

Automorphisms

S $K3$ surface.

$f: S \rightarrow S$ automorphism.

$$H^0(S, \omega_S^2) = \mathbb{C} \sigma \quad ; \quad f^* \sigma \in H^0(S, \omega_S^2)$$

$$\Rightarrow \exists \lambda \in \mathbb{C}^* \text{ s.t. } f^* \sigma = \lambda \sigma$$

Def f is symplectic if $f^* \sigma = \sigma$.

Ex: $S = V(x^4 + y^4 + z^4 + w^4) \subset \mathbb{P}^3$.

$S_4 \curvearrowright S$ by permutation.

$$\mathbb{A}_4^4 \curvearrowright S \text{ by } (x, y, z, w) \xrightarrow{(a, b, c, d)} (i^a x, i^b y, i^c z, i^d w)$$

Which ones are symplectic?

defn of $\mathcal{H}_{\text{ps}}^3(S) \neq 0$

✓
 $\downarrow \text{Res}$
 $\mathcal{H}_S^2$

$$\sigma_S = \text{Res} \left(\frac{\sum (-1)^i dx_0 \wedge \dots \wedge \widehat{dx_i} \wedge \dots \wedge dx_3}{f} \right)$$

e.g. in chart $x_0=1$: $\sigma_S|_{x_0=1} = \text{Res} \left(\frac{dx_1 \wedge dx_2 \wedge dx_3}{1+x_1^4+x_2^4+x_3^4} \right)$

$$\varphi_{(a,b,c,d)}^* \sigma_S = i^{b+c+d-3a} \sigma_S = i^{a+b+c+d} \sigma_S$$

$g \in S_4$: $g^* \sigma_S = \text{sgn}(g) \sigma_S$

$$\Rightarrow \mathbb{Z}_4^2 \subset \mathbb{Z}_4^4 / \mathbb{Z}_4$$

$\mathcal{A}_4 \subset S_4$ act symplectically.

$$\text{Aut}_{\text{symp}}(S) \cong \mathbb{Z}_4^{\times 2} \rtimes S_4$$

Res: let $z_1, \dots, z_n$ coords s.t. $D = (z_n = 0)$
f.d.v.

Write diff. form as

$$\omega = \frac{dz_n}{z_n} \wedge \alpha_{n-1}$$

$$\text{Res}(\omega) := \alpha_{n-1}|_D$$

Theorem: let $f: S \rightarrow S$ symplectic of prime order $n \geq 1$.

Then $1 \leq |\text{Fix}(f)| \leq 8$.

In fact:

$$|\text{Fix}(f)| = \frac{24}{n} \prod_{\substack{p|n \\ p \neq 1}} \left(1 + \frac{1}{p}\right)^{-1}$$

Proof

Claim 1:

Prop Let f symplectic of order n and $x \in \text{Fix}(f)$ fixed pt.
Then there are local coords x_1, x_2 s.t.

$$f(x_1, x_2) = (\lambda_x \cdot x_1, \lambda_x^{-1} x_2).$$

where λ_x primitive n -th root of unity.

Proof Assume $S = U \subset \mathbb{C}_{y_1, y_2}^2$, $x = 0$.

Define

$$g(y) = \frac{1}{n} \sum_{i=0}^{n-1} (df)_0^{-i} f^i(y)$$

$$\text{Note: } (dg)_0 = \frac{1}{n} \sum_{i=0}^{n-1} \underbrace{(df)_0^{-i}}_{\substack{= \\ (df)_0^i}} d(f^i)_0 = \text{id}.$$

$$\begin{aligned} (df)_0 g(y) &= \frac{1}{n} \sum_{i=0}^{n-1} (df)_0^{-i+1} f^{i-1}(f(y)) \\ &= \frac{1}{n} \sum_{j=1}^n (df)_0^{-j} f^j(f(y)) \\ &= g(f(y)). \end{aligned}$$

Set $x := g(y)$.

Then in coords x , we have

$$f(x) = (df)_0 \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$$

Link: This argument allows to show that if a finite acts on X smooth, then X^G is smooth.

(2) $(df)_0$ diagonalizable (since of order n).

$\Rightarrow$ After linear change of coords we may assume.

$$(df)_0 = \begin{pmatrix} \lambda_1 & \\ & \lambda_2 \end{pmatrix}, \quad \lambda_i \text{ } n^{\text{th}} \text{ roots.}$$

$$f(x) = (\lambda_1 x_1, \lambda_2 x_2)$$

(3) f symplectic:

$$(f^*(dx_1 \wedge dx_2))_0 = (dx_1 \wedge dx_2)_0$$

$\parallel$

$$\det((df)_0) dx_1 \wedge dx_2$$

$$\Rightarrow \lambda_1 \lambda_2 = 1$$

$$\Rightarrow \lambda_2 = \frac{1}{\lambda_1}$$

$$\Rightarrow \lambda_1 \text{ } n^{\text{th}} \text{ primitive root of unity.}$$

$\square$

Proof of Thm:

By Prop there are finitely many isolated fixed points.

Holomorphic Lefschetz fixed point formula:

$$\sum (-1)^i \text{Tr}(f^* | H^i(S, \mathbb{C})) = \sum_{x \in \text{Fix}(f)} \det((\text{id} - (df)_x)^{-1})$$

In our case:

$$f^* | H^0(S, \mathbb{C}) = \text{id}$$

$$f^* | H^2(S, \mathbb{C}) = \text{id}$$

$$\det((\text{id} - (df)_x)^{-1}) = (1 - \lambda_x)(1 - \bar{\lambda}_x)^{-1}$$

$$1 + 0 + 1 = \sum_{x \in \text{Fix}(f)} \frac{1}{(1 - \lambda_x)(1 - \bar{\lambda}_x)^{-1}}$$

$$|(1 - \lambda_x)| \leq 2$$

$$\Rightarrow 2 \geq \sum_{x \in \text{Fix}(f)} \frac{1}{2 \cdot 2}$$

$$\Rightarrow |\text{Fix}(f)| \leq 8$$

In fact: For $(k, n) = 1$,

$$\text{Fix}(f^k) = \text{Fix}(f).$$

$$\Rightarrow (df^k)_x = \begin{pmatrix} \lambda_x^k & \\ & \lambda_x^{-k} \end{pmatrix}$$

$$\Rightarrow \sum_{x \in \text{Fix}(f)} \frac{1}{(1 - \lambda_x^k)(1 - \lambda_x^{-k})} = 2.$$

$$\Rightarrow \sum_{x \in \text{Fix}(f)} \underbrace{\frac{1}{|\{k \mid (k, n) = 1\}|} \sum_{(k, n) = 1} \frac{1}{(1 - \lambda_x^k)(1 - \lambda_x^{-k})}}_{\stackrel{(*)}{=} \frac{1}{12} \prod_{p|n} \left(1 + \frac{1}{p}\right)} = 2$$

□

Rank's How to prove (*)?

One has

$$\frac{1}{(1-\lambda^i)(1-\lambda^{-i})} = -\frac{1}{\lambda^i-1} - \frac{1}{(\lambda^i-1)^2}$$

root of $(x+1)^n - x^n = 0$.

$$\Rightarrow \sum_{i=1}^{n-1} \frac{1}{(1-\lambda^i)(1-\lambda^{-i})} = \frac{n^2-1}{12}$$

Now: $f(n) = \sum_{i=1}^{n-1} \frac{1}{(1-\lambda^i)(1-\lambda^{-i})}$ where λ n^{th} root of unity.

$$g(n) = \sum_{\substack{i=1 \\ (i,n)=1}}^{n-1} 1$$

$$\Rightarrow f(n) = \sum_{d|n} \sum_{\substack{i=1 \\ (i,n)=d}}^{n-1} \frac{1}{(1-\lambda^i)(1-\lambda^{-i})} = \sum_{d|n} g\left(\frac{n}{d}\right)$$

$i = dj \Rightarrow (d, \frac{n}{d}) = 1$

Mobius inv.

$$\Rightarrow g(n) = \sum_{d|n} \mu(d) f\left(\frac{n}{d}\right)$$

$$= \sum_{d|n} \mu(d) \left(\left(\frac{n}{d}\right)^2 - 1 \right) \frac{1}{12}$$

//

In fact

Prop: Let $f: S \rightarrow S$ symplectic of order $n < \infty$, ~~S -projective~~

Then $n \in \{1, 2, \dots, 8\}$ and we have the table

n	2	3	4	5	6	7	8
$ Fix(f) $	8	6	4	4	2	3	2
$g(S) \geq$	9	13	15	17	17	18	18
	8	12	14	16	16	18	18

Proof: (Skipped in lecture).

Use Lefschetz trace formula and

$H^*(S, \mathbb{R})^{<f>}$ has $\dim \geq 5$

(contains $1, p, \bar{\sigma}, \bar{\sigma}$, fixed Kohler
or symplectic
class
by averaging)

For $g(S) \geq 0$, use that $f \in T(S) := Pic(S)^\perp$

as the identity^(*), so

$$\dim H^*(S, \mathbb{C})^{<f>} \geq \dim(T(S)) + 3.$$

(*) If not the identity, then for eigenvector v , we must have

$$v \perp \bar{\sigma}, \text{ so } v \in H^{1,1}.$$

(Of course I need to argue over $\mathbb{Q}$, so v generalized Eigenvector)

$pf(S) \vee \neq 0$
for p-incl. pf
still ok.

Corollary: If f is a homeomorphism of finite order,
and $f^*: H^*(S, \mathbb{Z}) \rightarrow H^*(S, \mathbb{Z})$ is the identity, then $f = \text{id}$.

Proof:

Topological Lefschetz fixed point formula:

$$\sum_i (-1)^i \text{Tr}(f^* | H^i(S, \mathbb{Z})) = |\text{Fix}(f)|$$

$\Rightarrow$ If $f^* = \text{id}$, get $24 = |\text{Fix}(f)|$.

By Thm, $f = \text{id}_S$. ✓

Thm Let $f: S \rightarrow S$ any automorphism s.t. $f^* = \text{id}$.
Then $f = \text{id}$.

$$\Rightarrow \text{Aut}(S) \xrightarrow{f \mapsto f^*} \text{Aut } H^*(S)$$

Remark: This is wrong for many surfaces.
 $\mathbb{P}^2, \dots$

Proof:

$$G := \text{Ker}(\text{Aut}(S) \longrightarrow \text{Aut } H^*(S))$$

Need to show $G = \{1\}$.

~~Step 1~~ G is a ~~finite~~ ^{discrete} group.

~~Proof~~ (~~wrt natural topology given by $\text{Aut}(S) \subset \text{Hilb}(S \times S)$~~)

$\text{Aut}(S)$
① is a complex Lie group with tangent space

$$T_{\text{id}} \text{Aut}(S) = H^0(S, T_S) \cong H^0(S, \Omega_S) = 0$$

$\Rightarrow \text{Aut}(S)$, and hence G discrete.

Claim: G is finite.

Assume Claim holds. Let $f \in G$.

$\Rightarrow$ (i) f has finite order.

$$\text{(ii)} \quad f^* = \text{id} \Rightarrow f^* \sigma = \sigma$$

$\Rightarrow f$ symplectic

$\Rightarrow f = \text{id}$ by Car.

So it suffices to prove Claim.

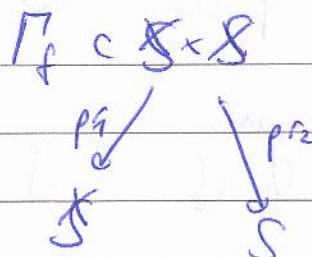
Proof of Claim

• If S is projective, let $H \in \text{Pic}(S)$ ample.

$$\Rightarrow f^*(H) = H \quad \forall f \in G.$$

$$\Rightarrow \chi(\Gamma_f, \mathcal{O}(p_1^*H + p_2^*H)^{\otimes m})$$

||



$$\chi(S, \mathcal{O}(mH + m f^*H))$$

$$= \chi(S, 2mH) = P_H(2m). \quad \text{indepd of } f.$$

$$\Rightarrow \text{Proj} \in \text{Hilb}(S \times S, P_H(2m)) \quad \text{projective.}$$

$$f \mapsto (\Gamma_f \subset S \times S)$$

$$\Rightarrow G \text{ finite}$$

• In general,

Thm (Yau) let X compact Kähler manifold with $c_1(X) = 0$.

For any Kähler class $\gamma \in H^{1,1}(X, \mathbb{R})$, there is a unique Ricci flat Kähler form ω s.t. $[\omega] = \gamma$.

$\exists \omega$ Kähler for $X \iff \exists ! g$ Riemannian metric ^{compatible with complex structure} s.t. $\omega = g(I \cdot, \cdot)$
 $d\omega = 0$

Let $\gamma \in H^{1,1}(X, \mathbb{R})$ Kähler.

ω Ricci flat for in γ with associated metric g

By uniqueness

$$f^* \omega = \omega \quad \text{for all } f \in G.$$

$$\Rightarrow f^* g = g$$

$$\Rightarrow G \subset O(M, g) \text{ compact}$$

↑ Riemannian mfd underlying X

□

Free Question: What are the largest finite groups that can act symplectically on a V_3 surface?

Thm (Mukai) TFAE for a finite group G :

- (1) G acts symplectically on a V_3 surface S .
- (2) G is a subgroup of the Mathieu group M_{23} and acts on $\Omega = \{1, \dots, 24\}$ with 75 orbits

M_{24} = finite simple ~~group~~ sporadic group.

$\curvearrowright \{1, \dots, 24\}$.

$M_{23} = \text{Stab}_{M_{24}}(24)$.

Largest G 's: order Associated V_3

$\text{PSL}_2(\mathbb{F}_7)$ 168 $V(X^3y + y^3z + z^3x + x^3 = 0)$ in $\mathbb{P}^3$

$M_{20} = 2^4 A_5$ 960 $X^4 + y^4 + z^4 + t^4 + 12xyzt = 0$ in $\mathbb{P}^3$

⋮