AUTOMORPHISMS OF HILBERT SCHEMES OF POINTS ON SURFACES

PIETER BELMANS, GEORG OBERDIECK, AND JØRGEN VOLD RENNEMO

Abstract. We show that every automorphism of the Hilbert scheme of n points on a weak Fano or general type surface is natural, i.e. induced by an automorphism of the surface, unless the surface is a product of curves and $n = 2$. In the exceptional case there exists a unique non-natural automorphism. More generally, we prove that any isomorphism between Hilbert schemes of points on smooth projective surfaces, where one of the surfaces is weak Fano or of general type and not equal to the product of curves, is natural. We also show that every automorphism of the Hilbert scheme of 2 points on $\mathbb{P}^n$ is natural.

Contents

1. Introduction

Let X be a non-singular complex projective surface and let $X^{[n]}$ be the Hilbert scheme of n points on X. Any isomorphism $g\colon X \xrightarrow{\sim} Y$ of smooth projective surfaces induces an isomorphism

$$g^{[n]}\colon X^{[n]} \xrightarrow{\sim} Y^{[n]}.$$

By [9, Definition 1] an isomorphism $\sigma\colon X^{[n]} \xrightarrow{\sim} Y^{[n]}$ is called *natural* if $\sigma = g^{[n]}$ for some g. In this paper we investigate which Hilbert schemes of points on surfaces have non-natural automorphisms and isomorphisms.

Consider the case of K3 surfaces. By a result of Beauville, the Hilbert scheme of points of a K3 surface is a hyperkähler variety [2, Théorème 3]. Isomorphisms of hyperkähler varieties can be controlled using the global

Date: January 17, 2020.

Torelli theorem. In particular, lattice arguments [21] show that there exist non-isomorphic K3 surfaces X_1 and X_2 such that $X_1^{[2]} \cong X_2^{[2]}$, see also [20, Example 7.2]. By construction these isomorphisms are not natural. Similarly, the involution of the Hilbert schemes of 2 points on a general quartic K3 that sends a subscheme to the residual subscheme of the line passing through it, does not preserve the diagonal and is hence not natural, see Beauville [3, §6]. The geometric construction and classification of auto- and isomorphisms of hyperkähler varieties of $\mathrm{K3}^{[n]}$-type is a rich and beautiful subject in its own right.

From now on we drop the condition on X to be Calabi–Yau. We first focus on the existence of non-natural automorphisms of $X^{[n]}$. By a computation of Boissière [9, Corollaire 1], the automorphism groups of $X^{[n]}$ and X have the same dimension and hence the same identity component. The question of whether non-natural automorphisms exist is therefore discrete in nature.

Our first result is the following. Recall that a surface X is called *weak Fano* if ω_X^{-1} is nef and big.

Theorem 1. *Let X be a smooth projective surface which is weak Fano or of general type, and let n be any integer. Except for the case $(C_1 \times C_2)^{[2]}$, where C_1 and C_2 are smooth curves, every automorphism of $X^{[n]}$ is natural.*

The second result deals with the case left open in the first theorem:

Theorem 2. *Let C_1, C_2 be smooth projective curves, either both rational or both of genus $g \geq 2$. Up to composing with natural automorphisms, there exists a unique non-natural automorphism of $(C_1 \times C_2)^{[2]}$.*

The non-natural automorphism on $(C_1 \times C_2)^{[2]}$ can be described as follows. On the complement of the diagonal it sends the cycle $(x_1, y_1) + (x_2, y_2)$ on $C_1 \times C_2$ to the cycle $(x_1, y_2) + (x_2, y_1)$. Formally, it is defined by lifting the covering involution of the natural map of symmetric products

$$(C_1 \times C_2)^{(2)} \to C_1^{(2)} \times C_2^{(2)}$$

to the Hilbert scheme.

Boissière and Sarti proved that if X is a K3 surface then an automorphism $f \in \mathrm{Aut}(X^{[n]})$ is natural if and only if it preserves the diagonal [10]. By a result of Hayashi the same holds if X is an Enriques surface [12, Theorem 1.2]. This gives some evidence in favour of a positive answer to the following question.

Question 3. *Suppose X is a smooth projective surface and $\sigma \colon X^{[n]} \xrightarrow{\sim} X^{[n]}$ is an automorphism preserving the diagonal. Excluding the case $X = C_1 \times C_2$ and $n = 2$, does it follow that σ is natural?*

For a smooth projective curve C of genus g the Hilbert scheme $C^{[n]}$ is isomorphic to the symmetric product $C^{(n)}$. In [7] Biswas and Gómez show that if $g > 2$ and $n > 2g-2$ then every automorphism of the nth symmetric product of C is natural. On the other hand non-natural automorphisms on $(\mathbb{P}^1)^{[n]} \cong \mathbb{P}^n$ for $n \geq 2$ are abundant.

If X is smooth of dimension ≥ 3 then the Hilbert scheme $X^{[n]}$ is smooth if and only if $n \leq 3$ [11, Theorem 3.0.1]. As a first step in understanding the situation in these cases we prove the following result.

Theorem 4. *Every automorphism of* $(\mathbb{P}^n)^{[2]}$ *is natural.*

The construction of the non-natural automorphism of $(C_1 \times C_2)^{[2]}$ generalizes to products of higher dimensionsional varieties, but we make no claims regarding the analogue of the uniqueness result of Theorem 2.

Bondal and Orlov [8, Theorem 2.5] proved that any derived equivalence between smooth projective varieties with one of them having ample or anti-ample canonical bundle is induced by an isomorphism of the underlying varieties. We obtain the following Hilbert scheme analog.

Corollary 5. *Let* X, Y *be smooth projective surfaces and assume that* Y *is weak Fano or of general type. Moreover if* Y *is a product of curves assume* $n \geq 3$*. Then for every isomorphism* $\sigma\colon X^{[n]} \xrightarrow{\sim} Y^{[n]}$ *there exist an isomorphism* $g\colon X \xrightarrow{\sim} Y$ *such that* $\sigma = g^{[n]}$.

After a first version of this paper appeared online, Hayashi made us aware of the preprint [13] in which he proves Theorems 1 and 2 and Corollary 5 for rational surfaces such that the Iitaka dimension of ω_X^{-1} is at least 1. Hayashi's arguments do not apply to surfaces with non-trivial fundamental group or in general type, while our arguments do not apply in Iitaka dimension 1. For simply connected surfaces with $\omega_X^{\pm 1}$ ample the arguments are parallel, see Section 3 for an outline of that case.

An interesting question beyond the scope of this paper is to describe the group of derived auto-equivalences of the Hilbert scheme of points of a smooth projective surface. In the weak Fano or general type case our results determine the group of *standard* auto-equivalences, that is the group generated by automorphisms of the variety, tensoring with line bundles, and shifts. But by a result of Krug [14, Theorem 1.1(ii)] there always exist non-standard auto-equivalences on the Hilbert scheme. For Hilbert squares and Hilbert cubes of surfaces with ample or anti-ample canonical bundle a proof of [14, Conjecture 7.5] combined with Theorems 1 and 2 would give a full description of the derived auto-equivalence group.

Hilbert schemes of points of Fano surfaces admit deformations which may be understood as Hilbert schemes of non-commutative deformations of Fano surfaces [15]. It would be interesting to compare the automorphism groups of these deformations with the automorphism groups of the underlying non-commutative surfaces which were computed in [6], and see whether they are all natural in the appropriate sense.

1.1. **Acknowledgements.** Chiara Camere was part of the collaboration at an earlier stage and we are very grateful for her insights and input. We would also like to thank Arnaud Beauville, Alberto Cattaneo, Daniel Huybrechts, Gebhard Martin and John Christian Ottem for their interest and useful discussions. We thank Taro Hayashi for useful comments and for sending us the preprint [13]. The project originated at the Max Planck Institute for Mathematics in Bonn and we thank the institute for support.

2. Preliminaries

Let X be a smooth complex projective surface. Let $X^{(n)}$ be the nth symmetric product of X obtained as the quotient of the cartesian product X^n under the permutation action by the symmetric group S_n. Let $\rho\colon X^n \to X^{(n)}$ be the quotient map and let $p_i\colon X^n \to X$ be the projection onto the ith factor. Recall also the Hilbert–Chow morphism

$$\epsilon\colon X^{[n]} \to X^{(n)}$$

which sends a subscheme $Z \subset X$ to its support. The notation is summarized in the following diagram:

$$\begin{tikzcd} X^{[n]} \arrow[dr, "\epsilon"] & X^n \arrow[r, "p_i"] \arrow[d, "\rho"] & X \\ & X^{(n)} & \end{tikzcd}$$

For any line bundle $\mathcal{L}$ on X the tensor product $\bigotimes_{i=1}^n p_i^*\mathcal{L}$ has a natural S_n-invariant structure, and taking S_n-invariants defines a line bundle $\mathcal{L}_{(n)}$ on $X^{(n)}$. If $\mathcal{L}$ is (very) ample, then $\mathcal{L}_{(n)}$ is (very) ample as well. We also define the pullback to the Hilbert scheme:

$$\mathcal{L}_{[n]} \coloneqq \epsilon^*\mathcal{L}_{(n)}$$

By arguments parallel to [2, §6], the canonical bundle of $X^{[n]}$ is

$$\omega_{X^{[n]}} = (\omega_X)_{[n]}.$$

The symmetric product $X^{(n)}$ is singular precisely at the diagonal Δ of cycles supported at less than n points. By [16, Theorem 18.18] the tangent space at $nx \in X^{(n)}$ for any $x \in X$ satisfies

$$\dim \mathrm{T}_{X^{(n)},nx} = \frac{n(n+3)}{2}.$$

This shows that the small diagonal $\Delta_{\text{small}} = \{nx \mid x \in X\}$ is distinguished in the symmetric product as the locus of points in $X^{(n)}$ where the Zariski tangent space is of maximal dimension.

For future use, we record the following lemma.

Lemma 6. *Let $f\colon X \to Y$ be a morphism of projective varieties, where Y is normal and f has connected fibres. Let $\mathcal{L}$ be an ample line bundle on Y. For any automorphism $\sigma\colon X \to X$ such that $\sigma^* f^* \mathcal{L} \cong f^* \mathcal{L}$ there exists an isomorphism $\tau\colon Y \xrightarrow{\sim} Y$ such that $\tau \circ f = f \circ \sigma$.*

Proof. By Stein factorization and our assumptions we have $f_*(\mathcal{O}_X) = \mathcal{O}_Y$, and so $\mathrm{H}^0(X, f^*\mathcal{L}^{\otimes m}) = \mathrm{H}^0(Y, \mathcal{L}^{\otimes m})$ for all $m \geq 0$. Applying the Proj construction to the corresponding graded algebra gives the isomorphism τ, and by construction $\tau \circ f = f \circ \sigma$. $\square$

3. The basic strategy

We first explain the proof of Theorem 1 under the assumption that

- ω_X or ω_X^{-1} is ample,
- X is simply connected,
- X is not a product of curves.

Let $\sigma\colon X^{[n]} \xrightarrow{\sim} X^{[n]}$ be an automorphism. Since the differential of σ is everywhere invertible we have

$$\sigma^* \omega_{X^{[n]}} \cong \omega_{X^{[n]}}.$$

Step 1. (Reduction to the symmetric product) Because $\omega_{X^{[n]}}$ is the pullback of an ample or anti-ample line bundle from the symmetric product $X^{(n)}$, by Lemma 6 there exists an automorphism $\tau\colon X^{(n)} \to X^{(n)}$ which makes the following diagram commute:

$$\begin{tikzcd} X^{[n]} \arrow[r, "\sigma"] \arrow[d, "\epsilon"] & X^{[n]} \arrow[d, "\epsilon"] \\ X^{(n)} \arrow[r, "\tau"] & X^{(n)}. \end{tikzcd}$$

Since ϵ is birational, τ is the identity if and only if σ is the identity. We are hence reduced to studying automorphisms of the symmetric product.

Step 2. (Lifting) Since τ preserves the singular points, the diagonal on the symmetric product is preserved:

$$\tau(\Delta) = \Delta.$$

Let $D \subset X^n$ denote the big diagonal in X^n and consider the restriction of the quotient map

$$\rho_D\colon X^n \setminus D \to X^{(n)} \setminus \Delta.$$

Since X is assumed to be simply connected and $D \subset X^n$ is of codimension 2 in a smooth ambient space, $X^n \setminus D$ is also simply connected. Hence ρ_D is the universal covering space of $X^{(n)} \setminus \Delta$. Applying the universal lifting property to the morphism $\tau \circ \rho_D$ we obtain an automorphism $f \in \mathrm{Aut}(X^n \setminus D)$ which makes the following diagram commute:

$$\begin{tikzcd}
X^n \setminus D \arrow[r, "f"] \arrow[d, "\rho_D"] & X^n \setminus D \arrow[d, "\rho_D"] \\
X^{(n)} \setminus \Delta \arrow[r, "\tau"] & X^{(n)} \setminus \Delta.
\end{tikzcd}$$

Step 3. (Extension to X^n) Since D is of codimension 2 in X^n and since X^n is smooth hence normal, every section of a line bundle on the complement of D extends. Applying this to ω_{X^n} and its powers, the automorphism f induces a graded ring automorphism of

$$\bigoplus_m \mathrm{H}^0(X^n, \omega_{X^n}^{\otimes m}).$$

Since ω_X is ample or anti-ample, this in turn induces an automorphism of X^n that extends f. We will denote this extension by f as well. We have constructed an automorphism

$$f \colon X^n \xrightarrow{\sim} X^n$$

such that $\rho \circ f = \tau \circ \rho$.

Step 4. (Splitting the automorphism) If $\mathcal{N}$ is a globally generated line bundle on a variety Z, its global sections induce a map from Z to projective space. The resulting morphism from Z to its image in projective space is called the morphism *associated with* $\mathcal{N}$.

Let $\mathcal{L}$ be a very ample line bundle on X, and let

$$\mathcal{L}_i \coloneqq p_i^* \mathcal{L}$$

be its pullback to X^n along the projection to the ith factor. The projection p_i is then naturally identified with the morphism associated with $\mathcal{L}_i$. We follow the arguments of [17, Theorem 4.1] and consider $f^*\mathcal{L}_i$.

Since X is simply connected, we have $\mathrm{H}^1(X, \mathbb{Z}) = 0$. On the one hand, this implies that $\mathrm{Pic}^0(X^n) = \mathrm{Pic}^0(X)^n = 0$. On the other hand, we have $\mathrm{H}^2(X^n, \mathbb{Z}) = \mathrm{H}^2(X, \mathbb{Z})^{\oplus n}$ and hence

$$\mathrm{Pic}(X^n) \cong \mathrm{H}^{1,1}(X^n, \mathbb{Z}) \cong \mathrm{H}^{1,1}(X, \mathbb{Z})^{\oplus n} \cong \mathrm{Pic}(X)^{\oplus n}.$$

We conclude that there exist line bundles $\mathcal{M}_j$ on X such that

$$f^*\mathcal{L}_i \cong p_1^*\mathcal{M}_1 \otimes \cdots \otimes p_n^*\mathcal{M}_n. \tag{1}$$

We have a natural Künneth identification

$$\mathrm{H}^0(X^n, p_1^*\mathcal{M}_1 \otimes \cdots \otimes p_n^*\mathcal{M}_n) \cong \mathrm{H}^0(X, \mathcal{M}_1) \otimes \cdots \otimes \mathrm{H}^0(X, \mathcal{M}_n).$$

Using this identification, we see that if $B_j \subset X$ is the base locus of $\mathcal{M}_j$, then the base locus of $p_1^*\mathcal{M}_1 \otimes \cdots \otimes p_n^*\mathcal{M}_n$ is $p_1^{-1}(B_1) \times \cdots \times p_n^{-1}(B_n) \subset X^n$. As $f^*\mathcal{L}_i$ is globally generated, it follows from (1) that this set is empty, hence all the B_j are empty and so all the $\mathcal{M}_j$ are globally generated.

Let now $g_j \colon X \to Y_j$ be the morphism associated with $\mathcal{M}_j$. Then the morphism associated with $p_1^*\mathcal{M}_1 \otimes \cdots \otimes p_n^*\mathcal{M}_n$ is

$$g_1 \times \cdots \times g_n \colon X^n \to Y_1 \times \cdots \times Y_n.$$

The morphism associated with $f^*\mathcal{L}_i$ is $p_i \circ f$, and so the isomorphism of line bundles in (1) implies that we have an isomorphism $h \colon \prod_i Y_i \to X$ such that $p_i \circ f = h \circ (g_1 \times \cdots \times g_n)$.

Since X is not a product of curves, one of the g_j is an isomorphism and the others are projections to a point. Let $j(i)$ be the number such that $g_{j(i)}$ is an isomorphism.

Then $p_i \circ f$ only depends on the corresponding $j(i)$th coordinate of a point $x = (x_1, \ldots, x_n) \in X^n$. After composing f with the automorphism of X^n induced by the permutation that sends i to $j(i)$, we may assume that $p_i \circ f$ only depends on the ith coordinate. We can then write

$$f = f_1 \times \cdots \times f_n$$

for some $f_i \colon X \to X$, and the f_i must be automorphisms since f is an isomorphism.

The automorphism $\tau \in \mathrm{Aut}(X^{(n)})$ preserves the small diagonal as the locus where the tangent space has maximal dimension. It follows that f preserves the small diagonal in X^n, so all the f_i are the same, and hence that σ is natural.

4. The general case

We present the proof of the main theorem. We proceed as in Section 3 but at every step we need to find an argument that works for weak Fano surfaces and for surfaces of general type. Our assumption throughout is that X is a smooth complex projective surface.

For a weak Fano surface X we will use that ω_X^{-1} is semiample, that is a power of it is basepoint free. Moreover, the morphism defined by the linear system of $\omega_X^{-\otimes m}$ is birational for appropriate $m \gg 0$, see [18, Theorem 4.3].

4.1. **Reduction to symmetric product.** We begin by giving a criterion for when an automorphism of the Hilbert scheme descends to the symmetric product. Let $\alpha \in \mathrm{H}_2(X^{[n]}, \mathbb{Z})$ be the class of a $\mathbb{P}^1$-fiber of the map $\epsilon\colon X^{[n]} \to X^{(n)}$. In particular, α is the unique primitive curve class such that

$$\epsilon_* \alpha = 0.$$

Proposition 7. *Let $\sigma \in \mathrm{Aut}(X^{[n]})$. If $\sigma_*\alpha = \alpha$ then there exists an automorphism $\tau \in \mathrm{Aut}(X^{(n)})$ such that the following diagram commutes:*

$$\begin{tikzcd} X^{[n]} \arrow[r, "\sigma"] \arrow[d, "\epsilon"] & X^{[n]} \arrow[d, "\epsilon"] \\ X^{(n)} \arrow[r, "\tau"] & X^{(n)}. \end{tikzcd}$$

Proof. We first show that the morphism $\epsilon\colon X^{[n]} \to X^{(n)}$ is the initial object in the category of morphisms $f\colon X^{[n]} \to Z$, where Z is a projective scheme and f contracts the $\mathbb{P}^1$-fibers of ϵ (or equivalently, $f_*\alpha = 0$).

Indeed, let $f\colon X^{[n]} \to Z$ be such a morphism and consider the scheme

$$Y = (\epsilon \times f)(X^{[n]}) \subset X^{(n)} \times Z.$$

We claim the projection to the first factor $p\colon Y \to X^{(n)}$ is an isomorphism. Since $X^{(n)}$ is normal (as the quotient of the normal space X^n by a finite group) and p is birational and proper, by Zariski's main theorem it suffices to show that p is finite. If p is not finite, it contracts a curve Σ. Then there exists a curve $\Sigma' \subset X^{[n]}$ such that its image under $\epsilon \times f$ is Σ (for example, take the preimage of Σ and if that is of dimension > 1 cut it down by sections of a relatively ample class). Since by assumption we have $(\epsilon \times f)_*\alpha = 0$, the class of Σ' is linearly independent (over $\mathbb{Q}$) of α and contracted by $\epsilon = p \circ (\epsilon \times f)$. But the kernel of

$$\epsilon_*\colon \mathrm{H}_2(X^{[n]}, \mathbb{Q}) \to \mathrm{H}_2(X^{(n)}, \mathbb{Q})$$

is 1-dimensional and spanned by α which gives a contradiction. We conclude that p is an isomorphism and hence that Y is the graph of a morphism $g\colon X^{(n)} \to Z$ with $g \circ \epsilon = f$. Since ϵ is birational, g is unique.

Applying the above universal property of ϵ to $\epsilon \circ \sigma$ with $Z = X^{(n)}$, we obtain a morphism $\tau\colon X^{(n)} \to X^{(n)}$ such that $\epsilon \circ \sigma = \tau \circ \epsilon$. On the other hand, the same argument also implies that $\epsilon \circ \sigma$ is initial, so τ is an isomorphism. □

We apply our criterion to the case at hand:

Proposition 8. *Let X be a smooth projective surface which is weak Fano or of general type. Then for every automorphism $\sigma \in \mathrm{Aut}(X^{[n]})$ we have $\sigma_*\alpha = \alpha$.*

Proof. We assume that X is of general type. The weak Fano case is parallel. Let Y be the canonical model of X, which is a surface with isolated singular points. The canonical line bundle on $X^{[n]}$ induces a morphism

$$\varphi\colon X^{[n]} \to Y^{(n)}$$

and since σ preserves this line bundle, by Lemma 6, there exists $\tau \in \mathrm{Aut}(Y^{(n)})$ such that

$$\begin{array}{ccc} X^{[n]} & \xrightarrow{\sigma} & X^{[n]} \\ \big\downarrow{\scriptstyle\varphi} & & \big\downarrow{\scriptstyle\varphi} \\ Y^{(n)} & \xrightarrow{\tau} & Y^{(n)}. \end{array}$$

commutes.

Let $y_1, \dots, y_r$ be the singular points of Y. The singular locus of $Y^{(n)}$ is

$$\operatorname{Sing} Y^{(n)} = \Delta \cup D_{y_1} \cup \ldots \cup D_{y_r} \tag{2}$$

where for a point $y \in Y$ the subscheme $D_y \subset Y^{(n)}$ is defined by

$$D_y = \{y + z \mid z \in Y^{(n-1)}\}. \tag{3}$$

Claim. The automorphism τ preserves the diagonal $\Delta \subset Y^{(n)}$.
Proof of the claim: The automorphism τ preserves the singular locus of $Y^{(n)}$. Since (2) is the decomposition of the singular locus into irreducible components we need to exclude the case that $\tau(\Delta) = D_{y_i}$ for some i. If $n \geq 3$ then the normalizations of D_y and Δ are

$$\tilde{D_y} = D_y \cong Y^{(n-1)}, \quad \tilde{\Delta} = Y \times Y^{(n-2)}. \tag{4}$$

To see this for the diagonal, we have a natural finite birational map $Y \times Y^{(n-2)} \to \Delta$. Since the source is normal it factors through a map to the normalization, which is an isomorphism by Zariski's main theorem. Since $Y^{(n-1)}$ and $Y \times Y^{(n-2)}$ are not isomorphic for $n \geq 3$ this completes the claim. In case $n = 2$ both Δ and D_{y_i} are isomorphic to Y. The corresponding inclusion maps factor as

$$\iota_\Delta\colon Y \xrightarrow{\Delta} D \subset Y \times Y \to Y^{(2)}$$
$$\iota_{D_{y_j}}\colon Y \cong y_j \times Y \subset Y^2 \to Y^{(2)}.$$

From this we get

$$\iota_\Delta^* \omega_{Y^{(2)}} = \omega_Y^{\otimes 2}, \quad \iota_{D_{y_j}}^* \omega_{Y^{(2)}} = \omega_Y.$$

Since $\omega_{Y^{(2)}}$ is preserved under pullback by τ, this excludes $\tau(\Delta) = D_{x_j}$. $\square$

We return to the proof of the proposition. Let $E \subset X^{[n]}$ denote the exceptional divisor. Let $C_{y_i} \subset X$ be the curve contracted to y_i under the canonical map $X \to Y$ and let $V_i \subset X^{[n]}$ be the preimage under ϵ of the subscheme

$$\{w_1 + w_2 \mid w_1 \in (C_{y_i})^{(2)}, w_2 \in X^{(n-2)}\} \subset X^{(n)}.$$

Then

$$\varphi^{-1}(\Delta) = E \cup \bigcup_{i=1}^{r} V_i.$$

By the claim σ preserves $\varphi^{-1}(\Delta)$. Since every V_i is of codimension ≥ 2 while E is a divisor, we conclude $\sigma(E) = E$. So we get a commutative diagram

$$\begin{tikzcd} E \arrow[r, "\sigma"] \arrow[d, "\varphi"] & E \arrow[d, "\varphi"] \\ \Delta \arrow[r, "\tau"] & \Delta. \end{tikzcd}$$

In particular, σ sends fibers of φ to fibers. Since the generic fiber of $E \to \Delta$ is precisely the $\mathbb{P}^1$ contracted by ϵ we are done. ☐

4.2. **Lifting.** Let X be a smooth projective surface. We show that every automorphism of $X^{(n)}$ lifts to an automorphism of $X^n \setminus D$.

Proposition 9. *For every* $\tau \in \mathrm{Aut}(X^{(n)})$ *there exists* $f \in \mathrm{Aut}(X^n \setminus D)$ *such that the following diagram commutes:*

$$\begin{tikzcd} X^n \setminus D \arrow[r, "f"] \arrow[d, "\rho"] & X^n \setminus D \arrow[d, "\rho"] \\ X^{(n)} \setminus \Delta \arrow[r, "\tau"] & X^{(n)} \setminus \Delta. \end{tikzcd}$$

Proof. The main idea is that since ρ is a normal covering space, by the standard lifting criterion we only have to show

$$(\tau \circ \rho)_*(\pi_1(X^n \setminus D)) \subset \rho_*(\pi_1(X^n \setminus D)).$$

We first make a simplification: Since the small diagonal is preserved by τ and isomorphic to X the restriction $\tau|_{\Delta_{\mathrm{small}}}$ defines an automorphism $g \in \mathrm{Aut}(X)$. Replacing τ by $(g^{-1})^{(n)} \circ \tau$ we may assume that

$$\tau|_{\Delta_{\mathrm{small}}} = \mathrm{id}_{\Delta_{\mathrm{small}}}.$$

Choose a point $x \in X$, a small open ball $U \subset X$ with $x \in U$, and n distinct points $x_1, \ldots, x_n \in U \setminus \{x\}$. Let $G = \pi_1(X, x)$, and note that we have canonical identifications $G \cong \pi_1(X, x_i)$ for every i, given by connecting x to x_i via a path in U. Then

$$\pi_1(X^n \setminus D, (x_i)) = \pi_1(X^n, (x_i)) = G^n.$$

The map

$$\rho\colon X^n \setminus D \to X^{(n)} \setminus \Delta$$

is obtained by taking the quotient by the free action of S_n, hence it is a normal covering space, and we have the exact sequence of groups

$$1 \to \pi_1(X^n \setminus D, (x_i)) \to \pi_1(X^{(n)} \setminus \Delta, \sum x_i) \to \mathrm{S}_n \to 1. \tag{5}$$

We define a splitting of this short exact sequence as follows. The inclusion of U in X induces an inclusion

$$U^{(n)} \setminus \Delta \hookrightarrow X^{(n)} \setminus \Delta.$$

By simply-connectedness of U we get $\mathrm{S}_n \cong \pi_1(U^{(n)} \setminus \Delta, \sum x_i)$, and the inclusion

$$\mathrm{S}_n \cong \pi_1(U^{(n)} \setminus \Delta, \sum x_i) \to \pi_1(X^{(n)} \setminus \Delta, \sum x_i)$$

splits (5). Thus we have

$$\pi_1(X^{(n)} \setminus \Delta, \sum x_i) \cong G^n \rtimes \mathrm{S}_n,$$

and one can check that the conjugation action of S_n on G^n is the standard permutation of factors.

The set $\tau(U^{(n)})$ is an open neighbourhood of $nx \in X^{(n)}$. Picking some sufficiently small open ball V with $x \in V \subset U$, we have $V^{(n)} \subset U^{(n)} \cap \tau(U^{(n)})$. We may assume that $x_i \in V$ for all i, hence $\sum x_i \in V^{(n)} \setminus \Delta$. We have the commutative diagram

$$\begin{tikzcd}
\pi_1(V^{(n)} \setminus \Delta, \sum x_i) \arrow[r, "a"] \arrow[d, "b"] & \pi_1(U^{(n)} \setminus \Delta, \sum x_i) \arrow[d, "c"] \\
\pi_1(\tau(U^{(n)}) \setminus \Delta, \sum x_i) \arrow[r, "d"] & \pi_1(X^{(n)} \setminus \Delta, \sum x_i),
\end{tikzcd}$$

where the three upper-left groups are isomorphic to S_n and where $c \circ a = d \circ b$ is injective. It follows that a and b are isomorphisms and that the three upper-left groups are equal as subgroups of $\pi_1(X^{(n)} \setminus \Delta, \sum x_i)$.

Picking a path in $\tau(U^{(n)}) \setminus \Delta$ from $\tau(\sum x_i)$ to $\sum x_i$ gives isomorphisms

$$s\colon \pi_1(X^{(n)} \setminus \Delta, \tau(\sum x_i)) \xrightarrow{\sim} \pi_1(X^{(n)} \setminus \Delta, \sum x_i)$$
$$s\colon \pi_1(\tau(U^{(n)}) \setminus \Delta, \tau(\sum x_i)) \xrightarrow{\sim} \pi_1(\tau(U^{(n)}) \setminus \Delta, \sum x_i)$$

We have the equality of subgroups of $\pi_1(X^{(n)} \setminus \Delta, \sum x_i)$

$$\begin{aligned}
(s \circ \tau_*)(\pi_1(U^{(n)} \setminus \Delta), \sum x_i) &= s(\pi_1(\tau(U^{(n)}) \setminus \Delta, \tau(\sum x_i))) \\
&= \pi_1(\tau(U^{(n)}) \setminus \Delta, \sum x_i) \\
&= \pi_1(U^{(n)} \setminus \Delta, \sum x_i).
\end{aligned}$$

Therefore, in the presentation $\pi_1(X^{(n)} \setminus \Delta, \sum x_i) = G^n \rtimes \mathrm{S}_n$ and the notation of §4.3, we have

$$(s \circ \tau_*)(e_{G^n} \rtimes \mathrm{S}_n) = e_{G^n} \rtimes \mathrm{S}_n,$$

and so by Lemma 11, we have

$$(s \circ \tau_*)(G^n \rtimes e_{\mathrm{S}_n}) = G^n \rtimes e_{\mathrm{S}_n}.$$

Since $G^n \rtimes e_{\mathrm{S}_n} = \rho_*(\pi_1(X^n \setminus D), (x_i))$, this implies

$$\tau_*(\rho_*(\pi_1(X^n \setminus D), (x_i))) = s^{-1}(\rho_*(\pi_1(X^n \setminus D), (x_i))) = \rho_*(\pi_1(X^n \setminus D, y)),$$

where $y \in X^n$ satisfies $\rho(y) = \tau(\sum x_i)$, and is the parallel transport of (x_i) along the path defining s. By the lifting criterion for covering spaces, it follows that $\tau \circ \rho$ lifts to an automorphism f as required. □

4.3. **Some group theory.**

Lemma 10. *Let $n \geq 3$, and let $\sigma \in \mathrm{S}_n$ be such that σ commutes with all its conjugates, and such that the centraliser $\mathrm{C}(\sigma)$ contains a subgroup isomorphic to S_{n-1}. Then $\sigma = e_{\mathrm{S}_n}$.*

Proof. By the first assumption on σ, the elements $g\sigma g^{-1}, g \in \mathrm{S}_n$ generate a normal, abelian subgroup H of S_n. If $n \geq 5$, then A_n is the only non-trivial normal subgroup of S_n ([19, §10.8.8]). As H is abelian, it must therefore be trivial, and so $\sigma = e_{\mathrm{S}_n}$. The remaining cases $n = 3, 4$ are checked directly. □

Let G be a group, and define $G^n \rtimes \mathrm{S}_n$ by the permutation action of S_n on the factors of G^n. We write $e \rtimes \mathrm{S}_n = e_{G^n} \rtimes \mathrm{S}_n$ and $G^n \rtimes e = G^n \rtimes e_{\mathrm{S}_n} \subset G^n \rtimes \mathrm{S}_n$ for the groups S_n, G^n thought of as subgroups of $G^n \rtimes \mathrm{S}_n$.

Lemma 11. *Let $\tau\colon G^n \rtimes \mathrm{S}_n \xrightarrow{\sim} G^n \rtimes \mathrm{S}_n$ be an automorphism. If $\tau(e \rtimes \mathrm{S}_n) = e \rtimes \mathrm{S}_n$, then $\tau(G^n \rtimes e) = G^n \rtimes e$.*

Proof. We need only show that $\tau(G^n \rtimes e) \subseteq G^n \rtimes e$, since applying this to τ^{-1} gives

$$\tau^{-1}(G^n \rtimes e) \subseteq G^n \rtimes e \Rightarrow G \rtimes e \subseteq \tau(G^n \rtimes e).$$

Let $g \in G$ be any element, and let $g_{(i)} \in G^n$ be the inclusion of g in the ith factor. The group $G^n \rtimes e$ is generated by elements of the form $(g_{(i)}, e_{\mathrm{S}_n})$, hence it suffices to show that $\tau(g_{(i)}, e_{\mathrm{S}_n}) \in G^n \rtimes e$.

Note first that $(g_{(i)}, e_{\mathrm{S}_n})$ commutes with all its $e \rtimes \mathrm{S}_n$-conjugates, hence so does $\tau(g_{(i)}, e_{\mathrm{S}_n})$. Note further that $(g_{(i)}, e_{\mathrm{S}_n})$ commutes with every element (e_{G^n}, δ) where δ fixes i. Hence the centraliser of $(g_{(i)}, e_{\mathrm{S}_n})$ contains a subgroup isomorphic to S_{n-1} inside $e \rtimes \mathrm{S}_n$, and so likewise the centraliser of $\tau(g_{(i)}, e_{\mathrm{S}_n})$ contains such a subgroup in $e \rtimes \mathrm{S}_n$. If now

$$\tau(g_{(i)}, e_{\mathrm{S}_n}) = (x, \sigma), \qquad x \in G^n, \sigma \in \mathrm{S}_n,$$

we have that σ commutes with all its conjugates, and $|\mathrm{C}_{\mathrm{S}_n}(\sigma)| \geq (n-1)!$, whence by Lemma 10 we have $\sigma = e_{\mathrm{S}_n}$ if $n \geq 3$.

Assume that $n = 2$. Let $\delta \in \mathrm{S}_2$ be the non-trivial element. By our assumption, $\tau(e_{G^2}, \delta) = (e_{G^2}, \delta)$, and so the centraliser $C = \mathrm{C}_{G^2 \rtimes \mathrm{S}_2}(e_{G^2}, \delta)$ is preserved by τ. By a direct check we have $C = \{((g,g), e_{\mathrm{S}_2}), ((g,g), \delta) \mid g \in G\}$. Next observe that elements $((g,g), e_{\mathrm{S}_2})$ can all be written as a product of an element x and its (e_{G^2}, δ)-conjugate, e.g. take $x = ((g, e_G), e_{\mathrm{S}_2})$. On the other hand, elements of the form $((g,g),\delta)$ cannot be written in such a way, since the product of an element with its (e_{G^2}, δ)-conjugate must have e_{S_2} in the S_2-factor. Therefore the set of elements of the form $((g,g), e_{\mathrm{S}_2})$ is preserved by τ.

Since these elements form a subgroup of $G^2 \rtimes \mathrm{S}_2$, the automorphism τ defines by restriction an automorphism ψ of G. After post-composing τ with the automorphism $((g,h),x) \mapsto ((\psi^{-1}(g), \psi^{-1}(h)), x)$, we may assume that τ in fact fixes each element $((g,g), e_{\mathrm{S}_2})$.

Now consider the element $((g, e_G), e_{\mathrm{S}_2})$. It satisfies the following equation in x:

$$x(e_{G^2}, \delta)x(e_{G^2}, \delta) = ((g,g), e_{\mathrm{S}_2}). \tag{6}$$

The same equation must therefore be satisfied by $\tau((g, e_G), e_{\mathrm{S}_2})$.

Assume now for a contradiction that $\tau((g, e_G), e_{\mathrm{S}_2}) = ((h,i),\delta)$. Since equation (6) is satisfied by $((h,i),\delta)$, we must have $(h^2, i^2) = (g,g)$. Therefore g is a square, hence so is $((g, e_G), e_{\mathrm{S}_2})$. But clearly $((h,i),\delta)$ is not a square, and so we have a contradiction. $\square$

4.4. **Extension.** Let $\tau\colon X^{(n)} \to X^{(n)}$ be an automorphism and assume that there exists a commutative diagram

$$\begin{array}{ccc} X^n \setminus D & \xrightarrow{f} & X^n \setminus D \\ \big\downarrow{\scriptstyle \rho} & & \big\downarrow{\scriptstyle \rho} \\ X^{(n)} \setminus \Delta & \xrightarrow{\tau} & X^{(n)} \setminus \Delta. \end{array}$$

for some automorphism $f \in \mathrm{Aut}(X^n \setminus D)$.

Proposition 12. *In the situation above the automorphism f extends to an automorphism $\tilde{f}\colon X^n \to X^n$ such that $\tau \circ \rho = \rho \circ \tilde{f}$.*

Proof. Let $U = X^n \setminus D$ and $V = X^{(n)} \setminus \Delta$ and consider the diagram

$$\begin{array}{ccc} U & \xrightarrow{\iota} & X^n \\ \big\downarrow{\scriptstyle \rho_U} & & \big\downarrow{\scriptstyle \rho} \\ V & \xrightarrow{j} & X^{(n)} \end{array}$$

where the horizontal maps ι and j are the inclusions and ρ_U is the restriction of ρ to U. Because U is of codimension 2 and X^n normal we have $\iota_* \mathcal{O}_U =$

$\mathcal{O}_{X^n}$. Both ρ and ρ_U are finite, therefore affine, so in particular we have

$$X^n = \underline{\operatorname{Spec}}\, \rho_* \mathcal{O}_{X^n}, \quad U = \underline{\operatorname{Spec}}\, \rho_{U*} \mathcal{O}_U.$$

The category of affine schemes over a base S is equivalent to the opposite of the category of quasi-coherent $\mathcal{O}_S$-algebras. Hence the V-morphism

$$\begin{tikzcd} U \arrow[rr, "f"] \arrow[dr, "\rho_U"'] & & U \arrow[dl, "\tau^{-1}\circ\rho_U"] \\ & V & \end{tikzcd}$$

corresponds to an isomorphism of quasi-coherent $\mathcal{O}_V$-algebras

$$\psi_f \colon (\tau^{-1})_* \rho_* \mathcal{O}_U \to \rho_* \mathcal{O}_U$$

By pushforward along j we obtain an isomorphism of $\mathcal{O}_{X^{(n)}}$-algebras

$$j_* \psi_f \colon j_* \tau_*^{-1} \rho_* \mathcal{O}_U \to j_* \rho_* \mathcal{O}_U.$$

We have $j_* \rho_* \mathcal{O}_U = \rho_* \iota_* \mathcal{O}_U = \rho_* \mathcal{O}_{X^n}$. Moreover, because the automorphism τ of U is the restriction of the automorphism τ of X^n we also have

$$j_* (\tau^{-1})_* \rho_* \mathcal{O}_U = (\tau^{-1})_* j_* \rho_* \mathcal{O}_U = (\tau^{-1})_* \rho_* \mathcal{O}_{X^n}.$$

The pushforward $j_* \psi_f$ thus corresponds to an isomorphism $\tilde{f}$ from $\rho \colon X^n \to X^{(n)}$ to $\tau^{-1} \circ \rho \colon X^n \to X^{(n)}$, hence to an isomorphism $\tilde{f} \in \operatorname{Aut}(X^n)$ with the desired properties. $\square$

4.5. **Splitting the automorphism.**

Proposition 13. *Assume the surface X is weak Fano or of general type, and let $f \in \operatorname{Aut}(X^n)$ be an automorphism. Then at least one of the following holds:*

(a) *$f = \alpha \circ (f_1 \times \cdots \times f_n)$ for some $f_i \in \operatorname{Aut}(X)$ and $\alpha \in \mathrm{S}_n$, or*
(b) *$X \cong C_1 \times C_2$ for smooth curves C_1, C_2. Moreover, if $C_1 \cong C_2$ then*

$$f = \alpha \circ (g_1 \times \cdots \times g_{2n})$$

for some $g_i \in \operatorname{Aut}(C_1)$ and some $\alpha \in \mathrm{S}_{2n}$. If $C_1 \not\cong C_2$ then under the isomorphism $X^n \cong C_1^n \times C_2^n$ we have

$$f \cong (\alpha_1 \times \alpha_2) \circ (g_1 \times \cdots \times g_n \times h_1 \times \cdots \times h_n)$$

for some $g_i \in \operatorname{Aut}(C_1), h_i \in \operatorname{Aut}(C_2)$ and $\alpha_1, \alpha_2 \in \mathrm{S}_n$.

For the proof we recall the general fact that the category of coherent sheaves on a smooth proper variety over an algebraically closed field is Krull–Schmidt, that is every object in it can be uniquely decomposed in irreducible components [1, Theorem 3].

Moreover, for a vector bundle $\mathcal{E}$ on X we will write

$$\mathcal{E}_i := p_i^*\mathcal{E}$$

for the pullback of $\mathcal{E}$ to X^n along the projection to the ith factor.

Proof. Let X be a surface of general type. We assume first that the cotangent bundle Ω_X is indecomposable. In this case the bundle Ω_{X^n} has the Krull–Schmidt decomposition

$$\Omega_{X^n} = \Omega_{X,1} \oplus \ldots \oplus \Omega_{X,n}.$$

Since this decomposition is unique and Ω_{X^n} is preserved under pullback by f, for every i we find $f^*\Omega_{X,i} = \Omega_{X,j(i)}$ for some $j(i)$. After composing f with a permutation we may assume $f^*\Omega_{X,i} \cong \Omega_{X,i}$ and thus

$$f^*\omega_{X,i} \cong \omega_{X,i}. \tag{7}$$

Let $\varphi\colon X \to Y$ be the map to the canonical model of X induced by a power of ω_X. From (7) and Lemma 6 we get an element $g_i \in \mathrm{Aut}(Y)$ such that the diagram

$$\begin{tikzcd} X^n \arrow[r, "f"] \arrow[d, "\varphi\circ p_i"] & X^n \arrow[d, "\varphi\circ p_i"] \\ Y \arrow[r, "g_i"] & Y \end{tikzcd}$$

commutes. Let $U \subset X$ be the open subset where φ is an isomorphism. Since i was arbitrary we conclude

$$\varphi^n \circ f \circ (\varphi^{-1})^n|_{\varphi(U)^n} = g_1 \times \cdots \times g_n|_{\varphi(U)^n}$$

For any fixed $(x_2, \ldots, x_n) \in U^{n-1}$ the composition

$$f_1 \colon X \hookrightarrow X^n \xrightarrow{f} X^n \xrightarrow{p_1} X,$$

where the first map is $x \mapsto (x, x_2, \ldots, x_n)$, defines a lift of $g_1 \in \mathrm{Aut}(Y)$ to $f_1 \in \mathrm{Aut}(X)$. Since the lift is unique if it exists, it is independent of the choice of the x_i. Using a parallel argument we find lifts $f_i \in \mathrm{Aut}(X)$ of g_i for any i. The equality

$$f = f_1 \times \cdots \times f_n$$

then holds on an non-empty open subset of X^n and hence holds everywhere.

Assume now that the cotangent bundle of X decomposes into line bundles:

$$\Omega_X \cong \mathcal{L} \oplus \mathcal{M}.$$

By a result of Beauville [4, §5.1, Proposition 4.3] the canonical bundle ω_X is ample and hence $W = X^n$ is canonically polarized. By [5, Theorem 1.3 and §4] it follows that the variety W can be decomposed into a product of irreducible factors and the decomposition is *unique* up to reordering the factors. Here a variety Z is called *irreducible* it does not admit a non-trivial product decomposition $Z \cong Z_1 \times Z_2$.

If X is not the product of curves then W has the following two factorizations. The standard one, induced by the projection maps p_i,

$$p = (p_1, \ldots, p_n)\colon W \xrightarrow{\sim} X^n$$

and the one obtained by mapping the first under the automorphism f,

$$p \circ f\colon W \xrightarrow{\sim} X^n.$$

Since both must coincide up to reordering (and factorwise isomorphism) we conclude there exist automorphisms $f_i \in \mathrm{Aut}(X)$ and a permutation $\alpha \in \mathrm{S}_n$ such that $p_i \circ f = f_i \circ p_{\alpha(i)}$ for all i. This yields the claim.

We therefore can assume that X is the product of curves

$$X = C_1 \times C_2$$

and hence that

$$\mathcal{L} = q_1^*\Omega_{C_1}, \quad \mathcal{M} = q_2^*\Omega_{C_2}$$

where we let q_j denote the projection from X to the jth factor.

If $C_1 \not\cong C_2$ then the pullback by f preserves the set of $\mathcal{L}_i$ and the set of $\mathcal{M}_i$ separately (since the image of the complete linear system defined by a power of $\mathcal{L}$ is precisely C_1, and likewise for $\mathcal{M}$). Hence there exists $g_i \in \mathrm{Aut}(C_1)$ and a permutation $\alpha_1 \in \mathrm{S}_n$ such that the following diagram commutes:

$$\begin{tikzcd} X^n \arrow[r, "f"] \arrow[d, "q_1 \circ p_i"'] & X^n \arrow[d, "q_1 \circ p_{\alpha_1(i)}"] \\ C_1 \arrow[r, "g_i"] & C_1. \end{tikzcd}$$

Since the parallel statement holds for the factor C_2, this yields the claim.

If $C_1 \cong C_2$ then we may determine $\mathrm{Aut}(C_1^{2n})$ as we determined $\mathrm{Aut}(X^n)$ when Ω_X is indecomposable, or again apply the result [5, Theorem 1.3].

Finally, we consider the case where X is weak Fano. If Ω_X is indecomposable, then we can argue as for general type. If Ω_X decomposes, then by the classification of weak Fano surfaces, or using Beauville [4, Theorem C(a)] and that X is rational so simply-connected, we have that X is isomorphic to $\mathbb{P}^1 \times \mathbb{P}^1$. The claim then follows as in case $C_1 = C_2$ above. $\square$

Corollary 14. *Assume the surface X is weak Fano or of general type. Let $f \in \mathrm{Aut}(X^n)$ and $\tau \in \mathrm{Aut}(X^{(n)})$ be automorphisms such that the diagram*

$$\begin{tikzcd} X^n \arrow[r, "f"] \arrow[d, "\rho"] & X^n \arrow[d, "\rho"] \\ X^{(n)} \arrow[r, "\tau"] & X^{(n)} \end{tikzcd}$$

commutes. Then one of the following holds:

(a) $f = \alpha \circ (g \times \cdots \times g)$ *for some* $g \in \mathrm{Aut}(X)$ *and* $\alpha \in \mathrm{S}_n$*, or*
(b) $n = 2$*,* $X = C_1 \times C_2$ *for smooth curves* C_1, C_2*, and we can write* $f = f_1 \circ f_2$*, where* $f_1 = \alpha \circ (g \times g)$ *for some* α, g *as in (a) and* f_2 *is the automorphism of* X^2 *given by*

$$((x_1, x_2), (y_1, y_2)) \mapsto ((x_1, y_2), (y_1, x_2)), \qquad x_i, y_i \in C_i.$$

Proof. Since the automorphism τ preserves the small diagonal in $X^{(n)}$ the map f preserves the small diagonal

$$D_{\text{small}} = \{(x, \dots, x) \mid x \in X\} \subset X^n.$$

If f is as in Proposition 13(a), this immediately implies that alternative (a) of our statement holds.

If f is as in Proposition 13(b), we consider the two cases more closely. In case $C_1 \cong C_2$ then, since f preserves the small diagonal, for all $(c_1, c_2) \in X$ there exists $(d_1, d_2) \in X$ such that

$$\alpha \circ (g_1(c_1), g_2(c_2), \dots, g_{2n-1}(c_1), g_{2n}(c_2)) = (d_1, d_2, \dots, d_1, d_2). \tag{8}$$

For a fixed $c_2 \in C_2$, there is a non-empty open subset of points $c_1 \in C_1$ such that, for any odd i, the point $g_i(c_1)$ is distinct from all the points

$$g_2(c_2), g_4(c_2), \dots, g_{2n}(c_2).$$

By (8), the set $\{g_1(c_1), \dots, g_{2n}(c_2)\}$ contains at most two points, and it follows that $g_1(c_1) = g_3(c_1) = \ldots = g_{2n-1}(c_1)$. Since c_1 was arbitrary in the open subset of C_1, we find that

$$g_1 = g_3 = \ldots = g_{2n-1}.$$

Similarly we have $g_2 = g_4 = \ldots = g_{2n}$. Moreover, α must preserve or invert parity.

Setting $g = g_1, h = g_2$, we thus find that f has the form

$$f = \alpha \circ (g, h)^{\times n},$$

with $\alpha \in \mathrm{S}_{2n}$ either preserving or inverting parity. If α preserves parity, then we define $\alpha_1 \in \mathrm{S}_n$ by $\alpha_1(i) = \frac{\alpha(2i-1)+1}{2}$, and set $f_1 = \alpha_1 \circ (g, h)^{\times n} \in \mathrm{Aut}(X^n)$. If α inverts parity, then we define $\alpha_1 \in \mathrm{S}_n$ by $\alpha_1(i) = \frac{\alpha(2i-1)}{2}$. We then let $\psi \in \mathrm{Aut}(X)$ be given by $\psi(c_1, c_2) = (c_2, c_1), c_i \in C_i$, and take $f_1 = \alpha_1 \circ (\psi \circ (g, h))^{\times n}$.

In either case, if we write $f = f_1 \circ f_2$, then under the identification $X^n = C_1^n \times C_2^n$ we have

$$f_2 = \mathrm{id}_{C_1^n} \times \alpha_2, \tag{9}$$

where $\alpha_2 \in \mathrm{S}_n$ acts on the factor C_2^n. It remains to show that either α_2 is trivial, in which case alternative (a) of our corollary holds, or that $n = 2$, in which case alternative (b) holds.

Since the automorphisms f and f_1 descend to the symmetric product, the same holds for $f_2 = f_1^{-1} \circ f$. In particular, for a point $z \in X^n$ the class of $f_2(z)$ in $X^{(n)}$ depends only on the class of $z \in X^{(n)}$. Hence for all permutations $\sigma \in \mathrm{S}_n$ there exists $\tilde{\sigma} \in \mathrm{S}_n$ such that

$$f_2(\sigma z) = \tilde{\sigma} f_2(z).$$

Using (9) and remembering that under the identification $X^n \cong C_1^n \times C_2^n$ the symmetric group S_n acts diagonally we find

$$(\tilde{\sigma} \times \tilde{\sigma})^{-1}(\mathrm{id}_{C_1^n} \times \alpha_2)(\sigma \times \sigma) = (\mathrm{id}_{C_1^n}, \alpha_2).$$

This gives $\sigma = \tilde{\sigma}$ and thus

$$\sigma^{-1}\alpha\sigma = \alpha$$

Since σ was arbitrary, α is in the center of S_n. Since the center of the symmetric group is non-trivial only for $n = 2$, we find as required that α_2 can be non-trivial only if $n = 2$.

In the case $C_1 \ncong C_2$, then Proposition 13 shows that we have $g_1, \ldots, g_n \in \mathrm{Aut}(C_1), h_1, \ldots, h_n \in \mathrm{Aut}(C_2)$ and $\alpha_1, \alpha_2 \in \mathrm{S}_n$ such that under the identification $X^n = C_1^n \times C_2^n$, we have

$$f = (\alpha_1 \times \alpha_2) \circ (g_1 \times \cdots \times g_n \times h_1 \times \cdots \times h_n).$$

A similar argument to the case of $C_1 \cong C_2$ shows that all the g_i are equal and that all the h_i are equal. Taking $f_1 = (\alpha_1)^{\times n} \circ (g_1, h_1)^{\times n}$, we find that $f = f_1 \circ f_2$ with $f_2 = \mathrm{id}_{C_1^n} \times \alpha_1^{-1}\alpha_2$. The same argument as in the case of $C_1 \cong C_2$ then completes the proof. $\square$

4.6. **Proof of Theorem 1 and 2.** Let X be weak Fano or of general type and let $\sigma \in \mathrm{Aut}(X^{[n]})$. By Proposition 8 applied to Proposition 7 the automorphism σ descends to the symmetric product. By Proposition 9 this automorphism of the symmetric product lifts to an automorphism of the complement of the big diagonal in X^n and by Proposition 12 it extends from there to an automorphism of X^n. Theorem 1 and the uniqueness part of Theorem 2 now follow from the classification in Corollary 14. For the existence part of Theorem 2 the automorphism in Corollary 14 (b) descends to $X^{(2)}$ and from there lifts to the Hilbert scheme by the universal property of the blow-up $X^{[2]} \to X^{(2)}$. $\square$

4.7. **Proof of Corollary 5.** Theorem 1 together with the following result immediately implies Corollary 5.

Proposition 15. *Let X, Y be smooth projective surfaces and let Y be weak Fano or of general type. If $X^{[n]} \cong Y^{[n]}$, then $X \cong Y$.*

Proof. We assume that Y is of general type, the case where Y is weak Fano is parallel. Let $\sigma\colon X^{[n]} \xrightarrow{\sim} Y^{[n]}$ be an isomorphism. Since $\sigma^*\omega_{Y^{[n]}} \cong \omega_{X^{[n]}}$ by Lemma 6 we have a commutative diagram

$$\begin{array}{ccc} X^{[n]} & \xrightarrow{\sigma} & Y^{[n]} \\ \downarrow & & \downarrow \\ X_{\mathrm{can}}^{(n)} & \xrightarrow{\tau} & Y_{\mathrm{can}}^{(n)} \end{array}$$

where τ is an isomorphism and we let X_{can} and Y_{can} denote the canonical models of X and Y respectively. Since $Y_{\mathrm{can}}^{(n)}$ is of dimension $2n$ we find that X is also of general type. Let x_i and y_j be the singular points of X_{can} and Y_{can} respectively. Then τ induces an isomorphism of the singular loci of $X_{\mathrm{can}}^{(n)}$ and $Y_{\mathrm{can}}^{(n)}$:

$$\tau\colon \bigcup_i D_{x_i} \cup \Delta_{X_{\mathrm{can}}} \xrightarrow{\sim} \bigcup_j D_{y_j} \cup \Delta_{Y_{\mathrm{can}}}.$$

where the D_x are defined as in (3). We claim that $\tau(\Delta_{X_{\mathrm{can}}}) = \Delta_{Y_{\mathrm{can}}}$. Indeed, if $n = 2$ then $D_{x_i} \cong \Delta_{X_{\mathrm{can}}} \cong X_{\mathrm{can}}$, so $X_{\mathrm{can}} \cong Y_{\mathrm{can}}$ and the claim follows as in the proof of Proposition 8. If $n \geq 3$, we can either use that D_{x_i} is normal while the diagonal is not, or argue as follows. Assume that

$$\tau(\Delta_{X_{\mathrm{can}}}) = D_{y_i}, \quad \tau(\Delta_{Y_{\mathrm{can}}}) = D_{x_i}.$$

Then from the description of their normalizations in (4) we have the equality of Betti numbers

$$2\mathrm{b}_i(X_{\mathrm{can}}) = \mathrm{b}_i(Y_{\mathrm{can}}) \quad \text{and} \quad \mathrm{b}_i(X_{\mathrm{can}}) = 2\mathrm{b}_i(Y_{\mathrm{can}})$$

for $i = 1$ so $\mathrm{b}_1(X_{\mathrm{can}}) = \mathrm{b}_1(Y_{\mathrm{can}}) = 0$, hence the same equations hold also for $i = 2$ which is impossible since $\mathrm{b}_2(X_{\mathrm{can}}) > 0$.

By the claim, τ preserves the diagonal, hence by an argument parallel to the proof of Proposition 8, we find that σ preserves the class of a $\mathbb{P}^1$-fiber of the Hilbert–Chow morphism $X^{[n]} \to X^{(n)}$. Then arguing as in Proposition 7 we find that σ descends to an isomorphism $X^{(n)} \xrightarrow{\sim} Y^{(n)}$. Since this automorphism sends the small diagonal to the small diagonal and the small diagonal is isomorphic to the underlying surface we are done. $\square$

5. The Hilbert scheme of 2 points of $\mathbb{P}^n$

The Hilbert scheme of 2 points of $\mathbb{P}^n$ is isomorphic to the quotient of the blow-up of $\mathbb{P}^n \times \mathbb{P}^n$ along the diagonal,

$$(\mathbb{P}^n)^{[2]} = \mathrm{Bl}_\Delta(\mathbb{P}^n \times \mathbb{P}^n)/\mathrm{S}_2.$$

Since $(\mathbb{P}^n)^{[2]}$ is rational we find

$$\mathrm{Pic}(\mathbb{P}^n)^{[2]} = \mathrm{H}^2((\mathbb{P}^n)^{[2]}, \mathbb{Z}) \cong \mathbb{Z}^{\oplus 2}.$$

Proof of Theorem 4. Let $\sigma\colon (\mathbb{P}^n)^{[2]} \xrightarrow{\sim} (\mathbb{P}^n)^{[2]}$ be an automorphism. The Hilbert scheme admits the following two contractions: the Hilbert–Chow morphism

$$f_1\colon (\mathbb{P}^n)^{[2]} \to (\mathbb{P}^n)^{(2)}$$

and the morphism

$$f_2\colon (\mathbb{P}^n)^{[2]} \to \mathrm{Gr}(2, n+1)$$

that sends a subscheme to the line passing through it. By pulling back polarizations of the targets along f_1, f_2 we obtain two divisors on the Hilbert scheme. Both maps contract curves so both divisors lie in the boundary of the nef cone of $(\mathbb{P}^n)^{[2]}$. Since the Picard group is rank 2 these divisors form precisely the extremal rays of the nef cone.

Since the automorphism σ preserves the nef cone, σ up to scaling either preserves these divisors or interchanges them. However, since the contractions above are non-isomorphic (they have non-isomorphic images), σ cannot interchange them hence must preserve them up to scaling. Since σ also preserves the divisibility, we find that σ fixes the two divisors, so σ fixed the Picard group. In particular, from

$$\sigma^* f_1^* L = f_1^* L$$

where L is an ample divisor on $(\mathbb{P}^n)^{(2)}$ we find by Lemma 6 that σ descends to an automorphism of the symmetric product $\tau\colon (\mathbb{P}^n)^{(2)} \to (\mathbb{P}^n)^{(2)}$. Arguing as in Section 3 this automorphism lifts to $\mathbb{P}^n \times \mathbb{P}^n$ where it has to be of the form $\alpha \circ (f, f)$ for some $\alpha \in \mathrm{S}_2$ and $f \in \mathrm{Aut}(\mathbb{P}^n)$. Hence σ is natural. $\square$

References

[1] M. Atiyah, M. *On the Krull–Schmidt theorem with application to sheaves*, Bull. Soc. Math. France **84** (1956), 307–317.

[2] A. Beauville, *Variétés Kähleriennes dont la première classe de Chern est nulle* J. Differential Geom. **18** (1983), no. 4, 755–782 (1984).

[3] A. Beauville, *Some remarks on Kähler manifold with* $\mathrm{c}_1 = 0$, Classification of algebraic and analytic manifolds, Progress in Mathematics, vol. 39, Birkhäuser, 1983, pp. 1–26.

[4] A. Beauville, *Complex manifolds with split tangent bundle*, Complex analysis and algebraic geometry, 61–70, de Gruyter, Berlin, 2000.

[5] B. Bhatt, W. Ho, Z. Patakfalvi, C. Schnell, *Moduli of products of stable varieties*, Compos. Math. **149** (2013), no. 12, 2036–2070.

[6] P. Belmans *Hochschild cohomology of noncommutative planes and quadrics*, J. Noncommut. Geom. **13** (2019), no. 2, 769–795.

[7] I. Biswas, T. Gómez, *Automorphisms of a symmetric product of a curve*, With an appendix by Najmuddin Fakhruddin. Doc. Math. **22** (2017), 1181–1192.

[8] A. Bondal, D. Orlov, *Reconstruction of a variety from the derived category and groups of autoequivalences*, Compositio Math. **125** (2001), no. 3, 327–344.

[9] S. Boissière, *Automorphismes naturels de l'espace de Douady de points sur une surface*, Canad. J. Math. **64** (2012), no. 1, 3–23.

[10] S. Boissière, A. Sarti, *A note on automorphisms and birational transformations of holomorphic symplectic manifolds*, Proc. Amer. Math. Soc. **140** (2012), no. 12, 4053–4062.

[11] J. Cheah, *Cellular decompositions for nested Hilbert schemes of points*, Pacific J. Math., **183** (1998), no. 1, 39–90.

[12] T. Hayashi, *Universal covering Calabi-Yau manifolds of the Hilbert schemes of n-points of Enriques surfaces*, Asian J. Math. **21** (2017), no. 6, 1099–1120.

[13] T. Hayashi, *Automorphisms of the Hilbert scheme of n points of a rational surface and the anticanonical Iitaka dimension*, Geometriae Dedicata (2020), published online ahead of print

[14] A. Krug, *$\mathbb{P}$-functor versions of the Nakajima operators*, Algebr. Geom., **6** (2019), no. 6, 678–715.

[15] C. Li, *Deformations of the Hilbert scheme of points on a del Pezzo surface* Mosc. Math. J., **17** (2017), no. 2, 291–321.

[16] E. Miller, B. Sturmfels, *Combinatorial commutative algebra*, Graduate Texts in Mathematics, **227.** Springer-Verlag, New York, 2005. xiv+417 pp.

[17] K. Oguiso, *A question of Doctor Malte Wandel on automorphisms of the punctural Hilbert schemes of K3 surfaces*, arXiv:1402.4228.

[18] F. Sakai, *Anticanonical models of rational surfaces*, Math. Ann. **269** (1984), no. 3, 389–410.

[19] W.R. Scott, *Group theory*, 2nd ed. Dover Publications, Inc., New York (1987).

[20] K. Yoshioka, *Moduli spaces of stable sheaves on abelian surfaces*, Math. Ann., **321** (2001), no. 4, 817–884.

[21] R. Zuffetti, *Strongly ambiguous Hilbert squares of projective K3 surfaces with Picard number one*, Rend. Semin. Mat. Univ. Politec. Torino **77**, 1 (2019), 113–130

Mathematisches Institut, Universität Bonn, Endenicher Allee 60, 53115 Bonn, Germany

E-mail address: pbelmans@math.uni-bonn.de

Mathematisches Institut, Universität Bonn, Endenicher Allee 60, 53115 Bonn, Germany

E-mail address: georgo@math.uni-bonn.de

Department of Mathematics, University of Oslo, PO Box 1053 Blindern, 0316 Oslo, Norway

E-mail address: jorgeren@uio.no