ON THE CHERN CHARACTER NUMBERS OF THE HILBERT SCHEME OF POINTS ON A K3 SURFACE

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Abstract. We propose a conjectural formula for the partition function of Chern character numbers of the Hilbert schemes of points on a K3 surface.

1. Conjecture

Let $S^{[n]}$ be the Hilbert scheme of points on a K3 surface S. Define the series

$$\mathcal{L}(q) = \sum_{r\geq 1} (-1)^r 2^{r-1} \sum_{\substack{k_1,\ldots,k_r \\ \ell_1,\ldots,\ell_r}} \frac{t_{2(k_1+\ell_r)} t_{2(k_2+\ell_1)} \cdots t_{2(k_r+\ell_{r-1})} q^{\sum_i (k_i+\ell_i)}}{\left(\prod_{i=1}^{r-1} k_i!(k_i-1)!\ell_i!(\ell_i-1)!(k_i+\ell_i)\right) k_r!^2 \ell_r!^2}$$

where the k_i, ℓ_i run over all non-negative integers, not all zero, such that

- $k_i \geq 1$ and $\ell_i \geq 1$ for all $i \leq r-1$

Similarly, define

$$\mathcal{M}(q) = \sum_{r\geq 1} \frac{2^{r-1}}{r} \sum_{\substack{k_1,\ldots,k_r \geq 0 \\ \ell_1,\ldots,\ell_r \geq 1}} \frac{t_{2(k_1+\ell_r)} t_{2(k_2+\ell_1)} \cdots t_{2(k_r+\ell_{r-1})} q^{\sum_i (k_i+\ell_i)}}{\left(\prod_{i=1}^{r} (k_i+\ell_i) k_i!^2 \ell_i!(\ell_i-1)!\right)} (\ell_1 \cdots \ell_r + (-1)^r k_1 \ldots k_r).$$

Both series define elements in the ring $\mathbb{Q}[t_2, t_4, t_6, \ldots][[q]]$.

The following is the main conjecture:

Conjecture A. *We have*

$$\sum_{n\geq 0} Q^n \int_{S^{[n]}} \exp\left(\sum_{k\geq 1} t_k \mathsf{ch}_k(\mathrm{Tan}) \right) = -\frac{Q^2}{q^2} \frac{dq}{dQ} \exp\Big(24 \mathcal{M}(q) \Big)$$

where q and Q are related by the variable change $Q = -q \exp(2\mathcal{L}(q))$.

Remarks. (i) By Lagrange inversion the conjecture is equivalent to:

$$\int_{S^{[n]}} \exp\left(\sum_{k\geq 1} t_k \mathsf{ch}_k(\mathrm{Tan}) \right) = (-1)^n \mathrm{Coeff}_{q^n} \Big[\exp((2-2n)\mathcal{L} + 24\mathcal{M}) \Big] \tag{1.1}$$

(ii) Consider the partition function of Chern character numbers of the Hilbert schemes:

$$Z_S(q) = \sum_n q^n \int_{S^{[n]}} \exp\left(\sum_{k\geq 1} t_k \mathsf{ch}_k(\mathrm{Tan}) \right)$$

Since $\exp(\sum_k \mathsf{ch}_k(-) t_k)$ is a multiplicative genus, by [2] there exist power series $f(q,t), g(q,t) \in \mathbb{Q}[[q, t_1, t_2, \ldots]]$ such that for all surface S we have

$$Z_S(q) = \exp\left(c_1(S)^2 f(q,t) + c_2(S) g(q,t) \right).$$

Conjecture A would determine $g(q,t)$.

Moreover, the partition function Z_S can be computed for the toric surfaces $\mathbb{P}^2$ and $\mathbb{P}^1 \times \mathbb{P}^1$ using Bott's residue formula to any fixed degree. This leads to an algorithm to compute f, g and hence the Chern numbers of $S^{[n]}$ for S a K3 surface, see [2]. This algorithm has been implemented by Jieao Song [10]. Using his program we have checked Conjecture A up to $n \leq 10$.

(iii) The formula of Conjecture A is quite unwieldy, and hopefully a much more efficient expression can

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be found eventually. We only remark that it is computationally slightly better than the [2] algorithm. (v) It is expected that all Chern numbers of $S^{[n]}$ are positive, see [2]. Unfortunately, the proposed formula does not yield any insight into this remarkable fact.

Example 1. We take the coefficient of the monomial $t_{2n}q^n$ in (1.1). Observe that:

$$\mathrm{Coeff}_{t_{2n}q^n}(\mathcal{L}) = (-1)\sum_{k+\ell=n}\frac{1}{k!^2\ell!^2} = (-1)\frac{(2n)!}{n!^4}$$

$$\begin{aligned}\mathrm{Coeff}_{t_{2n}q^n}(\mathcal{M}) &= \sum_{k+\ell=n}\frac{\ell(\ell-k)}{n\cdot k!^2\cdot \ell!^2} \\ &= \frac{2}{n}\sum_{k+\ell=n}\frac{\ell^2}{k!^2\cdot\ell!^2} - \sum_{k+\ell=n}\frac{\ell}{k!^2\cdot\ell!^2} \\ &= \frac{2}{n\cdot n!^2}\left(\frac{n^3}{2(2n-1)}\binom{2n}{n}\right) - \frac{1}{n!^2}\frac{n}{2}\binom{2n}{n} \\ &= \frac{(2n)!}{n!^4}\frac{n}{2(2n-1)}\end{aligned}$$

where we used the well-known combinatorial identities:

$$\sum_{i=0}^n\binom{n}{i}^2 = \binom{2n}{n}, \qquad \sum_{i=0}^n i\binom{n}{i}^2 = \frac{n}{2}\binom{2n}{n}, \qquad \sum_{i=0}^n i^2\binom{n}{i}^2 = \frac{n^3}{2(2n-1)}\binom{2n}{n}.$$

Hence, Conjecture A would via (1.1) imply

$$\begin{aligned}\int_{S^{[n]}}\mathsf{ch}_{2n}(\mathrm{Tan}) &= (-1)^n\mathrm{Coeff}_{t_{2n}q^n}\Big[\exp((2-2n)\mathcal{L}+24\mathcal{M})\Big] \\ &= (-1)^n\frac{(2n)!}{n!^4}\left(2n-2+12\frac{n}{2n-1}\right) \\ &= (-1)^n\frac{(2n+2)!}{n!^4(2n-1)}.\end{aligned}$$

This indeed holds as has been checked in [9].

Example 2. We specialize to $t_4 = t_6 = \ldots = 0$, that is the integrals over the top power of ch_2. Since the power of q is determined by the power of t_2 we may set $t_2 = q := t$. Under this specialization, one easily computes:

$$\begin{aligned}\mathcal{L} &= -2t+t^2 \\ Q &= -te^{2(-2t+t^2)} \\ \frac{dQ}{dt} &= e^{2(-2t+t^2)}(-1+4t-4t^2) \\ \mathcal{M} &= -\frac{1}{2}\ln(1-2t).\end{aligned}$$

Hence Conjecture A would imply the closed formula:

$$\sum_{n\geq 0}Q^{n-1}\int_{S^{[n]}}\frac{\mathsf{ch}_2(\mathrm{Tan})^n}{n!} = \frac{-1}{t(1-2t)^{14}}$$

under the variable change $Q = -te^{2(-2t+t^2)}$. It appears quite difficult to guess such a formula directly.

Example 3. The total Chern class $c(E) = 1 + c_1(E) + \ldots + c_{\mathrm{rk}(E)}(E)$ of a vector bundle E can be expressed in terms of its Chern characters by:

$$c(E) = \exp\left(\sum_{k\geq 1}(-1)^{k-1}(k-1)!\mathsf{ch}_k(E)\right).$$

If we specialize t_k to $(-1)^{k-1}(k-1)!$, Conjecture A hence would imply that

$$\prod_{n\geq 1}\frac{1}{(1-q^n)^{24}}=\sum_{n\geq 0}Q^n e(S^{[n]})=-\frac{Q^2}{q^2}\frac{dq}{dQ}\exp\Big(24\mathcal{M}(q)\Big)\Bigg|_{t_k=(-1)^{k-1}(k-1)!}$$

It would be interesting to see a proof of this algebraic identity.

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2. Origin of conjecture

In the remainder of this note we explain the origins of the above conjecture. The source is the comparison between the elliptic genus of $S^{[n]}$ (computed by Borisov-Libgober) with a formula for the Donaldson-Thomas partition function of $K3\times E$ obtained by the degeneration formula.

2.1. **Elliptic genus.** Let $z\in\mathbb{C}$ and $\tau\in\mathbb{H}$, and set $q=e^{2\pi i\tau}$. (Here $z=2\pi i z'$, where z' is the standard coordinate on the elliptic curve $\mathbb{C}/(\mathbb{Z}+\mathbb{Z}\tau)$). Consider the Jacobi theta function

$$\Theta(z)=z\exp\left(-2\sum_{k\geq 2}G_k(\tau)\frac{z^k}{k!}\right). \tag{2.1}$$

where $G_k(\tau)=-\frac{B_k}{2\cdot k}+\sum_{n\geq 1}\sum_{d|n}d^{k-1}q^n$ are the Eisenstein series.

Definition 4. The elliptic genus of a smooth compact complex manifold X is

$$\mathrm{Ell}(X)=\int_X\prod_i x_i\frac{\Theta(x_i-z)}{\Theta(x_i)}$$

where x_i are the Chern roots of the tangent bundle.

We refer to [4] for an introduction to the elliptic genus. By expressing the integrand as an exponential, the elliptic genus can be easily computed in terms of the Chern character numbers of X. Indeed, define functions for $k\geq 1$ by

$$F_k(z)=\frac{(-1)^k(k-1)!}{z^k}+2\sum_{\ell>k}G_\ell(\tau)\frac{z^{\ell-k}}{(\ell-k)!}$$

These function appear in the expansion

$$\log\left(\frac{x\Theta(x+z)}{\Theta(x)}\right)=\log(\Theta(z))-\sum_{k\geq 1}\frac{x^k}{k!}F_k(z) \tag{2.2}$$

which immediately leads to:

$$\mathrm{Ell}(X)=\Theta(-z)^{\dim X}\int_X\exp\left(\sum_{k\geq 1}(-1)^{k-1}F_k(z)\mathsf{ch}_k(\mathrm{Tan})\right).$$

Since the F_k are (meromorphic) quasi-Jacobi forms of weight k, and F_k for all $k\geq 2$ are Jacobi forms, we see that the elliptic genus is always a quasi-Jacobi form, and if X is Calabi-Yau, a Jacobi form.

The elliptic genera of the Hilbert schemes of points of a K3 surface is determined by the following beautiful formula of Borisov-Libgober:

Theorem 5 ([1])**.** *Let S be a K3 surface. Then we have*

$$\frac{1}{\Theta(z)^2\Delta(\tau)}\sum_{n\geq 0}\tilde{q}^{n-1}\,\mathrm{Ell}(S^{[n]})=\frac{1}{\chi_{10}\left(\begin{smallmatrix}\tau & z\\ z & \tilde{\tau}\end{smallmatrix}\right)}$$

where χ_{10} is the Igusa cusp form and $\tilde{q}=e^{2\pi i\tilde{\tau}}$ with $\tilde{\tau}\in\mathbb{H}$.

2.2. **Donaldson-Thomas theory.** Let $X = S \times E$ be the product of a K3 surface and an elliptic curve. Let $\mathsf{Hilb}_{k,(\beta,n)}(X)$ be the Hilbert scheme parametrizing 1-dimensional subschemes Z of X in class $[Z] = (\beta, n) \in H_2(X,\mathbb{Z})$ and with Euler characteristic $\chi(\mathcal{O}_Z) = k$. The elliptic curve acts on the Hilbert scheme by translation and the reduced Donaldson-Thomas invariants of X are defined by taking the orbifold Euler characteristic of the quotient stack:

$$\mathsf{DT}_{k,(\beta,n)} = e\left(\mathsf{Hilb}_{k,(\beta,n)}(X)/E, \nu\right)$$

weighted by Behrend's constructible function ν. We refer to [6] for the foundations of the theory. Given a primitive curve class $\beta_h \in H_2(S,\mathbb{Z})$ of square $\beta_h^2 = 2h-2$, the reduced DT invariant only depends on h, and we write $\mathsf{DT}_{k,(h,n)}$ for $\mathsf{DT}_{k,(\beta,n)}$. We form the generating series

$$Z(K3\times E) = \sum_{k,h,n} \mathsf{DT}_{k,(h,n)} (-1)^k e^{2\pi i z k} q^{h-1} \tilde{q}^{n-1}.$$

The following result gives a closed evaluation:

Theorem 6 ([8])**.** $Z(K3\times E) = \frac{-1}{\chi_{10}\begin{pmatrix} \tau & z \\ z & \tilde{\tau}\end{pmatrix}}$

In other words, up to constants in $\tilde{q}$, the DT partition function of $K3 \times E$ is the generating series of elliptic genera of the Hilbert schemes of points.

Theorem 6 has been proven by constraining the partition function using derived equivalences and holomorphic anomaly equations. A more direct path is to use the degeneration formula (for the degeneration of the elliptic curve to a nodal $\mathbb{P}^1$) which expresses $Z(K3\times E)$ as a trace-like series over the DT theory of $K3 \times \mathbb{P}^1$ relative to the fibers over two points in $\mathbb{P}^1$, see [7, Sec.1.3] for a formula. A conjectural expression for these relative DT invariants was found in [5] and made explicit recently [3]. This leads to an alternative expression for $Z(K3\times E)$ that we now derive.

To avoid technical details and somewhat annoying computations, the reader may also go directly to Section 2.4.

2.3. **Degeneration formula.** We will need two sets of functions, defined in [3].

Definition 7. For any $m \in \mathbb{Z}$ let $\varphi_m(z,\tau)$ be the unique solution to the Jacobi Kaneko-Zagier differential equation

$$\begin{cases} D_\tau^2 \varphi_m = m^2 F \varphi_m \\ \varphi_m = (e^{\pi i m z} - e^{-\pi i m z}) + O(q) \end{cases}$$

where $F = \frac{D_\tau^2 \Theta(z)}{\Theta(z)}$ and $D_\tau = q\frac{d}{dq}$.

Another characterizing property of these function is that their generating series $\Phi(y) = \sum_{m=1}^\infty \frac{\varphi_m}{m} y^m$ is the inverse of the ratio $\Theta(x)/\Theta(x+z)$, i.e. $\Phi\left(\frac{\Theta(x)}{\Theta(x+z)}\right) = x$.

Definition 8. For all $m,n \in \mathbb{Z}$ define $\varphi_{m,n}(z,\tau)$ as the unique solution to:

$$\begin{cases} D_\tau \varphi_{m,n} = mn\varphi_m\varphi_n F + (D_\tau\varphi_m)(D_\tau\varphi_n) \\ \varphi_{m,n} = O(q) \end{cases}$$

The degeneration formula [7, Sec.1.3] and the conjectural expression for the relative PT theory of $K3\times\mathbb{P}^1$ in [3] together with some tedious work computing the trace implies the following:

Proposition 9. *Assume that* [3, Conjectures 1.1 and 1.5] *hold. Then*

$$\operatorname{Coeff}_{q^{n-1}} Z(K3\times E) = -\operatorname{Coeff}_{q^{n-1}}\left(\frac{\exp((n-1)L)}{\exp(M)^{12}\Theta(z,\tau)^2\Delta(\tau)\Delta(\tilde{\tau})}\right). \tag{2.3}$$

where the functions $L(z,\tau,\tilde{\tau})$ and $M(z,\tau,\tilde{\tau})$ are defined by

$$L = \sum_{r\geq 1}(-1)^{r-1} \sum_{d_1,\ldots,d_r \in \mathbb{Z}\setminus 0} \frac{1}{|d_1|\cdots|d_r|}\frac{\tilde{q}^{|d_1|}}{1-\tilde{q}^{|d_1|}}\cdots\frac{\tilde{q}^{|d_r|}}{1-\tilde{q}^{|d_r|}} \varphi_{d_1}\varphi_{-d_1,d_2}\cdots\varphi_{-d_{r-1},d_r}\varphi_{-d_r}$$

$$M = \sum_{r\geq 1}\frac{(-1)^{r-1}}{r} \sum_{d_1,\ldots,d_r \in \mathbb{Z}\setminus 0} \frac{1}{|d_1|\cdots|d_r|}\frac{\tilde{q}^{|d_1|}}{1-\tilde{q}^{|d_1|}}\cdots\frac{\tilde{q}^{|d_r|}}{1-\tilde{q}^{|d_r|}} \varphi_{d_1,-d_2}\cdots\varphi_{d_r,-d_1}$$

To evaluate the right hand side of (2.3) further, we now use that φ_m and $\varphi_{m,n}$ are polynomial in m and n, that is there exists Laurent polynomials $b_i^r(s)$ and $a_{ij}^r(s)$ where $s = e^{\pi i z}$ such that

$$\varphi_m = (s^m - s^{-m}) \sum_{r\geq 0} \sum_{\substack{i=0 \\ i \text{ even}}}^{2r} b_i^r(s) m^i q^r$$

$$\varphi_{m,n} = (s^m - s^{-m})(s^n - s^{-n}) \sum_{r\geq 0} \sum_{\substack{i,j\geq 0 \\ i+j\leq 2r}} a_{ij}^r(s) m^i n^j q^r.$$

Note that $\varphi_{-m,-n} = \varphi_{m,n}$ we have $a_{ij}^r = 0$ unless $i+j$ even. Inserting into (2.3) we a bit of computation this leads to:[1]

$$-\operatorname{Coeff}_{q^{n-1}} Z(K3\times E) = \operatorname{Coeff}_{q^{n-1}} \left(\frac{\exp((2-2n)L^{\mathrm{red}})\exp(M^{\mathrm{red}})}{\Theta^2(z,\tau)\Delta(\tau)(s-s^{-1})^{2n-2}} \right) \frac{\Theta(z,\tilde{\tau})^{2n-2}}{\Delta(\tilde{\tau})} \tag{2.4}$$

where we let

$$M^{\mathrm{red}} = 24 \sum_{r\geq 1} \frac{2^r}{2r} \sum_{\substack{\ell_1,\dots,\ell_r \\ i_1,\dots,i_r \\ j_1,\dots,j_r}} a_{i_1 j_1}^{\ell_1} \cdots a_{i_r j_r}^{\ell_r} (-1)^{j_1+\dots+j_r} \widetilde{F}_{i_1+j_r} \widetilde{F}_{i_2+j_1} \cdots \widetilde{F}_{i_r+j_{r-1}} q^{\ell_1+\dots+\ell_r},$$

where we require $i_k, j_k, \ell_k \geq 1$ and $i_k + j_{k+1}$ even for all k, as well as

$$L^{\mathrm{red}} = \sum_{r\geq 1} \frac{2^r}{2} \sum_{\substack{\ell_0,\dots,\ell_r \\ i_1,\dots,i_r \\ j_0,\dots,j_{r-1}}} b_{j_0}^{\ell_0} a_{i_1 j_1}^{\ell_1} \cdots a_{i_{r-1} j_{r-1}}^{\ell_{r-1}} b_{i_r}^{\ell_r} (-1)^{i_1+\dots+i_r} \widetilde{F}_{j_0+i_1} \cdots \widetilde{F}_{j_{r-1}+i_r} q^{\ell_0+\dots\ell_r}$$

where we require $j_k + i_{k+1}$ even for all k, and if $r = 1$ then $(j_0, i_1) \neq (0,0)$; we also used the series:

$$\widetilde{F}_k = F_k(z) - \frac{(k-1)!}{(1-p)^k} + \zeta(-(k-1)).$$

One can show that for fixed r the polynomials a_{ij}^r and b_j^r that contribute the heighest degree in m and n are given by:

$$\begin{aligned} a_{i,j}^{(i+j)/2} &= (s-s^{-1})^{i+j}(-1)^{\frac{i+j}{2}} a_{ij} \\ b_j^{j/2} &= (s-s^{-1})^j (-1)^{j/2} \frac{1}{(j/2)!^2} \end{aligned} \tag{2.5}$$

where a_{ij} are define to vanish except for the following cases:

$$\begin{aligned} a_{2k,2\ell} &= \frac{1}{(k+\ell)} \cdot \frac{1}{k!(k-1)!\ell!(\ell-1)!} \quad &\text{if } k,\ell > 0 \\ a_{2k+1,2\ell+1} &= \frac{1}{(k+\ell+1)} \cdot \frac{1}{k!^2 \ell!^2} \quad &\text{if } k,\ell \geq 0. \end{aligned} \tag{2.6}$$

Finally, comparing (2.4) with Theorem 6 we expect that the quantity

$$C = \operatorname{Coeff}_{q^{n-1}} \left(\frac{\exp((2-2n)L^{\mathrm{red}})\exp(M^{\mathrm{red}})}{\Theta^2(z,\tau)\Delta(\tau)(s-s^{-1})^{2n-2}} \right)$$

is a Jacobi form of weight $2n$. From definition C is a linear combination of Jacobi forms of weight $\leq 2n$ with coefficients Laurent polynomials in s. Using the leading terms (2.5) the weight $2n$ term can be explicitly written down and one sees that the coefficients are rational numbers. Hence, for the statement to be true all lower order terms have to vanish. *We assume this vanishing in the following.*

[1] The key is the following evaluation: For $k \geq 1$ we have

$$\sum_{d\in\mathbb{Z}_{\neq 0}} \frac{q^{|d|}}{1-q^{|d|}} \frac{d^k}{|d|} (s^d - s^{-d})^2 = \begin{cases} 0 & \text{if } k \text{ is odd} \\ 2\widetilde{F}_k(z,\tau) & \text{if } k \text{ is even} \end{cases}$$

where $\zeta(-(k-1)) = (-1)^{k-1} B_k / k$.

2.4. **Conclusion.** We have obtained two different expressions for the coefficient $\tilde{q}^{n-1}$ of $1/\chi_{10}$: Using the elliptic genus computation we have:

$$\begin{aligned} \mathrm{Coeff}_{\tilde{q}^{n-1}}\left(\frac{1}{\chi_{10}}\right) &= \frac{1}{\Theta(z)^2\Delta(\tau)}\,\mathrm{Ell}(S^{[n]}) \\ &= \frac{\Theta(z)^{2n-2}}{\Delta(\tau)} \int_{S^{[n]}} \exp\left(\sum_{k\geq 1}(-1)^{k-1}F_k(z)\mathsf{ch}_k(\mathrm{Tan})\right). \end{aligned}$$

Using the degeneration formula and conjectures of [5, 3] we have

$$\begin{aligned} \mathrm{Coeff}_{\tilde{q}^{n-1}}\left(\frac{1}{\chi_{10}}\right) &= -\mathrm{Coeff}_{\tilde{q}^{n-1}} Z(K3\times E) \\ &= \frac{\Theta(z)^{2n-2}}{\Delta(\tau)}(-1)^n \mathrm{Coeff}_{\tilde{q}^n}\left(\exp((2-2n)\widehat{L} + 24\widehat{M}\right) \end{aligned}$$

where

$$\widehat{L} = \sum_{r\geq 1} 2^{r-1} \sum_{\substack{i_1,\ldots,i_r \\ j_1,\ldots,j_r \\ \text{all even} \\ (i_r,j_r)\neq(0,0) \text{ if } r=1}} \frac{a_{i_1j_1}\cdots a_{i_{r-1}j_{r-1}}}{(i_r/2)!^2(j_r/2)!^2} F_{i_1+j_r}F_{i_2+j_1}\cdots F_{i_r+j_{r-1}}\tilde{q}^{\frac{1}{2}\sum_k(i_k+j_k)}$$

$$\widehat{M} = \sum_{r\geq 1}\frac{2^{r-1}}{r} \sum_{\substack{i_1,\ldots,i_r \\ j_1,\ldots,j_r \\ \text{all even or all odd}}} (-1)^{j_1+\ldots+j_r} a_{i_1,j_1}\cdots a_{i_r,j_r} F_{i_1+j_r}F_{i_2+j_1}\cdots F_{i_r+j_{r-1}}\tilde{q}^{\frac{1}{2}\sum_k(i_k+j_k)}$$

with the coefficients a_{ij} given by (2.6).

Since these are two very similar looking formulas for $1/\chi_{10}$, it seems plausible that they should in fact be the same formula. Or in other words, this equality should hold when replacing F_k by the formal variables t_k. This is the motivation for Conjecture A.

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