19.12.2022

# Lecture 11: Proof of Torelli theorem.

## § Kähler and ample cone

$S$ algebraic K3 surface

$\mathrm{Amp}(S) :=$ Convex cone in $\mathrm{Pic}(S) \otimes \mathbb{R}$ generated by all ample classes.

$\tilde{\mathcal{C}}_S := \{ \alpha \in \mathrm{Pic}(S) \otimes \mathbb{R} \mid \alpha \cdot \alpha > 0 \}$ 2 connected comp.

$\cup$

$\mathcal{C}_S$ component containing $\mathrm{Amp}(S)$.

**Thm** $\mathrm{Amp}(S) = \{ \alpha \in \mathcal{C}_S \mid \alpha \cdot [C] > 0 \text{ for all } C \cong \mathbb{P}^1, C \subset S \}$

**Example:** Let $\pi : S \to \mathbb{P}^1$ elliptic fibration with section $B$.

Assume $\mathrm{Pic}(S) = \langle B, F \rangle$

$F$ ← fiber class.

$W := B + F$.

$\mathcal{C}_S = \{ aW + bF \mid 2ab > 0, \ a > 0 \}$

$\Rightarrow b > 0$

$(aW + bF)^2 = -2 \Rightarrow (a,b) = (1,-1)$ or $(-1,1)$

$\Rightarrow$ Only $\mathbb{P}^1$ we have is $B = W - F$.

$\Rightarrow \mathrm{Amp}(S) = \left\{ aW + bF \;\middle|\; \begin{array}{l} a, b > 0 \\ (aW+bF)\cdot(W-F) = b - a > 0 \end{array} \right\}$

$= \{ aW + bF \mid b > a > 0 \}$.

$S$ Kähler K3 surface.

$K_S \subset H^{1,1}(S, \mathbb{R})$ cone of Kähler classes.

$\mathcal{C}_S$ connected comp. of $\{\alpha \in H^{1,1}(S,\mathbb{R}) \mid \alpha \cdot \alpha > 0\}$ containing $K_S$.

Thm $K_S = \{\alpha \in \mathcal{C}_S \mid \alpha \cdot [\mathbb{P}^1] > 0$ for all $C \cong \mathbb{P}^1 \subset S\}$.

Torelli Theorem Let $S, S'$ K3 surfaces, let
$$f : H^2(S, \mathbb{Z}) \longrightarrow H^2(S', \mathbb{Z})$$
a Hodge isometry (i.e. $f(H^{2,0}(S)) = H^{2,0}(S')$).

Then $S \cong S'$.

If $f$ sends $K_S$ to $K_{S'}$ then $\exists! \, F : S' \to S$ s.t. $f = F^*$. $\square$

Last time:

$$\mathcal{M} = \left\{ \begin{matrix} \text{marked K3 surfaces} \\ (S,\varphi) \end{matrix} \right\} / \sim \qquad \text{non Hausdorff complex mfld}$$

$$\begin{array}{ccc} \mathcal{M} & & (S,\varphi) \\ \big\downarrow \lambda & & \big\downarrow \\ \mathcal{D}_L & \ni & \varphi(H^{2,0}(S,\mathbb{C})) \subset L \otimes \mathbb{C} \end{array}, \quad L = E_8(-1)^{\oplus 2} \oplus U^{\oplus 3}.$$

Prop If $x=(S,\varphi)$ and $y=(\tilde{S},\tilde{\varphi}) \in \mathcal{M}$ are inseparable ($x \sim y$), and $x \neq y$, then

(a) $S \cong \tilde{S}$

(b) $\lambda(x) = \lambda(y)$

(c) $\lambda(x) \perp \beta$ for some $\beta \neq 0$ in $\Lambda$.

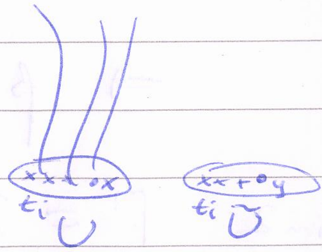
Proof Let $t_i \in \mathcal{M}$ sequence converging to $x$ and $y$.

$\mathcal{X} \longrightarrow U = Def(S)$, $\alpha$ with $\alpha_0 = \varphi$

$\tilde{\mathcal{X}} \longrightarrow \tilde{U} = Def(\tilde{S})$, $\tilde{\alpha}$ $\quad \tilde{\alpha}_0 = \tilde{\varphi}$ $\quad$ univ. def. families.

Can Assume: $U \subset \mathcal{M}$, $\tilde{U} \subset \mathcal{M}$, $t_i \in U \cap \tilde{U}$

Universality: $\exists!\, f_i : \mathcal{X}_{t_i} \longrightarrow \tilde{\mathcal{X}}_{t_i}$ s.t. $f_{t_i}^* = \alpha_{t_i}^{-1} \circ \tilde{\alpha}_{t_i}$




$\Rightarrow$ $\Gamma_{f_i} \subset \mathcal{X}_{t_i} \times \tilde{\mathcal{X}}_{t_i}$ graph has bounded volume.

$\Rightarrow$ $\exists$ limit cycle $\Gamma := \lim_{i \to \infty} \Gamma_{f_i}$ in $\mathcal{X}_0 \times \tilde{\mathcal{X}}_0 = S \times \tilde{S}$.

$$\Gamma_* = \alpha_0^{-1} \circ \tilde{\alpha}_0 = \varphi^{-1} \circ \tilde{\varphi}.$$

Also, ~~$\lambda(x) = \lambda(y)$ since $\mathcal{D}_L$ Hausdorff, so $\Gamma_*$ Hodge isometry.~~

$D_L$ Hausdorff $\Rightarrow \lambda(x) = \lambda(y)$

$$\Rightarrow \varphi(H^{2,0}(S)) = \tilde{\varphi}(H^{2,0}(\tilde{S}))$$

$$\Rightarrow \varphi^{-1} \circ \tilde{\varphi} \text{ Hodge isometry}$$

(This also follows directly, because the limit of a Hodge isometry is one.)

Option 1:

$$\Gamma = Z + \sum_{k=1}^{\ell} W_k.$$

$p_1 : Z \longrightarrow S$, $p_2 : Z \longrightarrow \tilde{S}$ both degree 1, so generically ~~inj~~ isom.

$p_1(W_k) = C_k$, $p_2(W_k) = C_k'$ irreducible curves.

$\Rightarrow$ $Z = \Gamma_g$ for $g : S \dashrightarrow S'$ birational.

By minimality of $S$ and $S'$ $\Rightarrow$ $g$ isomorphism.

If $\ell = 0 \Rightarrow g_* = \varphi^{-1} \circ \tilde{\varphi} \Rightarrow (S, \varphi) \cong (\tilde{S}, \tilde{\varphi})$, so $\ell > 0$,

and $\exists$ curve $C_k \subset S$.

$$\Rightarrow \beta = [C_k] \perp \sigma.$$

Option 2: $\Gamma = Z_1 + Z_2 + \sum W_k.$

$p_1 : Z_1 \longrightarrow S$ degree 1, $p_2 : Z_1 \longrightarrow \tilde{C} \subset \tilde{S}$.

$p_2 : Z_2 \longrightarrow \tilde{S}$ " , $p_1 : Z_2 \longrightarrow C \subset S$.

~~Use~~ ~~$\Phi_C$~~ $\Gamma_*[\sigma] = \underbrace{(Z_1)_*[\sigma]}_{\text{supported on } \tilde{C},\ S = \text{hyre}(1,1)} + \underbrace{(Z_2)_*[\sigma]}_{0 \text{ since } \sigma|_C = 0} + \underbrace{\sum W_{k*}[\sigma]}_{=0} = 0$ ↯

$\neq 0$ $\lambda \cdot [\tilde{\sigma}]$

**Corollary** (+ work) $\sim$ is an open equivalence relation and

$$\overline{\mathcal{M}} := \mathcal{M}/\sim \quad \text{Hausdorff complex mfld.}$$

**Cor:** The period map $\lambda : \mathcal{M} \longrightarrow \mathcal{D}_L$ factor as

$$\begin{tikzcd} \mathcal{M} \arrow[rr, "\lambda"] \arrow[dr, "p"'] & & \mathcal{D}_L \\ & \overline{\mathcal{M}} \arrow[ur, "\overline{\lambda}"'] & \end{tikzcd}$$

where • $p, \overline{\lambda}$ local isomorphisms.

• $p$ isomorphism over $\overline{\lambda}^{-1}\left(\mathcal{D}_L \setminus \bigcup_{\forall \beta \in L} \beta^{\perp}\right)$.

**Step 2:** (Skipped)

**Prop:** $\mathcal{D}_L \subset \mathbb{P}(L \otimes \mathbb{C})$ is connected and simply-connected.

**Proof:** $\mathcal{D}_L \cong O(3,19)/(SO(2) \times O(1,19))$

**Step 3:** **Thm:** $\overline{\lambda} : \overline{\mathcal{M}} \longrightarrow \mathcal{D}_L$ is a covering map.

In particular:

(i) $\overline{\lambda}$ is surjective, so every period $x \in \mathcal{D}_L$ is the period of some K3 surface

(ii) $\overline{\lambda}$ is isomorphism on each connected comp of $\overline{\mathcal{M}}$.

Period domain:

$$L = U^{\oplus 3} \oplus E_8(-1)^{\oplus 2}$$

$$\mathcal{D}_L = \{ x \in \mathbb{P}(L \otimes \mathbb{C}) \mid x \cdot x = 0,\ x \cdot \bar{x} > 0 \}.$$

Lemma $\mathcal{D}_L \cong Gr^{p.o.}(2, L \otimes \mathbb{R}) \cong O(3,19)/(SO(2) \times O(1,19).)$

$$\left\{ (U, o) \;\middle|\; \begin{array}{l} U \overset{real}{\subset} L \text{ s.t. } q|_U \text{ positive definite} \\ o \text{ orientation of } U \\ \in \Lambda^2 U / \mathbb{R}^+ \end{array} \right\}$$

Proof

$\Rightarrow$

$$x \longmapsto U := \mathrm{Span}(Re(x), Im(x))$$
with $o := [Re(x) \wedge Im(x)]$.

$\Leftarrow$

$$x \cdot x = 0 \iff (Re(x) + i Im(x)) \cdot (Re(x) + i Im(x))$$
$$= \underbrace{Re(x)^2 - Im(x)^2}_{0} + 2i \underbrace{Re(x) \cdot Im(x)}_{0}$$

$$x \cdot \bar{x} = Re(x)^2 + Im(x)^2 > 0$$

$$q|_U \cong \begin{pmatrix} a & \\ & a \end{pmatrix} \qquad a = \sqrt{Re(x)^2} > 0.$$

Conversely: let $v_1, v_2 \in U$ orthonormal basis,

$U \longmapsto x$ where $x := v_1 + i v_2$, $\quad x \cdot x = v_1^2 - v_2^2 = 0$
$\quad x \cdot \bar{x} = v_1^2 + v_2^2 > 0$.

a different ONB corresponds to $w_1, w_2$
$$\begin{pmatrix} w_1 \\ w_2 \end{pmatrix} = \begin{pmatrix} \cos\varphi & -\sin\varphi \\ +\sin\varphi & \cos\varphi \end{pmatrix} \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} \Rightarrow w_1 + i w_2 = (\cos\varphi + i \sin\varphi)(v_1 + i v_2)$$
$$= e^{i\varphi}(v_1 + i v_2) \quad \square$$

~~Cor~~ **Prop**: $\mathcal{D}_L$ is connected and simply connected

**Proof** proof $\mathcal{D}_L \subset Gr^{p.o.}(2, L\otimes\mathbb{R}) \cong O(3,19)/(SO(2)\times O(1,19))$

$$\begin{tikzcd} SO(2)\times O(1,19) \arrow[r] & O(3,19) \arrow[d] \\ & \mathcal{D}_L \end{tikzcd}$$

4 cap.

4 connected compon.

$\pi_1(SO(2)) = \mathbb{Z}$, $\pi_1(SO(k)) = \mathbb{Z}_2$ for $k>2$.

$\pi_1(O(1,19)^\circ) = \pi_1(SO(k)) \times \pi_1(SO(l))$.

$$\begin{tikzcd} & & \pi_1(\mathcal{D}_L) \arrow[dr] & & \\ \pi_1(SO(2))\times\pi_1(O(1,19)^\circ) \arrow[r] \arrow[d, phantom, "\cong"] & \pi_1(O(3,19)^\circ) \arrow[ur] \arrow[r] \arrow[d, phantom, "="] & \pi_0(SO(2)\times O(1,19)^\circ) \arrow[r] & \pi_0(-) \\ \mathbb{Z}\times\mathbb{Z}_2 \arrow[r] & \mathbb{Z}_2\times\mathbb{Z}_2 & & \end{tikzcd}$$

$\square$

$$\begin{tikzcd} \mathfrak{M} \arrow[rr, "\lambda"] \arrow[dr, "p"'] & & \mathcal{D}_L \\ & \overline{\mathfrak{M}} \arrow[ur, "\overline{\lambda}"'] & \end{tikzcd}$$

Marked K3's

Hausdorff

Idea of proof of surjectivity:

Let $W \subset L \otimes \mathbb{R}$ be a positive 3-plane.

$$T_W := \mathbb{P}(W \otimes \mathbb{C}) \cap \mathcal{D}_L \quad \text{twistor line.}$$

$$T_W = \left\{ x \in \mathbb{P}(W) \;\middle|\; \begin{array}{l} x \cdot x = 0 \\ x \cdot \bar{x} > 0 \end{array} \right\}.$$

$x = Re(x) + i\, Im(x) \quad \Rightarrow \quad x \cdot x = Re(x)^2 - Im(x)^2 + 2i\, Re(x) \cdot Im(x)$

~~$\Rightarrow Re(x) \perp Im(x)$.~~

$x \cdot \bar{x} = Re(x)^2 + Im(x)^2$.

If $x \cdot x = 0$, then $Re(x) \perp Im(x)$

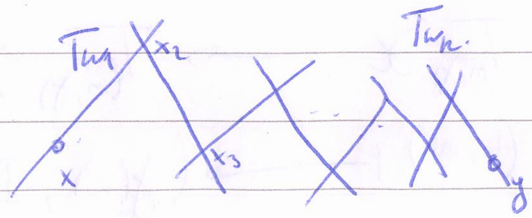
$\Rightarrow x \cdot \bar{x} > 0$ automatic.

$\Rightarrow T_W \cong \{ x \in \mathbb{P}(W) \mid x \cdot x = 0 \}$
$\cong \mathbb{P}^1$.

**Def** $W$ generic if $W^\perp \cap L = 0$.

Lemma: Any two points $x, y \in \mathcal{D}_L$ are connected by a chain of (generic) twistor lines, i.e. $\exists$ pos. 3 spaces $W_1, \ldots, W_k$ and points $x_1, \ldots, x_{k+1} \in \mathcal{D}_L$, $x_1 = x$, $x_{k+1} = y$ s.t. $x_i, x_{i+1} \in T_{W_i}$.

~~In fact, can choose $W_i$ generic,~~
~~i.e.~~



Lift of twistor lines:

$S$ K3 surface, $\alpha \in H^{1,1}(S, \mathbb{R})$ Kähler class.

Write $S = (M, I)$

diff mfld

complex structure $I \in \mathrm{End}(T_M)$ s.t. $I \circ I = -\mathbb{1}$ (+ integrality condition)

Yau's theorem: $\exists!$ Ricci flat Kähler metric $g$ s.t. $\omega_I = g(I-,-)$ has class $\alpha$.

$\mathrm{Holonomy}_g(S) = SU(2) = Sp(1)$

$\Rightarrow \exists$ complex structures $J, K \in \mathrm{End}(T_M)$ which define action

$$\mathbb{H} \circlearrowright T_M$$

Quaternions

$$i \longmapsto I$$
$$j \longmapsto J$$
$$k \longmapsto K$$

Prop [For any $(a,b,c) \in S^2 \subset \mathbb{R}^3$, the endomorphism $\gamma = aI + bJ + cK$ is a complex structure with Kähler form $\omega_\gamma = g(\gamma \circ, \circ)$.

Turn into family:

$$\mathcal{X} := (M \times \mathbb{P}^1, \mathbb{I})$$

$$\mathbb{I} : T_{(m,\gamma)}\mathcal{X} \longrightarrow T_{(m,\gamma)}\mathcal{X}$$
$$(v, w) \longmapsto (\gamma_m(v), I_{\mathbb{P}^1} w)$$

complex structure of $\mathbb{P}^1$.

$\pi_0 : \mathcal{X} \longrightarrow \mathbb{P}^1$ holomorphic family of K3 surfaces

s.t. $\pi^{-1}(\lambda) = (M, \lambda)$.

Let $\varphi : H^2(M, \mathbb{Z}) \longrightarrow L$ marking.

$$\lambda : \mathbb{P}^1 \longrightarrow \mathcal{D}_L \quad \text{Period map.}$$
$$\gamma \longmapsto \varphi(H^{2,0}(M, \gamma)) = [\sigma_\gamma]$$

what is this?

For $\gamma_I \in S^2$, let $\gamma_J, \gamma_K \in S^2$ ~~orthogonal~~ with $\gamma_I, \gamma_J, \gamma_K$ orthogonal s.t.

$$\gamma_I \gamma_J = \gamma_K, \quad \gamma_J \gamma_K = \gamma_I, \quad \gamma_K \gamma_J = \gamma_I.$$

Set $\sigma_{\gamma_I} := \omega_{\gamma_J} + i \gamma_K$.

<u>Claim</u>: $\sigma_{\gamma_I}$ is the holomorphic 2-form on $(M, \gamma_I)$.

<u>Proof</u>: Write $\gamma_I = I$, $\gamma_J = J$, $\gamma_K = K$.

$$\begin{aligned}
\sigma_I(Iu, v) &= \sigma_J(Iu, v) + i\,\sigma_K(\cancel{K}Iu, v) \\
&= g(\underbrace{JI}_{-IJ=-K}u, v) + i\,g(\underbrace{KI}_{J}u, v) \\
&= i\big(g(Ju, v) + i\,g(Ku, v)\big) \\
&= i \cdot \sigma_I(u, v)
\end{aligned}$$

$\Rightarrow \sigma_I$ closed $(2,0)$ form on $(M, \gamma_I)$, hence holomorphic $\square$

~~Ex~~:

Ex

For $I \in S^2$, we get $\sigma_I = \omega_J + i\omega_K$.

Let $W := \langle Re(\sigma), Im(\sigma), \alpha \rangle$

Upshot:

$$\Rightarrow \lambda_X : \begin{tikzcd} \mathbb{P}^1 \arrow[rr] \arrow[dr, "2"'] & & \mathcal{D}_L \\ & T_W \arrow[ur] & \end{tikzcd}$$

Cor (+some work) Given $(S,\varphi) \in \mathfrak{M}$ s.t. $\lambda(S,\varphi) \in T_W$ for a generic $W$. Then there exist a unique lift

$$\begin{tikzcd} \bar{\mathfrak{M}} \arrow[r, "\bar{\lambda}"] & \mathcal{D}_L \\ & T_W \arrow[u, hook] \arrow[ul, dotted, "\exists! \ell"] \end{tikzcd}$$

s.t. $(S,\varphi) \in \ell(T_W)$.

Proof $\bar{\lambda}$ local isomorphism, $\exists \Delta \subset T_W$ nbhd of $x = \lambda(S,\varphi)$ and lift

$$\begin{tikzcd} \bar{\mathfrak{M}} \arrow[r, "\bar{\lambda}"] & \mathcal{D}_L \\ & \Delta \arrow[u, hook] \arrow[ul, "\ell"] \end{tikzcd}$$

$\ell(t) = (S_t, \varphi_t)$

For generic $t \in \Delta$, $Pic(S_t) = 0$

$\Rightarrow \mathcal{C}_{S_t} = \mathcal{K}_{S_t}$

$\Rightarrow \varphi_t^{-1}(W_t) \cap H^{1,1}(S_t, \mathbb{R})$ spanned by Kähler class $\alpha_t$

~~s.t. $W_t$~~ $W = \langle \varphi_t(Re\sigma_t), \varphi_t(Im\sigma_t), \varphi_t(\alpha_t) \rangle$

$\Rightarrow$ Twistor construction for $(S_t, \alpha_t)$ given $\ell$.

$\square$

## Step 4: Understand connected components of $\mathfrak{M}$

**Lemma** $\mathfrak{M}$ has at least 2 components.

**Proof** Let $\overline{\mathfrak{M}}^\circ \subset \overline{\mathfrak{M}}$ connected compont $\rightsquigarrow$

$\rightsquigarrow \lambda|_{\overline{\mathfrak{M}}^\circ} : \overline{\mathfrak{M}}^\circ \longrightarrow \mathcal{D}_L$ iso

Let $(S, \varphi) \in \overline{\mathfrak{M}}^\circ$ marked K3.

Consider $(S, -\varphi)$.

- $(S, \varphi)$ and $(S, -\varphi)$ are not isomorphic.

  ($\nexists f \in \mathrm{Aut}(S)$ s.t. $f^* = -\mathrm{id}$, since $-\alpha$ of Kähler class is not Kähler)

- $\lambda(S, -\varphi) = -\varphi(H^{2,0}(S)) = \lambda(S, \varphi)$

- If $\mathrm{Pic}(S) = 0$, then $(S, \varphi) \neq (S, -\varphi)$ in $\mathfrak{M}$

  (e.g. let $(S, \varphi)$ preimage of $x \in \mathcal{D}_L \setminus \bigcup_\beta \beta^\perp$) $\Rightarrow (S, \varphi) \neq (S, -\varphi)$ in $\overline{\mathfrak{M}}$.

$\Rightarrow (S, -\varphi)$ must live in different compont. of $\overline{\mathfrak{M}}$. $\square$

Lemma: $\mathcal{M}$ has at most 2 components.

Facts:

(a) Any two K3 surfaces are deformation equivalent.

(b) $\mathrm{Mon}(S) \subseteq O(H^2(S,\mathbb{Z}))$ subgroup generated by monodromy operators.

Def A monodromy operator of $S$ is

$$\alpha_1^* \circ PT_\gamma \circ (\alpha_0^*)^{-1} \in O(H^2(S,\mathbb{Z})).$$

where
- $\mathcal{X} \to B$ smooth family of K3 surfaces
- $\gamma : [0,1] \longrightarrow B$ path,
- $\alpha_i : S \xrightarrow{\sim} \mathcal{X}_{\gamma(i)}$ isomorphisms, $i=0,1$.
- $PT_\gamma : H^2(\mathcal{X}_0,\mathbb{Z}) \longrightarrow H^2(\mathcal{X}_1,\mathbb{Z})$ parallel transport.

Thm $\mathrm{Mon}(S) \subset O(H^2(S,\mathbb{Z}))$ has index 2, with

$$O(H^2(S,\mathbb{Z}))/\mathrm{Mon}(S) = \langle -1 \rangle$$

Proof of Lemma: Let $(S,\varphi), (T,\psi) \in \mathcal{M}$.

By (a), $(T,\psi) \underset{\text{def}}{\sim} (S,\varphi')$ for some $\varphi$.

By (b), $\exists h \in \mathrm{Mon}(S)$ s.t. $\varphi \circ (\varphi')^{-1} = \pm h$.

$\Rightarrow \quad (S,\varphi') \overset{\text{monodromy}}{\sim} (S, h\circ\varphi') = (S, \pm\varphi)$

$\Rightarrow (T,\psi)$ or $(T,-\psi)$ lies in compt of $(S,\varphi)$ $\square$

## § Proof of Torelli thm

Let $S, S'$ K3 surfaces with Hodge isometry $f: H^2(S,\mathbb{Z}) \xrightarrow{f} H^2(S',\mathbb{Z})$

let $\varphi$ marking of $S$. $\quad \varphi \downarrow L$

Let $\varphi' = \varphi \circ f^{-1}$

$$\Rightarrow \lambda(S',\varphi') = \varphi'(H^{2,0}(S',\mathbb{C}))$$
$$\overset{HI}{=} (\varphi\circ f^{-1})(f(H^{2,0}(S,\mathbb{C})))$$
$$= \varphi(H^{2,0}(S,\mathbb{C}))$$
$$= \lambda(S,\varphi).$$

If $(S,\varphi)$ and $(S',\varphi')$ lie in same compont of $\mathcal{M}$

$$\Rightarrow (S,\varphi) \sim (S',\varphi')$$
$$\Rightarrow S \cong S'.$$

If not, then use $(S,\varphi)$ and $(S,-\varphi')$.

(b) If $f$ sends Kähler class to Kähler class:
$$f(\alpha) = \alpha'.$$

By before $f = \Gamma_*$ for cycle $\Gamma = \Gamma_g + \sum_{k=1}^{\ell} C_k \times C_k'$,
$g: S \to S'$ isom.

$$\Gamma_*(\alpha) = g_*(\alpha) + \sum \underbrace{\langle C_k,\alpha\rangle}_{>0} C_k' \quad (*)$$

$$\Rightarrow \Gamma_*(\alpha) - g_*(\alpha) = \sum \langle C_k,\alpha\rangle C_k' \quad (*),$$

Kähler classes form a cone, so $\tilde{\alpha} = \Gamma_*(\alpha) + g_*(\alpha)$ Kähler.