Lecture 13 / Ω, 16. Januar 2023.

Last time:

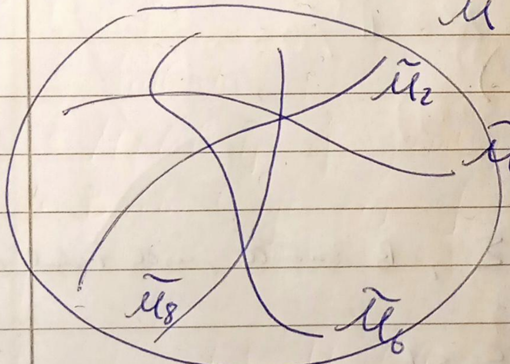
Thm There is a 19-dim'l irreducible quasiprojective variety M_d which is a coarse moduli space of polarized K3 surfaces (S, H) of degree $d = H^2$. In particular,

$$M_d(\mathbb{C}) = \{(S,H) \text{ polarized of deg } d\}/\sim$$

Idea of proof

$$\mathcal{M} = \{(S,\alpha) \mid S \text{ K3},\ \alpha : H^2(S,\mathbb{Z}) \longrightarrow L\}$$

$\mathcal{M}$ (or $\overline{\mathcal{M}}$).



$$M_d = \underset{\substack{\uparrow \\ \text{nice space}}}{\tilde{\mathcal{M}}_d} \Big/ \underset{\substack{\uparrow \\ \text{nice quot.}}}{\Gamma_d}$$

$\square$.

§ Rational curves on K3 surfaces

Def A rational curve on S is an irreducible ~~(reduced)~~ curve $C \subset S$ s.t. $\tilde{C} \cong \mathbb{P}^1$.

Prop For any $L \in \mathrm{Pic}(S)$, there are at most finitely many rational curves in $|L|$.

Proof:

with reduced scheme structure

$$\Sigma := \{ Z \in |L| \mid \text{all irreducible components are rational curves} \}$$

- **Fact:** $\Sigma \subset |L|$ closed.

(E.g. Image of $M_0(S, \mathrm{Pic}(L)) \longrightarrow |L|$.)

- Claim: Σ is of dimension 0.

If not, there is a curve $B \longrightarrow \Sigma$, B smooth, with 1-dim'l image.

$\Rightarrow \exists\ \mathcal{C} \subset S \times B$ flat over B smooth, s.t.

- $\mathcal{C}_b \subset S$ rational $\forall b \in B$.
- $p_S(\mathcal{C})$ 2-dim'l.

$\Rightarrow \tilde{\mathcal{C}} :=$ normalization of $\mathcal{C}$.

$\tilde{\mathcal{C}}$ normal, so regular in codim 1.
$\Rightarrow$ Sing. locus = pts.
$\Rightarrow \tilde{\mathcal{C}}$ proper over B, so apply generic smoothness

$$\begin{tikzcd}
\tilde{\tilde{\mathcal{C}}} \arrow[r, "res."] \arrow[ddrr, bend right, "q"'] & \tilde{\mathcal{C}} \arrow[d] & \\
& \mathcal{C} \arrow[r, phantom, "\subset"] \arrow[dr] & S \times B \arrow[d] \\
& & B
\end{tikzcd}$$

$\Rightarrow$ q generically $\mathbb{P}^1$-bdle.

$\Rightarrow \tilde{\tilde{\mathcal{C}}}$ ruled surface, i.e. birational to $B \times \mathbb{P}^1$.

$\Rightarrow \kappa(\tilde{\tilde{\mathcal{C}}}) = -\infty$

but $\kappa(S) = 0$ ↯.

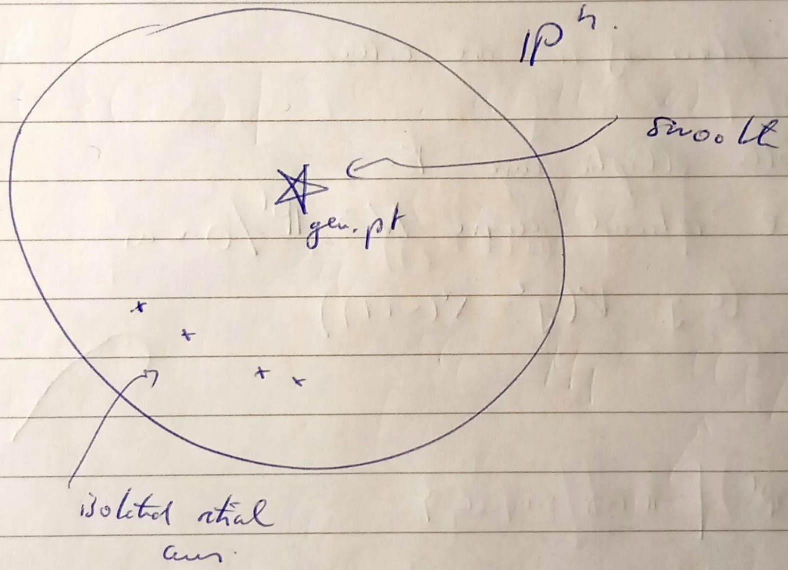
$\square$

Ex: Let $H \in \mathrm{Pic}(S)$ primitive & (very) ample.

Write

$$H \cdot H = \underset{\substack{\uparrow \\ \text{magic number}}}{2h} - 2.$$

Exercise:
$$\begin{cases} |H| = \mathbb{P}^h & (\text{show } H^0(S,H) = \mathbb{C}^{h+1}) \\ \text{For } C \in |H|,\ C \text{ has \underline{arithmetic} genus } h. \end{cases}$$



Q1: Does every alg K3 surface have a rational curve?

Answer (Bogomolov–Mumford 1980's) Yes.

Q2: Does " " have ∞-ly many rational curves?

Answer: Bogomolov–Tschinkel 2005, B-Hassett-T 2011, Li-Liedtke 2012, Chen-Liedtke-Gounelas 2020): Yes

Idea: Construct rational curve on special K3 and deform.

Q3: How many rational curves are there in $|L|$ precisely?

Ex

$$\begin{tikzcd} S \arrow[d, "\pi \text{ elliptic}"] \\ \mathbb{P}^1 \end{tikzcd} \qquad L = \pi^* \mathcal{O}(1) = \mathcal{O}(F), \quad F \text{ a fiber.}$$

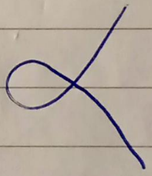
~~red~~ $\mathbb{P}^1 \xrightarrow{\sim} |L|$

$x \longmapsto \pi^{-1}(x)$.

Assume that all fibers of π are either

- smooth elliptic curve E.
- nodal rational curve $N \cong \mathbb{P}^1/0\sim\infty$

(e.g. $V(y^2 = x^2(x+1))$



Let $U = \{x \in \mathbb{P}^1 \mid \pi^{-1}(x) \text{ smooth}\}$

$Z = \{x \in \mathbb{P}^1 \mid \pi^{-1}(x) \text{ singular}\}$

Compute.

$$\chi_{top}(S) = \chi_{top}(\pi^{-1}(U)) + \chi_{top}(\pi^{-1}(Z))$$

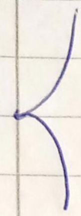
$$= \chi_{top}(U) \cdot \underbrace{\chi(E)}_{0} + \chi_{top}(Z) \cdot \underbrace{\chi_{top}(N)}_{1}$$

$$= \chi_{top}(Z)$$

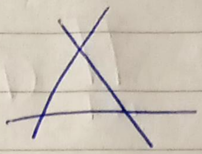
$$= \# Z$$

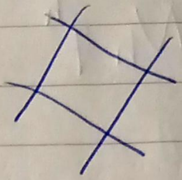
$= \#$ rational curves in $|L|$.

In general, assumption fails. Can have fibers


cusp

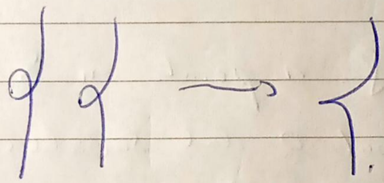

banana


triangle


wheel

. .

Also K3 satisfying assumption can deform to one who does not s.t.



Two options to count:

(a) Assign "weights" ⇝ GW / DT theory

(b) restrict to generic case.

~~Avoid~~ require $\begin{cases} C \in |L| & \text{reduced} \\ C \in |L| & \text{irreducible} \end{cases}$ I

$\Rightarrow$ Ideally: $\mathrm{Pic}(S) = \mathbb{Z}L$.

Q3: Let $(S,H) \in M_d$ generic. How many rational curves lie in $|H|$?

Write $H^2 = 2h-2$, $N_h :=$ answer.

Classical computations: $N_0 = 1$, $N_1 = 24$, $N_2 = 324$, $N_3 = 3200$, ..

$N_4 = 25650$

Theorem (Yau-Zaslow formula)

$$\sum_{h=0}^{\infty} N_h q^h = \prod_{n\geq 1} \frac{1}{(1-q^n)^{24}}$$

$$\|$$

$$1 + 24q + 324q^2 + 3200q^3 + \ldots$$

Two interpretations

① $S^{[n]}$:= Hilbert scheme of n points on S.

$$= \left\{ Z \subset S \;\middle|\; \begin{array}{l} Z \text{ closed } 0\text{-dim subscheme} \\ \dim(H^0(Z, \mathcal{O}_Z)) = n \end{array} \right\}$$

smooth, projective, dim $2n$.

Göttsche: $$\sum_{n=0}^{\infty} \chi_{top}(S^{[n]}) = \prod_{n\geq 1} \frac{1}{1-q^n}$$

Thm $\Rightarrow$ $$N_h \overset{(*)}{=} \chi_{top}(S^{[h]})$$

Proof of Thm: Prove (*) by Method of example and use Göttsche's calculation. //

(2) Modular forms.

$$\mathbb{H} = \{\tau \in \mathbb{C} \mid \mathrm{Im}(\tau) > 0\} \circlearrowleft SL_2(\mathbb{Z})$$

Def A modular form of weight k is a holomorphic function $f: \mathbb{H} \to \mathbb{C}$ s.t.

$$f(\gamma \cdot \tau) = (c\tau + d)^k f(\tau)$$

for all $\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL_2(\mathbb{Z})$ & satisfying growth condition:

Mod_k vector space of modular fms of wt k.
$\rightarrow$ fin. dim'l

$Mod = \bigoplus_k Mod_k$ alg.

Examples

(1)
$$\Delta(\tau) = q \prod_{n \geq 1} (1 - q^n)^{24} \in Mod_{12}$$

Upshot: $F_0 = \sum N_n q^{n-1}$ is the q-expansion of modular fm $1/\Delta(\tau)$.

(2) $k \geq 4$ even
$$G_k(\tau) = -\frac{B_k}{2k} + \sum_{n \geq 1} \sum_{d \mid n} d^{k-1} q^n. \in Mod_k$$

$$\Rightarrow Mod = \mathbb{C}[G_4, G_6]$$

Higher genus:

$$\begin{tikzcd} \mathcal{C} \arrow[r, "f"] \arrow[d, "\pi"] & S \\ \overline{M}_g(S,H) & \end{tikzcd}$$

$$\overline{M}_g(S,H) = \left\{ f : C \longrightarrow S \;\middle|\; \begin{array}{l} C \text{ nodal genus } g \text{ curve} \\ f_*C = H \\ |\mathrm{Aut}(f)| < \infty \end{array} \right\}$$

Natural classes: $p \in S$ $k \geq 0$ ~~$p \in H^4(S,\mathbb{Z})$~~ class of point.

$$I_k(p) := \overset{\pi}{\cancel{p}}\left(f^*(p_i) \cdot \psi_i^{k_i} \right) \in H^*(\overline{M}_g(S,h))$$

degree $1+k$ — cotangent line constant.

~~$F_g(I_{k_1}(p) \dots I_k$~~

$$F_{\cancel{g}\,k_1,\dots,k_n}\left(\cancel{I_{k_1}(p) \dots I_{k_n}(p)}\right) = \sum q^{h-1} \int_{[\overline{M}_g(S,\cancel{p}H_h)]^{vir}} I_{\cancel{g}k_1}(p_1) \dots I_{k_n}(p_n)$$

g determined by $g+n = \sum(k+h_i)$

$$\Rightarrow \boxed{g = \sum k_i + 1}$$

Thm (Bryan Leung) ~~$\forall g$~~

Yau–Zaslow: $F() = \cancel{\frac{1}{\Delta}} \sum N_h q^{h-1} = \frac{1}{\Delta}$.

Bryan–Leung (96): $F(0,\dots,0) = \frac{G_2^n}{\Delta}$

Maulik-P.-Th: $F(k_1 \dots, k_n) \in \frac{\mathrm{QMod}}{\Delta}$ ✓, poly in $G_2, G_4,$

$$F_g(k_1, \dots, k_n) = \frac{F_{k_1} \cdots F_{k_n}}{\Delta(q)}$$

$$F_k = \mathrm{Res} \quad \wp(z,\tau) = \frac{(-1)^k}{(2k+3)!} \mathrm{Res}_{z=0}\Big[(\wp - 4G_2)^{k+\frac{3}{2}} (\wp + 2$$

$$\wp(z,\tau) = \sum_{\substack{\omega \neq 0 \\ \omega \in \Lambda = \mathbb{Z} \oplus \mathbb{Z}\tau}} \frac{1}{(z+\omega)^2 - z^2}$$

$$= \frac{1}{z^2} + \sum_{\substack{\omega \neq 0 \\ \omega \in \Lambda = \mathbb{Z} \oplus \mathbb{Z}\tau}} \frac{1}{(z-\omega)^2 - \omega^2}$$