Practice problem for the exam

Lecturer: Georg Oberdieck, Course: K3 surfaces, Winter 2022

1. Give an example of a K3 surface (with proof). Does your example S have an elliptic fibration? Does your example admit a non-trivial automorphism? If not, can you modify your example so it has one?

2. Let D be a divisor on a K3 surface S. Compute $\chi(\mathcal{O}(D))$.

3. Let $C \subset S$ be a smooth connected curve on S. Compute $C \cdot C$ and the dimension of $H^0(C, N_{C/S})$ in terms of the genus of g.

4. Let H be an ample divisor on S (so that $H^1(S, \mathcal{O}(H)) = H^2(S, \mathcal{O}(H)) = 0$ by Kodaira vanishing). Determine the dimension of the linear system $|H|$ in terms of h where $H^2 = 2h-2$. Determine the arithmetic genus of a curve $C \in |H|$.

Remark: This exercise reduces to Problem 3 if we know that there always exists a smooth curve $C \in |H|$ in a given linear system. Does this always hold? Hint: Consider a generic elliptic K3 surface with a section, let $H = B + dF$ where B is the class of the section and F is a class of a fiber, and determine the linear system $|H|$.

5. Let $S \to \mathbb{P}^2$ be a degree two cover branched along a smooth conic curve $C \subset \mathbb{P}^2$. Give a more explicit description of S. To which well-known surface is S isomorphic to?

6. Let $U \subset \mathbb{P}(H^0(\mathbb{P}^3, \mathcal{O}_{\mathbb{P}^3}(4)))$ be the locus of all smooth quartics in $\mathbb{P}^3$. Show that there exists a point $f \in U$ such that the Picard group of $S = V(f)$ is generated by the hyperplane class (and hence is rank 1).

(You may use that the Kodaira spence map $T_{[S]}U \to H^1(S, T_S)$ can be identified with the map $(*)$ in the following sequence obtained from $0 \to T_S \to T_{\mathbb{P}^3}|_S \to N_{S/\mathbb{P}^3} \to 0$ by taking cohomology:

$$0 \to H^0(T_{\mathbb{P}^3}|S) \to H^0(S, N_{S/X}) \xrightarrow{(*)} H^1(S, T_S) \to H^1(T_{\mathbb{P}^3}|_S) = \mathbb{C} \to 0.)$$

Conclude that $\mathrm{Aut}(S) = \{\mathrm{id}\}$ for any such K3 surface S, and that it does not admit an elliptic fibration.

7. (Automorphisms)

(i) Does there exist a K3 surface S with a non-symplectic involution g such that $\mathrm{Fix}(g)$ contains an isolated point? Same question but with g of order 3.

(ii) Find a quartic K3 surface $S \subset \mathbb{P}^3$ that admits an automorphism of order i for each $2 \leq i \leq 8$.

(iii) Does there exists a K3 surface S with an automorphism of order 691?

8. What is the maximal Picard rank that a complex K3 surface can have? Construct an example in two different ways: (i) Using the global Torelli theorem. (ii) By an explicit construction (e.g. using Kummer surfaces).

9.(More a homework problem) Let S be a K3 surface with $\mathrm{Pic}(S) = \mathbb{Z}L$ where $L^2 = 4$. Show that S is a quartic K3 surface as follows.

(i) Compute $\chi(L)$. Show that (after replacing L by $L^\vee$ if necessary) we have $H^2(L) = 0$ and $H^0(L) \neq 0$.
(ii) Let $C \in |L|$ be a curve, which is necessarily reduced and irreducible. Show $\omega_C = L|_C$ and $H^0(S, L) \twoheadrightarrow H^0(C, L|C)$.
(iii) Assume for now that there exists $C \subset |L|$ which is smooth. Since ω_C is generated by its sections, conclude that L on S is generated by its sections.
(iv) With assumption as in (iii), show $H^1(L) = 0$ and show that the induced morphism $f : S \to \mathbb{P}^3$ defined by L is finite and $4 = \deg(f) \cdot \deg(f(S))$.
(v) [more difficult] With assumptions as in (iii), show that $\deg(f) = 1$ and hence f is an isomorphism onto a quartic. (Hint: If $\deg(f) = 2$ we have that $Q := f(S)$ is a quadric in $\mathbb{P}^3$. If Q is smooth, then $Q = \mathbb{P}^1 \times \mathbb{P}^1$ and $\mathrm{Pic}(Q) = \mathbb{Z}^2$ which yields a contradiction; if Q is a cone over $\mathbb{P}^1$ with vertex s use that $\mathcal{O}(1)$ restricted to $Q \setminus \{s\}$ is divisible by 2 in the Picard group).
(v) [hard] Remove the assumption made in (iii).