# Lecture 7

Advertisement:
Thursday, 4pm
Kilian Bönisch – Periods of Calabi-Yau mflds

## Back to K3 surfaces

**Def** A complex K3 surface $S$ is a complex compact mfld of dim 2

s.t. $\begin{cases} H^1(S, \mathcal{O}_S) = 0 \\ \Lambda^2 \Omega_S \cong \mathcal{O}_S \end{cases}$

Ex: Not every complex K3 surface is algebraic:

~~$T$ complex $\overset{\text{2-dim}}{\text{torus}}$ with $\mathrm{Pic}(T) = 0$.~~

$T$ complex 2-dim torus s.t. $NS(T) = \mathrm{Im}(\mathrm{Pic}(T) \xrightarrow{c_1} H_2(T, \mathbb{Z}))$
is zero.

($\rightsquigarrow$ $T$ non-algebraic).

$\Rightarrow$ $\mathrm{Km}(T)$ is a complex, non-algebraic K3 surface.

(If $\mathrm{Km}(T)$ algebraic, then it has an ample class

$\Rightarrow$ $\widehat{T}$ projective

$\Rightarrow$ $T$ projective.

you can contract the (-1)-curves in the cat. of projective varieties.)

$$\begin{array}{ccc} \widehat{T} & \to & \mathrm{Km}(T) \\ \downarrow & & \\ T & & \end{array}$$

Thm Every complex K3 surface is Kähler

Proof We use

Thm (Lamari) A compact complex surface $X$ is Kähler if and only if $b_1(X)$ is even.

Here $0 \longrightarrow \mathbb{Z} \longrightarrow \mathcal{O}_S \longrightarrow \mathcal{O}_S^\times \longrightarrow 0$.

$\Rightarrow \quad H^1(S,\mathbb{Z}) \hookrightarrow H^1(S,\mathcal{O}_S) = 0$.

$\Rightarrow \quad b_1(S) = 0$ $\square$

We have some results as before:

Hodge diamond , $H^i(S,\mathbb{Z})$ torsion-free.

$$\begin{matrix} & & 1 & & \\ & 0 & & 0 & \\ 1 & & 20 & & 1 \\ & 0 & & 0 & \\ & & 1 & & \end{matrix}$$

$H^2(S,\mathbb{Z}) \cong U^{\oplus 3} \oplus E_8(-1)^{\oplus 2}$

Prop Every deformation of a K3 surface is a K3 surface.

Proof Let

$$\begin{array}{ccc} X & \supset & X_0 = S \\ \downarrow \pi & & \downarrow \\ B & \ni & 0 \end{array}$$

deformation family of a K3 surface $S$.

~~We show that $\exists$ open $0 \in U \subset B$ s.t. $X_t$ K3 for all $t \in U$.~~

- $b_1(X_t) = b_1(X_0) = 0 \Rightarrow$ All $X_t$ are Kähler.

$\Rightarrow h^1(\mathcal{O}_{X_t}) + h^0(\omega_{X_t}) = b_1(X_t) = 0$

$\Rightarrow H^1(X_t, \mathcal{O}_{X_t}) = 0$.

__Lemma__ $S$ complex K3 surface. Then

$$H^0(S, T_S) = 0$$
$$H^1(S, T_S) \cong \mathbb{C}^{20}$$
$$H^2(S, T_S) = 0.$$

__Proof__ The symplectic form $\sigma : T_S \to \Omega_S$ is an iso. Hence

$$H^i(S, T_S) \cong H^i(S, \Omega_S) \qquad \square$$

__Cor__ Any complex K3 surface $S_0$ has a smooth, ~~locally universal~~ deformation family

$$X \to \mathrm{Def}(S_0)$$

which is locally universal for all of th

__Cor__ Any complex K3 surface $S_0$ has a smooth, locally universal deformation family

$$X \longrightarrow \mathrm{Def}(S_0)$$

where $\mathrm{Def}(S_0)$ is smooth of dim 20.

__Rmk:__ There is no explicit description of any $X \to \mathrm{Def}(S_0)$ known.

- $\chi(\mathcal{O}_{X_t}) = \overset{1}{\overset{\|}{h^0(\mathcal{O}_{X_t})}} + h^2(\mathcal{O}_{X_t})$

$\|$ since $\pi$ flat.

$\chi(\mathcal{O}_{X_0}) = 2$

$\Rightarrow h^0(\omega_{X_t}) = h^2(\mathcal{O}_{X_t}) = 1.$

- $\chi(\omega^*_{X_t}) = h^0(\omega^*_{X_t}) - h^1(\omega^*_{X_t}) + h^2(\omega^*_{X_t}).$

$\|$

$\chi(\omega^*_{X_0}) = \chi(\mathcal{O}_{X_0}) = 2.$

$t \longmapsto h^1(\omega^*_{X_t})$

$t \longmapsto h^2(\omega^*_{X_t})$ semi upper continuous.

$\Rightarrow \exists$ open $U \subset B$ s.t. $\forall t \in U$:

$\Rightarrow h^1(\omega^*_{X_t}) \leq h^1(\omega^*_{X_0}) = h^1(\omega^{\otimes 2}_{X_0}) = 0$ ~~for $t \in U$~~

$h^2(\omega^*_{X_t}) \leq h^2(\omega^2_0) = 1$ ~~for some open $U \subset B$.~~

For $t \in U$:

$\Rightarrow h^0(\omega^*_{X_t}) = 2 + \underbrace{h^1(\omega^*_{X_t})}_{0} - \underbrace{h^2(\omega^*_{X_t})}_{\leq 1} \geq 1$

$\Rightarrow \omega^*_{X_t}$ and $\omega_{X_t}$ both have sections, so $\omega_{X_t} \cong \mathcal{O}_{X_t}$.

$\Rightarrow \exists$ open $U \subset B$ s.t. $X_t$ K3 surface for $t \in U$.

But $h^0(\omega_{X_t}), h^0(\omega^*_{X_t})$ upper semicontinuous, so

$\tilde{B} = \{ t \mid h^0(\omega_{X_t}) \geq 1$ and $h^0(\omega^*_{X_t}) \geq 1 \}$

is ~~both~~ also closed

$\Rightarrow \tilde{B} = B \Rightarrow X_t$ K3 for all $t \in B$

$\square$

## Periods

$S$ complex K3 surface

$$H^2(S,\mathbb{C}) = \underset{\substack{\| \\ H^0(\Lambda^2\Omega_S) \\ \Downarrow \\ \sigma_S}}{H^{2,0}(S)} \oplus H^{1,1}(S) \oplus H^{0,2}(S).$$

$\sigma_S$ symplectic form.
unique up to scalar.

$\Rightarrow [\sigma_S] \in \mathbb{P}(H^2(S,\mathbb{C}))$ distinguished element.

Concretely, if $\alpha_1,\dots,\alpha_{22} \in H_2(S,\mathbb{Z})$ is ~~gi~~ a basis of cycles.
then $\sigma_S$ det. by

$$\left(\int_{\alpha_1}\sigma, \dots\dots, \int_{\alpha_{22}}\sigma\right) \in \mathbb{C}^{22}.$$

<u>Obvious relations of $\sigma_S$:</u>

<u>Lemma:</u> $\sigma_S \cdot \sigma_S = 0.$
$\sigma_S \cdot \overline{\sigma_S} > 0.$

where $\cdot$ is the intersection pairing on $H^2(S,\mathbb{C})$.

<u>Rmk</u> $X$ compact Kähler.

$$H^k(X,\mathbb{C}) = \bigoplus_{p+q=k} H^{p,q}(X,\mathbb{C})$$

Two descriptions of $H^{p,q}$:

(a) (alg) $H^{p,q}(X) = H^q(X,\Omega^p)$

(b) (Dolbeault) $H^{p,q}(X) = \cancel{\mathrm{Ker}(\bar\partial : A^{p,q}} \; H^q(A^{p,\bullet}(X),\bar\partial).$

$$= \frac{\mathrm{Ker}(\bar\partial : A^{p,q} \to A^{p,q+1})}{\mathrm{Im}(\bar\partial : A^{p,q-1} \to A^{p,q})}$$

$A^{p,q}(X) = C^\infty$-Sections of $\Lambda^p T^{1,0}(X)^* \otimes \Lambda^q T^{0,1}(X)^*$.

$= \{ \cancel{\alpha = \sum \alpha_{IJ}\, dz^I} \mid$ i.e. locally of the form (i-coords $z_1 \ldots z_n$)

$\alpha = \sum_{\substack{|I|=p \\ |J|=q}} \alpha_{IJ}\, dz_I \wedge d\bar{z}_J$

$=$

$$= \left\{ \alpha \in C^\infty(\Lambda^{p+q}(T_{\mathbb{R}}X \otimes \mathbb{C})) \;\middle|\; \begin{array}{l} \text{locally in coords } z_1, \ldots, z_n \\ \alpha = \sum_{\substack{|I|=p \\ |J|=q}} \alpha_{IJ}\, dz_I \wedge d\bar{z}_J \end{array} \right\}$$

$dz_I = \prod_{i \in I} dz_i$

$d\bar{z}_J = \prod_{j \in J} d\bar{z}_j$.

$\sigma \in H^{2,0}(S) \subset H^2(S, \mathbb{C})$

$\bar{\sigma} \in H^{0,2}(S) \subset H^2(S, \mathbb{C})$

<u>Proof</u> For $\alpha, \beta$ 2-forms on $S$,

$$\alpha \cdot \beta = \int_S \alpha \wedge \beta.$$

Let $z_1, z_2$ local coords of open $U \subset S$, $z_1 = x_1 + iy_1$, $z_2 = x_2 + iy_2$, <u>$x_i, y_i$ real.</u>

With $\sigma|_U = f(z_1, z_2)\, dz_1 \wedge dz_2$.

$$\Rightarrow \int\!\!\int \sigma|_U \wedge \bar{\sigma}|_U = \big(f(z_1,z_2)(dx_1 + i dy_1) \wedge (dx_2 + i dy_2)\big) \wedge \big(\bar{f}(z)(dx_1 - i dy_1)(dx_2 - i dy_2)\big)$$

$$= |f|^2 (-1)(-2i\, dx_1 \wedge dy_1) \wedge (-2i\, dx_2 \wedge dy_2)$$

$$= |f|^2 \cdot 4\, dx_1 \wedge dy_1 \wedge dx_2 \wedge dy_2.$$

$\Rightarrow$ Contribution of $U$ is $\int_U |f|^2 4 \cdot \underbrace{dx_1\, dy_1\, dx_2\, dy_2}_{\text{positive orientation}} > 0$ $\square$

## Period domain:

$L$ lattice (over $\mathbb{Z}$ or $\mathbb{C}$)

$$\mathcal{D}_L = \left\{ x \in \mathbb{P}(L \otimes \mathbb{C}) \;\middle|\; \begin{array}{l} x \cdot x = 0 \\ x \cdot \bar{x} > 0 \end{array} \right\}.$$ period domain.

If $L = H^2(S, \mathbb{C})$, we write $\mathcal{D}_S$ instead of $\mathcal{D}_{H^2(S,\mathbb{Z})}$.

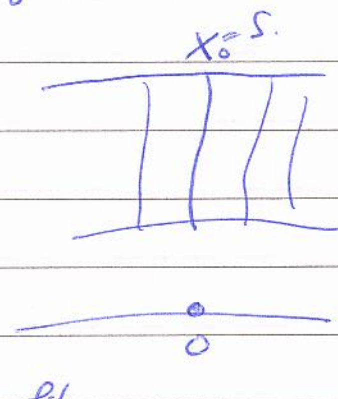
Cor: $[\sigma_S] \in \mathcal{D}_L$

## Local period map

Let ($\pi : X \to B$, $0 \in B$, $X_0 = S$) be deformation of $S$.

~~By~~ Ehressmann: $\pi$ differentiably loc. trivial $\Rightarrow$ $R^2_{\pi_X}\mathbb{Z}$ is local system.

Assume that $B$ is contractible.




~~Local system~~

$\Rightarrow R^2_{\pi_X}\mathbb{Z}$ is <u>trivial</u>.

(classified / given by representation of $\pi_1 \to GL_n(\mathbb{Z})$).

$\Rightarrow \alpha : R^2 \pi_X \mathbb{Z} \cong \underline{L}$, $\underline{L}$ constant sheaf with fiber $H^2(X_0, \mathbb{Z})$

$\Rightarrow \forall t \in B$, $\alpha_t : H^2(X_t, \mathbb{Z}) \longrightarrow H^2(X_0, \mathbb{Z})$ isometry

__Def__ The period map of $\pi$ is:

$$\lambda : B \longrightarrow \mathcal{D}_{X_0} \subset \mathbb{P}(H^2(X_0, \mathbb{C}))$$

$$t \longmapsto [\alpha_t(\sigma_{X_t})] = \alpha_t(H^{2,0}(X_t))$$

$$\sigma_{X_t} \in H^2(X_t, \mathbb{C}) \xrightarrow[\text{isometry}]{\alpha_t} H^2(X_0, \mathbb{C}).$$

__Thm:__ $\lambda$ is a holomorphic map.

# Local Period map

$\pi : X \to B$, $0 \in B$, $X_0 = S$ deformation family of $S$

Ehresmann: $R^2\pi_*\mathbb{Z}$ local system (locally constant sheaf)

Assume $B$ contractible.

$\Rightarrow$ $R^2\pi_*\mathbb{Z}$ trivial, i.e.

$$\alpha : R^2\pi_*\mathbb{Z} \xrightarrow{\sim} \underline{L}$$ constant sheaf with fibre $L = H^2(X_0, \mathbb{Z})$

$\Rightarrow$ $\forall t \in B : \alpha_t : H^2(X_t, \mathbb{Z}) \longrightarrow H^2(X_0, \mathbb{Z})$

$$\alpha_t : H^2(X_t, \mathbb{C}) \longrightarrow H^2(X_0, \mathbb{C})$$

$$\cup$$

$$H^{2,0}(X_t, \mathbb{C}) = \mathbb{C}\sigma_{X_t}$$

**Def** $\lambda : B \longrightarrow \mathcal{D}_{X_0} \subset \mathbb{P}(H^2(X_0, \mathbb{C}))$

$$t \longmapsto \alpha_t(H^{2,0}(X_t)) = [\alpha_t(\sigma_{X_t})]$$

Period map of family.

Thm $\lambda$ is holomorphic.

Lemma Let $\pi: X \to B$ smooth family of K3 surfaces.
There is a natural inclusion of holomorphic vector bundles.

$$\pi_* \Omega^2_{X/B} \longrightarrow (R^2\pi_* \mathbb{C}_X) \otimes \mathcal{O}_S$$

which on each fiber induces the inclusion $H^{2,0}(X_t) \subset H^2(X_t, \mathbb{C})$.

Proof
For $S$ fixed K3 surface:

$$H^{2,0}(S) \longleftrightarrow H^2(S, \mathbb{C})$$

can be obtained as follows:

$$\begin{array}{ccccccc} \Omega^2_S[-2] & \xrightarrow{\sim} & [\,0 & \longrightarrow & 0 & \longrightarrow & \Lambda^2\Omega_S\,] \\ \downarrow & & \downarrow & & \downarrow & & \downarrow \\ \mathbb{C}_S & \xrightarrow{\sim} & [\,\mathcal{O}_S & \xrightarrow{d} & \Omega_S & \longrightarrow & \Lambda^2\Omega_S\,] = \Omega^\bullet_S \end{array} \qquad \mathbb{C}\text{-linear}$$

$$\Rightarrow \quad H^2(\Omega^2_S[-2]) \longrightarrow H^2(S, \Omega^\bullet_S) = H^2(S, \mathbb{C})$$

$$\| $$

$$H^0(S, \Lambda^2\Omega_S).$$

[Why inclusion according to Hodge decomp? This is how you define Hodge filtration of $\mathbb{H}(X, \Omega^\bullet_X) = H^\bullet(X, \mathbb{C})$:

$F^p$ is image of $\mathbb{H}(X, \Omega^{\geq p}_X)$. ]

In relative case:

$$\begin{array}{ccccccc}
\Lambda^2\Omega_{X/B}[-2] \cong [ & 0 & \longrightarrow & 0 & \longrightarrow & \Lambda^2\Omega_{X/B} & ] \\
\downarrow & \downarrow & & \downarrow & & \downarrow & \quad f^{-1}\mathcal{O}_S\text{-linear.} \\
f^{-1}\mathcal{O}_S \simeq [ & \mathcal{O}_X & \xrightarrow{d} & \Omega_{X/B} & \xrightarrow{d} & \Lambda^2\Omega_{X/B} & ]
\end{array}$$

$$\Rightarrow \quad \begin{array}{ccc}
R^2\pi_{X*}(\Omega^2_{X/B}[-2]) & \longrightarrow & R^2\pi_{X*}(f^{-1}\mathcal{O}_S) \\
\wr\!\!\mid & & \wr\!\!\mid \text{ adjunction} \\
\pi_{X*}\Omega^2_{X/B} & & R^2\pi_{X*}(\mathbb{C}_X)\otimes\mathcal{O}_S
\end{array}$$

$\mathcal{O}_S$-linear. morphism of $\mathcal{O}_S$-modules.

By coh & base chge: $R^2\pi_{X*}(\Omega^2_{X/B}$ $\pi_{X*}(\Omega^2_{X/B})_t = H^0(X_t, \Omega^2_{X_t})$.

$$(R^2\pi_{X*}(\mathbb{C}_X)\otimes\mathcal{O}_S)_t = R^2\pi_{X*}(\mathbb{C}_X)_t$$
$$= H^2(X_t, \mathbb{C}). \quad \square$$

Pf Thm:

$$\pi_{X*}\Omega^2_{X/B} \xrightarrow{hol} (R^2\pi_{X*}\mathbb{Z})\otimes\mathcal{O}_S \xrightarrow[\otimes \mathrm{id}]{hol.} L\otimes\mathcal{O}_S.$$

$\Rightarrow$ $\lambda$ induced by hol. subbdles $\pi_{X*}\Omega^2_{X/B} \hookrightarrow L\otimes\mathcal{O}_S$.

$\Rightarrow$ hol. by univ. prop. of $\mathbb{P}(L)$

$\square$

# Lecture 8

## Derivative of $\lambda$

What is the derivative

$$(d\lambda)_0 : T_0 B \longrightarrow T_{\lambda(0)} \mathcal{D}_{x_0} \quad ?$$

Tangent space of ~~$\mathcal{D}_L$~~, ~~$\mathcal{D}_L$~~ ~~$\mathbb{C}$~~ $L$ vector space $/\mathbb{C}$ with non-deg. inner pairing.

- $\mathcal{D}_L = \left\{ x \in \mathbb{P}(L) \;\middle|\; \begin{matrix} x \cdot x = 0 \\ x \cdot \bar{x} > 0 \end{matrix} \right\}$

**Lemma** If $\ell \in \mathbb{P}(L)$, there is canonical isomorphism

$$T_\ell \mathbb{P}(L) \cong \mathrm{Hom}(\ell, L/\ell)$$

$$T_\ell \mathcal{D}_L \cong \mathrm{Hom}(\ell, \ell^\perp/\ell) \quad \text{if } \ell \in \mathcal{D}_L$$

**Proof** (1) For $v \in T_\ell \mathbb{P}(L)$, let

- We define $\varphi$ $T_\ell \mathbb{P}(L) \longrightarrow \mathrm{Hom}(\ell, L/\ell)$.
$v \longmapsto \varphi_v$

$$\gamma : \Delta \longrightarrow \mathbb{P}(L)$$
$\Delta \subset \mathbb{R}$

path such that $\gamma(0) = \ell$, $\dot{\gamma}(0) = v$.

For $x \in \ell$,

let $\tilde{\gamma} : \Delta \to L \setminus \{0\}$ lift of $\gamma(t)$ s.t.

$$\tilde{\gamma}(0) = x.$$

Define $\varphi_v(x) = \dot{\tilde{\gamma}}(0) \bmod \ell$.

$\Rightarrow$

If $\tilde{\gamma}, \tilde{\tilde{\gamma}}$ two lifts, then

$$\tilde{\gamma}(t) - \tilde{\tilde{\gamma}}(t) = \mu(t) \cdot \tilde{\gamma}(t), \quad \mu(0) = 0.$$

$$\Rightarrow \dot{\tilde{\gamma}}(0) - \dot{\tilde{\tilde{\gamma}}}(0) = \dot{\mu}(0)\, \tilde{\gamma}(0) \in \ell.$$

$\Rightarrow \varphi_v$ well defined

Check: $\varphi_v$ linear and injective, hence an isomorphism.

~~Observe: ⇒ d~~

(2) If $v \in T_\ell D_L$, can choose $\gamma: \Delta \to D_L$.

$$\Rightarrow \tilde{\gamma}(t) \cdot \tilde{\gamma}(t) = 0. \quad \left| \frac{d}{dt} \right|_{t=0}$$

$$\Rightarrow \dot{\tilde{\gamma}}(0) \cdot \tilde{\gamma}(0) = 0$$

$$\Rightarrow \varphi_v(x) \cdot x = 0$$

$$\Rightarrow \varphi_v(\ell) \perp \ell. \qquad \square$$

Rmk If $L = H^2(X_0, \mathbb{C})$, $\ell = H^{2,0}(X_0, \mathbb{C})$.

$$\ell^\perp = H^{2,0}(X_0) \oplus H^{1,1}(X_0)$$

$$\ell^\perp / \ell \cong H^{1,1}(X_0) = H^1(\Omega_{X_0}).$$

<u>Thm</u> We have the following commutative ~~triangle~~ diagram:

$$\begin{tikzcd}
T_0 B \arrow[r, "(d\lambda)_0"] \arrow[d, "KS_0"'] \arrow[dr, phantom, "\circlearrowleft"] & T_{\lambda(0)} \mathcal{D}_{X_0} \cong \mathrm{Hom}\left(H^{2,0}(X_0), H^{1,1}(X_0)\right) \\
H^1(X_0, T_{X_0}) \arrow[ur, "\varphi"] & 
\end{tikzcd}$$

$$\varphi_v = v \lrcorner \qquad v \mapsto \varphi_v$$

$$\varphi_v(\sigma) = v \lrcorner \sigma \quad , \quad H^1(X_0, T_{X_0}) \otimes H^0(X_0, \Omega^2_{X_0}) \longrightarrow H^1(X_0, \Omega^1)$$
$$(v \ , \ \sigma) \longmapsto v \lrcorner \sigma.$$

Rmk 1

<u>Rmk</u>: This is a special case of Griffith's transversality.

<u>Cor</u>: Let $S$ a K3 surface.

Let $\mathcal{S} \to \mathrm{Def}(S)$ universal def. family of $S$

Assume $\mathrm{Def}(S)$ contractible and $\mathcal{S} \to \mathrm{Def}(S)$ univ. for all of its fibers

Let $\lambda : \mathrm{Def}(S) \longrightarrow \mathcal{D}_S$ period map.

Then $\lambda$ is a local ~~hom~~ isomorphism.

<u>Proof</u> We need to check $(d\lambda)_t$ is an isomorphism for all $t$.

We have

Morphism $H^1(S, T_S) \longrightarrow H^1(S, \Omega_S)$

$$(d\lambda)_0 = \varphi \circ KS_0 \qquad v \longmapsto v \lrcorner \sigma.$$

is an isomorphism $\qquad$ induced by isom $\sigma : T_S \xrightarrow{\sim} \Omega_S$

$KS_0$ = isomorphism, $\cancel{\varphi = H^1\left(T_{X_0} \xrightarrow{\sim} T_{X_0} \otimes \Omega^2_{S_0} \xrightarrow{\sim} \Omega_{S_0}\right)}$ $\Rightarrow \varphi$ isom.

For any $t_0 \in B$, write $\lambda$ as composition.

$$\lambda : Def(S) \xrightarrow[\substack{\text{local}\\ \text{hom}\\ \text{at } t_0}]{\lambda_{t_0}} D_{X_{t_0}} \xrightarrow[\text{isom.}]{\alpha_{t_0}} D_{X_0}$$

$\Rightarrow (d\lambda)_{t_0}$ isomorphism. $\square$

<u>Rmk 1</u>: Pick $\sigma \in H^{2,0}(X_0)$:

$$\begin{tikzcd}
H^1(X_0, T_{X_0}) \arrow[r, "\varphi"] \arrow[ddr, "\tilde{\varphi} : v \mapsto v \lrcorner \sigma"'] & Hom(H^{2,0}(X_0), H^{1,1}(X_0)) \arrow[d, "\cong\ \sigma"] \\
 & H^{1,1}(X_0) \arrow[d, equal] \\
 & H^1(\Omega_X)
\end{tikzcd}$$

If $v = (v_i) \in \prod_i H^0(U_i, T_X)$ Čech cocycle, then $v \lrcorner \sigma = (v_i \lrcorner \sigma) \in \prod H^0(U_i, \Omega)$.

$\Rightarrow \quad \tilde{\varphi} = H^1(T_{X_0} \xrightarrow{\sim} \Omega_{X_0})$

$\Rightarrow \quad \tilde{\varphi}$ isomorphism.

<u>Proof of Thm</u>:

$\rightarrow$ Special case of Griffiths transversality (Voisin, Hodge theory I)

$\rightarrow$ Direct computation with forms (Kondo's book).

Proof

(1)

Assume $B = \Delta \subset \mathbb{C}_t$ small disc

$\frac{\partial}{\partial t} \in T_0 B$.

~~Choose cov~~

After shrinking $B$, $\exists$ open cover $\mathcal{U}_\alpha$ of $X$

open cover $U_\alpha$ of $X_0$

$$z_1^\alpha, z_2^\alpha, t$$

s.t.
$$\begin{tikzcd}
\mathcal{U}_\alpha \arrow[r, phantom, "\cong"] \arrow[d] & U_\alpha \times \Delta \arrow[dl] \\
\Delta &
\end{tikzcd}$$

$$z^\alpha = \varphi_{\alpha\beta}(z^\beta, t) \quad , \varphi_{\alpha\beta} \; \mathcal{U}_{\beta\alpha} \longrightarrow \mathcal{U}_{\alpha\beta}.$$

$$\theta_{\alpha\beta} = \frac{\partial \varphi_{\alpha\beta}}{\partial t} \Rightarrow \theta = (\theta_{\alpha\beta}) \in H^1(X_0, T_{X_0})$$

$$\overset{\text{def.}}{=} KS_0\left(\frac{\partial}{\partial t}\right)$$

line bdle

(2) Let $\sigma(t) \in H^0(\Delta, \pi_* \Omega^2_{X/B})$ nonvanishing section

$$\sigma(t) \in H^0(X_t, \Omega^2_{X_t})$$

$$\tilde{\lambda}(t) := \alpha_t(\sigma(t)) : \Delta \longrightarrow H^2(X_0, \mathbb{C}) \quad \text{lift of } \lambda$$

$$(d\lambda)_0\left(\frac{\partial}{\partial t_0}\right) : \sigma(0) \longmapsto \dot{\tilde{\lambda}}(0) \mod H^{2,0}(X_0)$$

Ehresmann:

$\exists$ diffeo $\varphi : X \xrightarrow{\sim} X_0 \times \Delta$.

(after shrinking $\Delta$) $\quad z_1^\alpha, z_2^\alpha, t \qquad x_1^\alpha, x_2^\alpha, t$

$$\begin{cases} z_1 = z_1(x_1, x_2, t) \\ z_2 = z_2(x_1, x_2, t) \end{cases}$$

- $z_i$ depend $C^\infty$-ly on $x_1, x_2, t$.
- Can assume: depend holomorphically on $t$.

$x_i = z_i(x_1, x_2, 0)$

$$\varphi_t^* \sigma(t) = f(x,t)\, dz_1 \wedge dz_2$$

$$= f(x,t)\left((\partial(z_1) + \bar{\partial}(z_1)) \wedge (\partial(z_2) + \bar{\partial}(z_2))\right)$$

$$\partial(g) := \sum \frac{\partial g_\alpha}{\partial x_i} dx_i$$

$$\frac{d}{dt}\Big|_{t=0} \varphi_t^* \sigma(t) = \dot{f}(x,0)\, dx_1 \wedge dx_2$$

$$+ f(x,t)\left(\partial\left(\frac{\partial z_1}{\partial t}\right) + \bar{\partial}\left(\frac{\partial z_1}{\partial t}\right)\right) \wedge dx_2$$

$$+ f\,(dx_1) \wedge \left(\partial\left(\frac{\partial z_2}{\partial t}\right) + \bar{\partial}\left(\frac{\partial z_2}{\partial t}\right)\right)$$

$$\overset{\text{mod } H^{2,0}}{=} f\,\bar{\partial}\left(\frac{\partial z_1}{\partial t}\right) \wedge dx_2 + f\, dx_1 \wedge \bar{\partial}\left(\frac{\partial z_2}{\partial t}\right)$$

# Dolbeault vs. Čech

$\mathcal{E}$ hol. vector bundle on $M$

$$H^{p,q}_{Dolbeault}(X,\mathcal{E}) := \frac{\mathrm{Ker}\left(\bar\partial : \mathcal{A}^{p,q}(\mathcal{E}) \longrightarrow \mathcal{A}^{p,q+1}(X,\mathcal{E})\right)}{\mathrm{Im}\left(\bar\partial : \mathcal{A}^{p,q-1}(\mathcal{E}) \longrightarrow \mathcal{A}^{p,q}(\mathcal{E})\right)}$$

$\mathcal{A}^{p,q}(\mathcal{E}) = C^\infty$-sections of $\Lambda^p T^{1,0} \otimes \Lambda^q T^{0,1} \otimes \mathcal{E}$

(locally of form $\alpha = \sum_{|I|=p,\ |J|=q} s_{IJ}\, dz_I \wedge d\bar z_J$

$s_{IJ}$ section of $\mathcal{E}$

$dz_I = dz_{i_1} \wedge \dots \wedge dz_{i_p}$, $I = (i_1, \dots, i_p)$ )

We have $H^{0,1}_{Dolbeault}(X,\mathcal{E}) \cong \check{H}^1(X,\mathcal{E})$:

$$\eta \longleftarrow\!\!| \ (\theta_{ij}) \in \Phi$$

- Write ~~locally~~ $\theta_{ij} = s_i - s_j$, $s_i \in C^\infty(U_i, \mathcal{E})$ (exist by partition of unity Griffiths–Harris)

$\Rightarrow 0 = \bar\partial(\theta_{ij}) = \bar\partial(s_i)|_{U_{ij}} - \bar\partial(s_j)|_{U_{ij}}$

$\Rightarrow \exists \eta \in \mathcal{A}^{0,1}(\mathcal{E})$ s.t. $\eta|_{U_i} = \bar\partial(s_i)$.

- Inverse map: For $\alpha \in H^{0,1}_{Dolbeault}$ find locally $\beta_i \in H^0(U_i, \mathcal{E})$

s.t. $\bar\partial(\beta_i) = \alpha|_{U_i}$ ⟵ exist by Poincaré $\bar\partial$-Lemma: If $\alpha \in \mathcal{A}^{p,q}(B_\varepsilon)$ s.t. $\bar\partial\alpha = 0$ $\Rightarrow \exists \beta$ s.t. $\bar\partial(\beta) = \alpha$.

$\Rightarrow \bar\partial(\beta_i - \beta_j) = 0$ on $U_{ij}$

$\Rightarrow \beta_i - \beta_j \in H^0(\mathcal{E}_{U_{ij}})$

In our case:

$$z^\alpha(x^\alpha,t) = \varphi_{\alpha\beta}\left(z^\beta(x^\beta,t),t\right).$$

$$\frac{\partial z^\alpha}{\partial t} = \frac{\partial \varphi_{\alpha\beta}}{\partial t} + d\varphi_{\alpha\beta}\,\frac{\partial z^\beta}{\partial t}$$

Set $\xi_\alpha := \dfrac{\partial z^\alpha}{\partial t}$

$$\Rightarrow\ \theta_{\alpha\beta} = \frac{\partial \varphi_{\alpha\beta}}{\partial t} = \xi_\alpha - \xi_\beta .$$

$\Rightarrow\ \theta \in H^1(X_0, T_{X_0})$ represented by $\eta = \bar\partial(\xi_\alpha)$

$$= \bar\partial\left(\frac{\partial z^\alpha}{\partial t}\right) = \bar\partial\left(\frac{\partial z_1^\alpha}{\partial t}\frac{\partial}{\partial z_1} + \frac{\partial z_2^\alpha}{\partial t}\frac{\partial}{\partial z_2}\right)$$

$$\overset{\text{at } t=0}{=} \bar\partial\left(\frac{\partial z_1^\alpha}{\partial t}\right)\frac{\partial}{\partial x_1} + \bar\partial\left(\frac{\partial z_2^\alpha}{\partial t}\right)\frac{\partial}{\partial x_2}$$

Finally, $\varphi(\theta)(\sigma(0)) = \theta \lrcorner\, \sigma(0)$

$$= \bar\partial\left(\frac{\partial z^\alpha}{\partial t}\right) \wedge f(x,0)\,dx_1 \wedge dx_2 .$$

$$= \bar\partial\left(\frac{\partial z_1^\alpha}{\partial t}\right) f(x,0)\,dx_2 - \bar\partial\left(\frac{\partial z_2^\alpha}{\partial t}\right) f(x,0) \wedge dx_1$$

$$= \frac{d}{dt}\Big|_{t=0}\left(\varphi_t^* \sigma(t)\right)$$

$\square$

# Global Period map

$$L = E_8(-1)^{\oplus 2} \oplus U^{\oplus 3}$$

**Def** A marked K3 surface is a pair $(S, \alpha)$ where $S$ is a complex K3 surface, $\alpha : H^2(S,\mathbb{Z}) \xrightarrow{\sim} L$ lattice isomorphism.

**Def** An isomorphism of marked K3 surfaces $(S, \alpha_S)$, $(\tilde{S}, \alpha_{\tilde{S}})$ is a marked isomorphism

$$\varphi : S \longrightarrow \tilde{S}$$

s.t.

$$\begin{tikzcd} H^2(S,\mathbb{Z}) \arrow[r, "\alpha_S"] & L \\ & H^2(\tilde{S},\mathbb{Z}) \arrow[ul, "\varphi^*"] \arrow[u, "\alpha_{\tilde{S}}"'] \end{tikzcd}$$

commutes.

$$\mathcal{M} \overset{\text{set}}{:=} \{\text{marked K3 surfaces}\} / \text{isomorphism}$$

Global Period Map :

$$\lambda : \mathcal{M} \longrightarrow \mathcal{D}_L$$

$$(S, \alpha) \longmapsto \alpha(H^{2,0}(S,\mathbb{C})) = [\alpha(\sigma_S)]$$

Goal:

~~(1) Construct~~ ~~/define~~ ~~complex structure on $\mathcal{M}$~~

(1): Give $\mathcal{M}$ structure of complex manifold (non-Hausdorff)

(2) Show $\mathcal{M}$ fine moduli space.