Lecture 12.12.2022 Lecture 10

§ Torelli theorem for K3 surfaces.

Thm A Let S, S' (complex) K3 surfaces for which there exist an isometry

$$\varphi : H^2(S,\mathbb{Z}) \longrightarrow H^2(S',\mathbb{Z}),$$

with $\varphi(H^{2,0}(S)) = H^{2,0}(S')$. Then $S \cong S'$.

Thm B: If moreover φ sends a Kähler class to a Kähler class, then $\exists!\ f: S' \to S$ s.t. $f^* = \varphi$.

(If S, S' projective, ~~can ost~~ can replace Kähler class by ample class)

3 Proof Strategies:

(1) 1971: Pjateckiĭ-Šapiro, Šafarevič / Burns-Rapoport 1975
- Torelli thm for abelian surfaces ("easy")
  $\Rightarrow$ Torelli thm for Kummer surfaces.
- Kummer surfaces are dense in moduli of K3 surfaces.
  (use local Torelli + lattice theory)
- Take limit of isomorphisms.

(2) 2013: Verbitsky *, Huybrechts
- Moduli-theoretic approach
- Twistor lines.
- Works for HK manifolds

Reference: Huybrechts, A global Torelli thm for HK manifolds (Bourbaki)

(3) Bakker-Lehn: Ergodic theory
2020.
- Works for singular HK's

Recall: Local Torelli thm:

$L = E_8(-1)^{\oplus 2} \oplus U^{\oplus 3}$

(S, φ) marked K3 surface

$\varphi: H^2(S,\mathbb{Z}) \xrightarrow{\sim} L$ isometry.

$$\begin{tikzcd} X \arrow[d, "\pi"', "\text{univ. def.}"] & \supset & S \arrow[d] \\ Def(S) & \ni & 0 \end{tikzcd}$$

$\Rightarrow \exists! \; \alpha: R^2_{\pi_*}\mathbb{Z} \xrightarrow{\sim} \underline{L}$

s.t. $\alpha_0 = \varphi$.

$Def(S)$ contractible

$\Rightarrow \lambda: Def(S) \longrightarrow \mathcal{D}_L = \left\{ x \in \mathbb{P}(L\otimes\mathbb{C}) \;\middle|\; \begin{array}{l} x \cdot x = 0 \\ x \cdot \bar{x} > 0 \end{array} \right\}$

$t \longmapsto [\alpha_t(H^{2,0}(X_t))]$

Thm (Local Torelli): λ local isomorphism ($(d\lambda)_t$ invertible)

Thm (Last time): $f: S \to S$ automorphism, s.t. $f^* = id$ on $H^*(S,\mathbb{Z})$.

Then $f = id$.

Cor: If there is an isomorphism f between two marked K3 surfaces (S,φ) and $(\tilde{S},\tilde{\varphi})$, then it is unique.

Proof: $f : (S, \varphi) \longrightarrow (\tilde{S}, \tilde{\varphi})$ consists of

$f : S \to \tilde{S}$ s.t.

$$\begin{tikzcd} H^2(S,\mathbb{Z}) \arrow[r, "\varphi"] & L \\ H^2(\tilde{S},\mathbb{Z}) \arrow[u, "f^*"] \arrow[ur, "\tilde{\varphi}"'] & \end{tikzcd}$$ commutes.

~~If there are two~~ $\cancel{f_1, f_2}$ ~~satisfying this~~

For $i = 1, 2$, let $f_i : S \to \tilde{S}$ s.t. $\varphi \circ f_i^* = \tilde{\varphi}$

$$\Rightarrow f_2^{-1} \circ f_1 : S \to S$$

$$(f_2^{-1} \circ f_1)^* = f_1^* \circ (f_2^*)^{-1} = id$$

$$\Rightarrow f_1 = f_2$$

$\square$.

Thm There is a fine moduli space $\mathcal{M}$ of marked K3 surfaces, i.e. there is a (non-Hausdorff) complex mfld $\mathcal{M}$, univ. family $\pi : \mathcal{X} \to \mathcal{M}$, $\alpha : R^2\pi_* \mathbb{Z} \to \underline{L}$ s.t.

for any deformation family $X \xrightarrow{\pi_B} B$ of K3 surfaces and for any $\alpha_B : R^2\pi_{B*} \mathbb{Z} \to \underline{L}$, there is a unique morphism $f : B \to \mathcal{M}$ s.t.

$$(X \to B, \alpha_B) \cong (f^*\mathcal{X} \to B, f^*\alpha)$$

(We allow B to be a complex space).

Proof:

For any marked K3 surface (S_i, φ_i) let

$$\{ X_i \xrightarrow{\pi_i} U_i = \mathrm{Def}(S_i)$$

deformation family s.t.

(1) U_i is contractible, so $\exists!\ \alpha_i : R^2\pi_* \mathbb{Z} \longrightarrow \underline{L}$ with $\alpha_0 = \varphi_i$

(2) π_i is universal for all its fibers

(3) $\lambda : U_i \longrightarrow \mathcal{D}_L$ is injective.

Obs1: For $t, t' \in U_i$,

$$(X_t, \alpha_t) \cong (X_{t'}, \alpha_{t'})$$

$$\cancel{\Rightarrow \lambda(X_t, \alpha_t) = \lambda(X_{t'}, \alpha_{t'})}$$

$$\Rightarrow \lambda(t) = \alpha_t(H^{2,0}(X_t)) = \alpha_{t'}(H^{2,0}(X_{t'})) = \lambda(t')$$

$$\overset{(3)}{\Rightarrow} t = t'$$

Obs2: Let $t \in U_i$, $t' \in U_j$, $i \neq j$ s.t. $\underset{\text{unique}}{F_0} : (X_t, \alpha_t) \xrightarrow{\cong} (X_{t'}, \alpha_{t'})$

Universality: $\exists\ t \in V_i \underset{\text{open}}{\subset} U_i$, $t' \in V_j^* \subset U_j$ ~~s.t. and unique~~ $\cancel{f : V_i \xrightarrow{\cong} V_j}$ s.t. $\cancel{F : X|}$

$X|_{V_i}$
$\downarrow$

and <u>unique</u> isomorphism of marked families extending F_0.

$$\begin{array}{ccc} X_i|_{V_i} & \xrightarrow[\sim]{F} & X_j|_{V_j} \\ \downarrow & & \downarrow \\ V_i & \xrightarrow[\sim]{f} & V_j \end{array}$$

Set: $\mathcal{M} := \bigsqcup_{i \in I} U_i / \sim$, $t \sim t'$ if $(X_t, \alpha_t) \cong (X_{t'}, \alpha_{t'})$

$\overset{\text{Obs 1+2}}{=}$ (Complex Manifold (not nec. Hausdorff))

$\overset{\text{set}}{=} \{ \text{marked K3 sufaces} \}/\sim$

Univ. family:

$\mathcal{X} = (\bigsqcup X_i)/\sim$, where $\sim$ glues X_t to $X_{t'}$, with $t \sim t'$

$\downarrow \pi$ via unique isom.

$\mathcal{M}$ $(X_t, \alpha_t) \xrightarrow{\sim} (X_{t'}, \alpha_{t'})$

Define $\tilde{\alpha} : R^2_{\pi_X} \mathbb{Z} \longrightarrow \underline{L}$ by $\tilde{\alpha}_t := \alpha_t$. (rhs defined by (*))

For any family $(X_B \xrightarrow{\pi_B} B, \alpha_B)$ set.

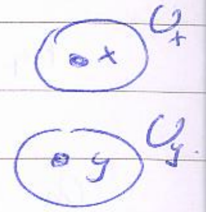
$$f : B \longrightarrow \mathcal{M}$$
$$t \longmapsto [(X_{B,t}, \alpha_t)].$$

$F : X_B \longrightarrow f^* \mathcal{X}$ unique iso $F_t : X_{B,t} \longrightarrow \mathcal{X}_{f(t)}$
of marked K3s.

On $f^{-1}(U_i)$ the maps f, F are induced by univ. property of U_i, so hol. $\square$

Example: $\mathcal{M}$ is not Hausdorff.

Rmk: A Topological space T is Hausdorff if $\forall x, y \in T$ $\exists$ open $x \in U_x \subset T$, $y \in U_y \subset T$ s.t. $U_x \cap U_y = \emptyset$



Example: Let $\pi : X \to \Delta \subset \mathbb{C}_t$ proper flat map s.t.

(1) $X \setminus X_0 \longrightarrow \Delta^* = \Delta \setminus \{0\}$ family of K3 surfaces.

(2) X_0 singular with one singularity locally of the form
$$(x^2+y^2+z^2=0) \subset \mathbb{C}^3_{x,y,z}.$$

(3) X smooth, so locally near singular part
$$X \cong (x^2+y^2+z^2+t=0) \subset \mathbb{C}^3.$$

Concrete: $X = V\left(x^2(x^2-2) + y^2(y^2-2) + z^2(z^2-2) = 2t \right)$

Base change:

$$\begin{tikzcd}
X_0 \subset X' \arrow[rr] \arrow[d] & & X \arrow[d] \\
0 \in \mathbb{C}_t \arrow[rr, "t \mapsto t^2"'] & & \mathbb{C}_t
\end{tikzcd}$$

Locally, $X' \cong (x^2+y^2+z^2+t^2=0)$ singular!

Blowup of $x_0^2 + x_1^2 + x_2^2 + x_3^2 = 0$ at $0 \in \mathbb{C}^4$:

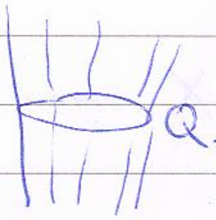
Proj coords $U_0, \ldots, U_3$ s.t. $U_i x_j = U_j x_i$

On $U_0 = 1$: $x_1 = x_0 U_1$
$x_2 = x_0 U_2$
$x_3 = x_0 U_3$.

$\Rightarrow \pi^*(x_0^2 + \cdots + x_3^2) = x_0^2 (1 + U_1^2 + U_2^2 + U_3^2) = 0$.

$\Rightarrow$ Blowup is preimage of $Q = V(U_0^2 + \cdots + U_3^2) \subset \mathbb{P}^3$ under natural map

$$\begin{array}{c} Bl_0(\mathbb{C}^4) \longrightarrow \mathbb{P}^3 \\ \cong \\ \mathrm{Tot}(\mathcal{O}(-1)) \end{array}$$



If $t = x_3$, fibre over $t=0$ on $U_0 = 1$ is:

$\pi^*(x_3) = x_0 \cdot U_3$.

$\Rightarrow$ Two components: Exceptional divisor $\cong Q$
+ Proper transform of $x_3 = 0$.

proper transform of X_0 smooth

$\mathbb{P}(\text{Tangent cone})$ of $p \in X'$ $= \mathbb{P}(C_p X')$.

$$
\begin{tikzcd}
\overline{X}_0 \underset{C \cong \mathbb{P}^1}{\cup} (\mathbb{P}^1 \times \mathbb{P}^1) \arrow[r] \arrow[d] & Bl_p(X') = \tilde{X} \arrow[d] \\
0 \in & \mathbb{C}_t
\end{tikzcd}
$$

, $p \in X'$ singular point.

$N_{\mathbb{P}^1 \times \mathbb{P}^1 / \tilde{X}} = \mathcal{O}(-1,-1)$, $C \subset \mathbb{P}^1 \times \mathbb{P}^1$ is $(1,1)$-curve

$\Rightarrow$ Can contract all $\mathbb{P}^1 \times \{s\}$ or $\{s\} \times \mathbb{P}^1$.

$\Rightarrow$ $p_i : X_i \to \mathbb{C}_t$ when $\tilde{X} \to X_i$ contraction of the i^{th} ruling ~~$(\mathbb{P}^1 \times \{s\})$ ruling~~

$$
\begin{tikzcd}
\overline{X}_0 \arrow[d] & \subset X_i \arrow[d, "p_i"] & \supset \tilde{X} \setminus \tilde{X}_0 \arrow[d, "\tilde{p}"] \\
0 & \in \mathbb{C}_t & \supset \mathbb{C}_t \setminus \{0\}
\end{tikzcd}
$$

$$
\begin{cases}
p_i|_{\mathbb{C}_t \setminus \{0\}} = \tilde{p} & \text{for } i = 1,2. \\
p_i^{-1}(0) = \overline{X}_0 & \text{smooth} \\
\text{Let } \alpha_i : R^2 \pi_* \mathbb{Z} \to \underline{L} & \text{s.t. } \alpha_{1,t=1} = \alpha_{2,t=1}
\end{cases}
$$

$\Rightarrow$ • $(p_i : X_i \to \mathbb{C}_t, \alpha_i)$ isomorph as marked families over $\mathbb{C}_t \setminus \{0\}$

• NOT isomorph over $\mathbb{C}_t$

~~Assume $\exists\ f: \mathbb{C}X_1 \longrightarrow X_2$~~

~~Assume $\exists\ (F, f): (X_1 \to B_1)$~~

If
$$\begin{tikzcd} \tilde{X} \arrow[r, "\tilde{F}"] \arrow[d] & \tilde{X} \arrow[d] \\ X_1 \arrow[r, "F"] \arrow[d, "p_1"] & X_2 \arrow[d, "p_2"] \\ \mathbb{C}_t \arrow[r, "f"] & \mathbb{C}_t \end{tikzcd}$$

isomorphism

~~$\Rightarrow f = id$~~

~~& $F|_{X \setminus \{0\}}$~~

~~& $F|_{X_1 \setminus X_0} = id$~~

$\Rightarrow\ f = id, \quad F|_{\mathbb{C}_t \setminus \{0\}} = id.$

but $\tilde{F}$ interchanges rulings of $\mathbb{P}^1 \times \mathbb{P}^1$

so acts ~~acts~~ nontrivially on $\mathbb{P}(C_p X')$ $\lightning$.

$\diamond$

We see:

$f_1 : \Delta_t \longrightarrow \mathcal{M}$

$f_2 : \Delta_t \longrightarrow \mathcal{M}$

agree on $t \neq 0$, but not at $t = 0$

In fact: $\alpha_1(t=0) = S_\delta \circ \alpha_2(t=0)$, $\delta = [C]$ (-2)-class

$$S_\delta(x) = x + \langle x, \delta \rangle \delta. \text{ reflection.}$$

(S_δ is also the monodromy of $X \to \mathbb{C}_t$, the original family)

Hausdorffication of $\mathcal{M}$:

Def $x \sim y$ if $\forall\, x \in U \underset{op}{\subset} T$, $y \in V \underset{op}{\subset} T$ we have $U \cap V \neq \emptyset$.

We say x and y are <u>inseparable</u>.

Want to define
$$\overline{\mathcal{M}} = \mathcal{M}/\sim \qquad \text{"Hausdorffization of } \mathcal{M}\text{"}.$$

In general $\sim$ not an equivalence relation:

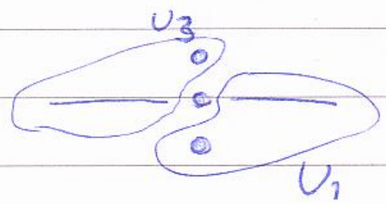
Ex

$U_1 = \circ\!\!-\!\!-\!\!-$ $\mathbb{R}_{\geq 0}$

$U_2 = -\!\!-\!\!\circ\!\!-\!\!-$ $\mathbb{R}$

$U_3 = -\!\!-\!\!-\!\!\circ$ $\mathbb{R}_{\leq 0}$

$T := U_1 \sqcup U_2 \sqcup U_3 /_{\approx}$ glue ~~along~~ away from origin



$O_1 \sim O_2 \sim O_3$

but $O_1 \not\sim O_3$.

Prop Suppose that $x = (S, \alpha)$ and $y = (\tilde{S}, \tilde{\alpha})$ are inseparable in $\mathcal{M}$, then

$$\text{(1)} \quad S \cong \tilde{S}, \quad \text{(2)} \quad \lambda(x) = \lambda(y) \in \mathcal{D}_L$$

and (3) $\lambda(x) \perp \beta$ ~~(for some~~ $\beta \in \Lambda$.

Proof.

Prop: Suppose that $x=(S,\varphi)$ and $y=(\tilde{S},\tilde{\varphi})$ are inseparable and $x\neq y$, then

(1) $S\cong\tilde{S}$

(2) $\lambda(x)=\lambda(y)\in\mathcal{D}_L$.

(3) $\lambda(\alpha)\perp\alpha$ for some $\alpha\in\Lambda$.

Proof:

Let $t_i\in\mathcal{M}$ sequence converging to both x and y.

Univ. deformation families:

$\mathcal{X}\longrightarrow U=\mathrm{Def}(S,\varphi)$, $\alpha: R^2\pi_*\mathbb{Z}\longrightarrow \underline{L}$, $\alpha_0=\varphi$

$\tilde{\mathcal{X}}\longrightarrow \tilde{U}=\mathrm{Def}(\tilde{S},\tilde{\varphi})$, $\tilde{\alpha}:\ \dots$, $\tilde{\alpha}_0=\tilde{\varphi}$

Assume $U\subset\mathcal{M}$, $\tilde{U}\subset\mathcal{M}$, $t_i\in U\cap\tilde{U}$.

$\Rightarrow\ \exists!\ f_i:\mathcal{X}_{t_i}\longrightarrow\tilde{\mathcal{X}}_{t_i}$ with $f_{t_i}^*=\alpha_{t_i}^{-1}\circ\tilde{\alpha}_{t_i}$

$$\begin{tikzcd} H^2(\mathcal{X}_{t_i},\mathbb{Z}) \arrow[r, "\alpha_{t_i}"] & L \\ H^2(\tilde{\mathcal{X}}_{t_i},\mathbb{Z}) \arrow[u, "f_{t_i}^*"] \arrow[ur, "\tilde{\alpha}_{t_i}"'] & \end{tikzcd}$$

If $\mathcal{X},\tilde{\mathcal{X}}$ projective, use properness of Hilbert scheme of $\mathrm{Hilb}(\mathcal{X}\times\tilde{\mathcal{X}}/U\cap\tilde{U})$ to pass to limit of Γ_{f_i}

Instead here: Kodaira: $\nexists H$

$\exists$ Kähler metric ω_t on $\mathcal{X}_t$ varying continuously in t. $\forall t\in U$.
$\tilde{\omega}_t$ on $\tilde{\mathcal{X}}_t$

$$\mathcal{X}_{t_i} \times \tilde{\mathcal{X}}_{t_i}$$
$$\cup$$

$$\text{Volume}(\Gamma_{f_i}) = \int_{\Gamma_{f_i}} \left(pr_1^*(\omega_{t_i}) + pr_2^*(\tilde{\omega}_{t_i}) \right)^2$$

$$= \int_{\mathcal{X}_{t_i}} \left(\cancel{pr_1^*}\omega_{t_i}) + f_i^* \tilde{\omega}_{t_i} \right)^2$$

$$= \int_{\mathcal{X}_{t_i}} \Big([\omega_{t_i}] + \underbrace{(\alpha_{t_i}^{-1} \circ \tilde{\alpha}_{t_i})}_{\text{continuous}} \underset{\uparrow \atop \text{continuous}}{([\tilde{\omega}_{t_i}])} \Big)^2$$

Continuous in t

$\Rightarrow$ bounded as $i \to \infty$.

Bishop

$\Rightarrow$ $\Gamma_{f_i} \subset \mathcal{X}_{t_i} \times \tilde{\mathcal{X}}_{t_i}$ converges to cycle $\Gamma \subset \mathcal{X}_0 \times \tilde{\mathcal{X}}_0$ s.t.

$$\Gamma_* = \alpha_{t_0}^{-1} \circ \tilde{\alpha}_{t_0} = \varphi^{-1} \circ \tilde{\varphi}.$$

$\Big\lceil$ $f_{t_i}^* = \alpha_{t_i}^{-1} \circ \tilde{\alpha}_{t_j}$ $\Rightarrow$ Γ_{f_i} have bounded volume in $\mathcal{X}_{t_i} \times \tilde{\mathcal{X}}_{t_i}$ as $i \to \infty$

$\Rightarrow$ Converges to a cycle $\Gamma \subset \mathcal{X}_0 \times \tilde{\mathcal{X}}_0$ $\Big\rfloor$

<u>Two options:</u>

① $$\Gamma = Z + \sum_{k=1}^{\ell} Y_k$$

where $pr_1 : Z \longrightarrow \mathcal{X}_0$, $pr_2 : Z \longrightarrow \tilde{\mathcal{X}}_0$ generically degree 1

$\Rightarrow$ Z graph of birational morphism $g : \mathcal{X}_0 \dashrightarrow \tilde{\mathcal{X}}_0$

$pr_1(Y_k) = C_k$ curve

$pr_2(Y_k) = \tilde{C}_k$ curve $\Rightarrow Y_k = C_k \times \tilde{C}_k$

② $$\Gamma = Z + R$$

$pr_1(Z) = C \subset \mathcal{X}_0$ curve

$pr_2(Z) = \tilde{\mathcal{X}}_0$

$pr_2(R)$ supported on curves

$\left(pr_{2*}[\Gamma] = 1 \cdot [\tilde{\mathcal{X}}_0] \right)$

<u>But:</u>

$$0 \neq \Gamma_*([\sigma_{\mathcal{X}_0}]) = \underbrace{[Z]_*([\sigma])}_{\substack{\text{image of } j^*(\sigma) = 0 \\ j : C \to \tilde{\mathcal{X}}_0}} + \underbrace{\overset{H^{2,0}(\tilde{\mathcal{X}}_0)}{\downarrow}[R]_*([\sigma])}_{\substack{\text{supported on algebraic} \\ \text{curves so type } (1,1)}} = 0 \quad ↯$$

In case 1:

~~$\mathcal{X}_0, \tilde{\mathcal{X}}_0$ minim~~

K3 surfaces are minimal surfaces $\Rightarrow g$ isomorphism.

If $l=0$, g isomorphism of marked K3 surfaces.

If $l>0$, $C_k \subset \mathcal{X}_0$ curve, so

$[C_k] \perp [\sigma_{x_0}]$.

Also: $\quad \Gamma_*([\sigma_{x_0}]) = [\sigma_{\tilde{x}_0}]$. $\square$

<u>Cor (+ some work)</u>

$\sim$ - equivalence relation on $\mathcal{M}$

$\sim$ open.

<u>Cor</u> $\overline{\mathcal{M}} := \mathcal{M}/\sim$ is a complex Hausdorff manifold.

- We have commutative diagram

$$\begin{tikzcd} \mathcal{M} \arrow[rr, "\lambda"] \arrow[dr, "h"'] & & \mathcal{D}_L \\ & \overline{\mathcal{M}} \arrow[ur, "\overline{\lambda}"'] & \end{tikzcd}$$

where ph, $\overline{\lambda}$ local isomorphisms

- ph is an isomorphism over $\overline{\lambda}^{-1}\left(\mathcal{D}_L \setminus \bigcup_{\alpha \in \Lambda} \alpha^{\perp}\right)$

M