (72) gives

$$(dx+by)^3 - 3\beta(dx+by)(cx+ay)^2 - n(cx+ay)^3 = x^3 - 3\beta' xy^2 - n'y^3.$$

We obtain the equations

$$\begin{aligned} d^3 - 3\beta c^2 d - nc^3 &= 1, \\ bd^2 - \beta(2acd + bc^2) - nac^2 &= 0, \\ \beta' &= \beta(a^2 d + 2abc) - b^2 d + a^2 cn, \\ n' &= a^3 n + 3\beta a^2 d - b^3. \end{aligned} \tag{74}$$

Since $ad - bc = 1$, comparing with the first equation of (74) gives

$$a = d^2 - 3\beta c^2 + mc, \ \ b = nc^2 + md \tag{75}$$

for some $m \in \mathbb{Z}$. By substituting this into the second equations of (72), we obtain $m = 2\beta c$. By substituting (75) into the third and fourth equation of (74), and simplifying, we obtain (73). □

Let $C_{\beta,n} \in \mathbb{Q}$ be the conjectural value of $\mathrm{DT}_{\beta,n}$ defined by the right hand side of (7). By Lemma 5.6 and an elementary check we have

$$C_{\beta,n} = C_{\beta',n'}$$

whenever (β, n) and (β', n') are related as in (72). We therefore obtain the following evidence for Conjecture 1.4.

Corollary 5.7. *If $4\beta^3 - n^2 \geq 0$ and (β', n') is as in (72), then*

$$\mathrm{DT}_{\beta,n} = \mathrm{DT}_{\beta',n'}\,.$$

In particular, $\mathrm{DT}_{\beta,n} = C_{\beta,n}$ if and only if $\mathrm{DT}_{\beta',n'} = C_{\beta',n'}$.

Proof. Since $\Delta(1, 0, -\beta, -n) = 4\beta^3 - n^2$ this follows from Theorem 1.1 and Proposition 1.2. □

Suppose that $(\beta, n) \in \mathbb{Z}^2$ satisfies

$$(1, 0, -\beta, -n) = \gamma_1 + \gamma_2, \ \ \gamma_i \in \mathcal{C}, \ \ \Theta(\gamma_1) < \Theta(\gamma_2).$$

We address the following question:

Conjecture 5.8. *Suppose that $\beta \neq 0$ or $n > 0$. For any integer solution (c, d) of $d^3 - 3\beta c^2 d - nc^3 = 1$, we have*

$$-\frac{d}{c} \notin (\Theta(\gamma_1), \Theta(\gamma_2)).$$

Example 5.9. If $\beta = 0$ and $n > 0$, then we have

$$(1, 0, 0, -n) = \gamma_1 + \gamma_2, \ \ \gamma_1 = (1, 0, 0, 0), \gamma_2 = -n(0, 0, 0, 1)$$

and $\Theta(\gamma_1) = 0$, $\Theta(\gamma_2) = \infty$. In this case, for any integer solution (c, d) of $d^3 - nc^3 = 1$ we have $-d/c \notin (0, \infty)$. Moreover $-d/c = 0$ only if $n = 1$ and $(c, d) = (-1, 0)$. In this case, (β', n') given by (73) is $(0, -1)$.

We have the following lemma:

Lemma 5.10. *Conjecture 5.8 is equivalent to the following: for $\beta \neq 0$ or $n > 0$ and an integer solution (c, d) of $d^3 - 3\beta c^2 d - nc^3 = 1$, if we have*

$$-\frac{d}{c} \in [\Theta(\gamma_1), \Theta(\gamma_2)) \tag{76}$$

then $\beta' = 0$ and $n' \leq 0$. Here (β', n') is given by (73).

Proof. By Example 5.9, we may assume that $\beta \neq 0$. By writing $\theta_i = \Theta(\gamma_i)$, the computation in Lemma 5.2 shows

$$\theta_1 + \theta_2 = \frac{n}{\beta},\ \theta_1\theta_2 = \beta. \tag{77}$$

It follows that

$$\left(\theta_1 + \frac{d}{c}\right)\left(\theta_2 + \frac{d}{c}\right) = \frac{1}{c^2} \cdot \frac{\beta'}{\beta}. \tag{78}$$

Suppose that Conjecture 5.8 is true. Then the condition (76) implies $-d/c = \theta_1$, hence $\beta' = 0$ follows. Suppose by a contradiction that $n' > 0$. Note that

$$g^{-1}(1, 0, 0, -n') = (1, 0, -\beta, -n). \tag{79}$$

We write

$$g^{-1} = \begin{pmatrix} a' & b' \\ c' & d' \end{pmatrix} = \begin{pmatrix} d & -b \\ -c & a \end{pmatrix}.$$

Then the condition (79) implies $(d')^3 - n'(c')^2 = 1$, and the condition (76) implies that $-d'/c' \in [0, \infty)$ (see Remark 5.9). By Remark 5.9, this implies that $d' = a = 0$. By (75), we have $a = d^2 - \beta c^2 = 0$, thus $\beta = \theta_1^2$ follows. By (77), we have $\theta_1^2 = \theta_1\theta_2$. Since $\theta_1 \neq \theta_2$, we have $\theta_1 = 0$ and $\beta = 0$, a contradiction.

The converse statement follows from (78). □

Remark 5.11. If Conjecture 5.8 is false, then by Lemma 5.10 we have $\mathrm{DT}_{\beta,n} \neq \mathrm{DT}_{\beta',n'}$ for $\beta' \neq 0$ or $n' > 0$, while $C_{\beta,n} = C_{\beta',n'}$. So either (β, n) or (β', n') would give a counter-example to Conjecture 1.4.

Theorem 1.3 (i) and Lemma 5.10 immediately implies the following:

Corollary 5.12. *For $\beta \neq 0$ or $n > 0$, suppose that Conjecture 5.8 is true. Then for any integer solution (c, d) of $d^3 - 3\beta c^2 d - nc^3 = 1$ with either $\beta' \neq 0$ or $n' > 0$, we have $\mathrm{DT}_{\beta,n} = C_{\beta,n}$ if and only if $\mathrm{DT}_{\beta',n'} = C_{\beta',n'}$ holds.*

By Example 5.9, we can apply the above corollary for $\beta = 0$ and $n > 0$. Since $\mathrm{DT}_{0,n} = C_{0,n}$ holds by [She15], we obtain the following:

Corollary 5.13. *For $n > 0$ and any integer solution (c, d) of $d^3 - nc^3 = 1$, except $n = 1$ and $(c, d) = (-1, 0)$, we have*

$$\mathrm{DT}_{cdn, n+2c^3n^2} = (-1)^{n-1}\frac{1}{n}\sum_{k \geq 1, k|n} k^2.$$

Appendix A. Spin representations and the discriminant

Let U be a $\mathbb{Q}$-vector space with basis $x_1, \ldots, x_n$. The algebra of endomorphisms of the exterior algebra $\bigwedge^\bullet U$ is the exterior algebra generated by multiplication by x_i and differentiation (i.e. interior product) $\partial/\partial x_i$:

$$\mathrm{End}_{\mathbb{Q}}\left(\bigwedge^\bullet U\right) = \bigwedge^\bullet \left\langle x_1\wedge\,, \ldots, x_n\wedge\,, \frac{\partial}{\partial x_1}, \ldots, \frac{\partial}{\partial x_n} \right\rangle.$$

The Lie subalgebra of $\mathrm{End}_{\mathbb{Q}}(\wedge^\bullet U)$ generated by

$$x_i \wedge x_j, \quad x_i \wedge \frac{\partial}{\partial x_j} - \frac{1}{2}\delta_{ij}, \quad \frac{\partial^2}{\partial x_i \partial x_j}, \quad 1 \le i < j \le n \tag{80}$$

is isomorphic to $\mathfrak{so}(2n)$, and the induced action of $\mathfrak{so}(2n)$ on $\bigwedge^\bullet U$ is called the *spin representation*. This Lie algebra action integrates to a representation of the spin group $\mathrm{Spin}(2n)$.

The action by $\mathfrak{so}(2n)$ preserves the decomposition

$$\bigwedge^\bullet U = \bigwedge^{\mathrm{even}} U \oplus \bigwedge^{\mathrm{odd}} U.$$

where $\bigwedge^{\mathrm{even/odd}} U$ is the subspace spanned by all even/odd wedge products. The induced action of the spin group on $\bigwedge^{\mathrm{even/odd}} U$ is irreducible and called the even/odd half-spin representation.

There exist a unique (up to scalar) invariant bilinear form β on $\bigwedge^{\mathrm{even}} U$. If n is even, we normalize β by $\beta(1, \prod_{i=1}^n x_i) = 1$.

Remark A.1. If A is an abelian variety of dimension g, then $H^1(A, \mathbb{Q})$ is of dimension $2g$ and

$$H^*(A, \mathbb{Q}) = \bigwedge^\bullet H^1(A, \mathbb{Q}).$$

The action of the group of derived autoequivalences on $H^*(A, \mathbb{Q})$ factors through the spin representation of $\mathrm{Spin}(4g)$, see [Muk, Section 3]. Every function on $H^*(A, \mathbb{Q})$ invariant under Spin(4g) is therefore invariant under all autoequivalences. For instance the invariant bilinear form β is the Euler pairing:

$$\forall E, F \in \mathrm{Coh}(A): \quad \chi(E, F) = \beta(\mathrm{ch}(E), \mathrm{ch}(F)).$$

Theorem A.2. *Assume* $\dim(U) = 6$*.*

a) *There exist a unique homogeneous degree* 4 *polynomial function*

$$\Delta : \bigwedge^{even} U \to \mathbb{Q}$$

which is invariant under the action of $\mathrm{Spin}(12)$*. We normalize* Δ *by* $\Delta(1 + \prod_{i=1}^6 x_i) = -1$*.*

b) *We have* $\Delta(e^\omega) = 0$ *for all* $\omega \in \bigwedge^2 U$*.*

c) *For all* $r_1, r_2 \in \mathbb{Z}$ *and* $\omega_1, \omega_2 \in \bigwedge^2 U$ *we have*

$$\Delta(r_1 e^{\omega_1} + r_2 e^{\omega_2}) = -\beta(r_1 e^{\omega_1}, r_2 e^{\omega_2})^2.$$

Remark A.3. Let $A = E_1 \times E_2 \times E_3$ where E_1, E_2, E_3 are very general elliptic curves. The subalgebra of algebraic classes $\Gamma \subset H^*(A, \mathbb{Q})$ is generated by

$$L_i = \pi_i^*[\mathbf{p}_i] \in H^2(A, \mathbb{Z}), \quad i = 1, 2, 3$$

where $\mathbf{p}_i \in H^2(E_i)$ is the point class and $\pi_i : A \to E_i$ is the projection. If

$$\gamma = (r, b_1 L_1 + b_2 L_2 + b_3 L_3, d_1 L_2 L_3 + d_2 L_1 L_3 + d_3 L_1 L_2, n) \in \Gamma$$

is a general element, then the discriminant of γ is

$$\begin{aligned} \Delta(\gamma) = & -n^2 r^2 - 4(r d_1 d_2 d_3 + b_1 b_2 b_3 n) \\ & - (b_1^2 d_1^2 + b_2^2 d_2^2 + b_3^2 d_3^2) \\ & + 2 b_1 b_2 d_1 d_2 + 2 b_1 b_3 d_1 d_3 + 2 b_2 b_3 d_2 d_3 \\ & + 2rn(b_1 d_1 + b_2 d_2 + b_3 d_3). \end{aligned}$$

Proof of Theorem A.2. Let $V = \bigwedge^{\mathrm{even}} U$. By a calculation in [SAGE] the tensor product $V^{\otimes 4}$ contains 4 copies of the trivial representation.[13] Three of them arise from $\beta \otimes \beta$ by permuting factors, and hence are not S_4 invariant. This shows uniqueness. We prove existence. Consider a general element

$$\gamma = \sum_{\substack{I \subset \{1,2,3,4,5,6\} \\ |I| \text{ even}}} a_I x_I$$

where $a_I \in \mathbb{Q}$ and $x_I = \prod_{i \in I} x_i$. We make the ansatz

$$\Delta(\gamma) = \sum_{I = (I_1, I_2, I_3, I_4)} c_I a_{I_1} a_{I_2} a_{I_3} a_{I_4} \tag{81}$$

for some $c_I \in \mathbb{Q}$, where the I_j run over all even subsets of $\{1, \ldots, 6\}$ such that every $1 \leq i \leq 6$ appears in the subsets $I_j, j = 1, 2, 3, 4$ exactly twice. A computer calculation[14] shows that there exist unique (up to scaling) c_I such that Δ is invariant under the generators (80). This proves (a).

Multiplication by $\omega \in \wedge^2 U$ is an element of the Lie algebra $\mathfrak{so}(12)$, hence multiplication by e^ω is an element of $\mathrm{Spin}(12)$. It follows

$$\Delta(e^\omega) = \Delta(1) = 0$$

where the last equality follows from (81). This shows part (b).

Finally, (c) follows again by a direct computer calculation. $\square$

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[13]The submission line to SAGE is:

```
chi=WeylCharacterRing("D6")(1/2,1/2,1/2,1/2,1/2,1/2); chi^4
```

with result `chi^4 = 4*D6(0,0,0,0,0,0) + (...)`.

[14]The code for this computation is available on the first author's webpage.

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