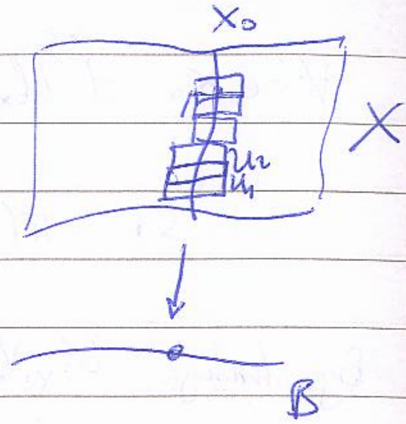
Lecture 6

Last time

$$\begin{array}{ccc} X & \supset & X_0 \\ \downarrow{\scriptstyle \pi \text{ proper, smooth}} & & \downarrow \\ B & & \ni 0 \end{array}$$

compact complex manifolds



$$KS_0 : T_0 B \longrightarrow H^1(X_0, T_{X_0})$$ Kodaira-Spencer map.

Recall: Thm of Ehresmann.

<u>Idea</u>: $\exists\ V \underset{\text{open}}{\subset} B$, $0 \in V$

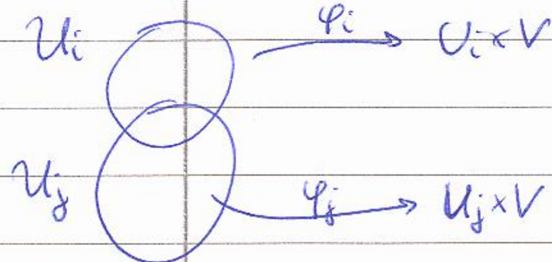
$\exists$ open cover $\pi^{-1}(V) = \mathcal{U}_1 \cup \dots \cup \mathcal{U}_\ell$

s.t. (i) $\mathcal{U}_i \overset{\varphi_i}{\cong} \underset{t}{V} \times U_i$, $\quad U_i = \mathcal{U}_i \cap X_0$

(ii) $\pi|_{\mathcal{U}_i} = pr_1 \circ \varphi_i$ under this isom.

$\Rightarrow$ X_t encoded by $\varphi_{ij}(-,t) = \varphi_i(\varphi_j^{-1}(z_j, t))$

$: \varphi_j(\mathcal{U}_i \cap \mathcal{U}_j) \longrightarrow \varphi_i(\mathcal{U}_i \cap \mathcal{U}_j)$



Proof of existence of such coordinates:

$\forall x \in X_0 \ \exists \overset{x\in}{\mathcal{U}_x} \underset{\text{open}}{\subset} X$ with coords $z_1, \dots, z_{m+n}$ and $\overset{0\in}{V_x} \underset{\text{open}}{\subset} B$

s.t. $\pi(z_1, \dots, z_{m+n}) = (z_1, \dots, z_m) \in V_x \cong B_\varepsilon^{\mathbb{C}^m}(r)$

By shrinking $\mathcal{U}_x, V_x$ to a polydisc, we can assume

$$\mathcal{U}_x \cong V_x \times \underbrace{(\mathcal{U}_x \cap X_0)}_{U_x}$$

Let $x_1, \dots, x_s \in X_0$ s.t. U_{x_i} cover X_0.

$U_i := U_{x_i}$

$V := V_{x_1} \cap \dots \cap V_{x_s}$

$\mathcal{U}_i = \pi|_{\mathcal{U}_{x_i}}^{-1}(V) \cong V \times U_i$

Since π is proper, $\pi((\mathcal{U}_1 \cup \dots \cup \mathcal{U}_s)^c)$ is closed, so

$\exists$ open $0 \in \tilde{V} \subset V$ s.t. $\pi^{-1}(\tilde{V}) \cap (\mathcal{U}_1 \cup \dots \cup \mathcal{U}_s)^c = \phi$.

$\Rightarrow$ Replace V by $\tilde{V}$.

$\mathcal{U}_i$ by $\mathcal{U}_i \cap \pi^{-1}(\tilde{V})$, U_i the same.

$\Rightarrow$ we get required chart

$\square$

Def

(i) Let $X \to B$ smooth family of compact manifolds, $f: \tilde{B} \to B$ morphism.

$$\begin{array}{c} f^*X = X \times_B B' = \{(x,b') \in X \times B' \mid \pi(x) = f(b')\} \\ \Big\downarrow \quad \text{pull back family.} \\ B' \end{array}$$

(ii) Let Y compact complex mfld.
A deformation family of Y is the triple
- $\pi: X \to B$ smooth family, B connected.
- $0 \in B$ base point.
- $\varphi: X_0 \xrightarrow{\sim} Y$ isomorphism.

We say X_t for $t \in B$ is a deformation of Y.

(iii) A morphism $(f,F): (\pi: X \to B, 0, \varphi) \longrightarrow (\tilde{\pi}: \tilde{X} \to \tilde{B}, 0, \tilde{\varphi})$
of deformation families of Y is morphism $f: B \to \tilde{B}$, $F: X \to \tilde{X}$ s.t.

$$\begin{cases} \tilde{\pi} \circ F = f \circ \pi \\ f(0) = \tilde{0} \\ \tilde{\varphi} \circ F|_{X_0} = \varphi \end{cases} \qquad \begin{array}{ccc} X & \xrightarrow{F} & \tilde{X} \\ \downarrow & & \downarrow \\ B & \xrightarrow{f} & \tilde{B} \end{array}$$

A deformation family $(\pi : X \to B, 0, \varphi)$ of Y is locally complete if for any def. family $(\tilde{\pi} : \tilde{X} \to \tilde{B}, \tilde{0}, \tilde{\varphi})$ of Y there exist an open nbhd $\tilde{0} \in U \subset \tilde{B}$ and morphism $f : U \to B$ s.t.

$$\left(\underset{\tilde{\pi}^{-1}(U)}{\underset{\shortparallel}{\tilde{X}_U}} \xrightarrow{\tilde{\pi}} U, \tilde{0}, \tilde{\varphi} \right) \text{ is isomorphic over } U \text{ to } \left(\begin{array}{c} f^*X \\ \downarrow f^*\pi \\ U \end{array}, \tilde{0}, \varphi \right)$$

$\pi = (\pi : X \to B, 0, \varphi)$ is locally universal if f is (locally) unique, i.e. given two $f_1 : U_1 \to B$, $f_2 : U_2 \to B$ as above then

$$f_1|_U = f_2|_U \quad \text{for some open } U \subset U_1 \cap U_2$$

We do not require that the isom between $\tilde{X}_U$ and f^*X is unique, but this is true if $\mathrm{Aut}(X_0)$ is discrete, since the isomorphism over 0 is fixed.

π is versal if $(df)_0$ is unique.

Case $n=3, d=4$:

$$\begin{array}{ccccccccc} 0 \to & H^0(T_{\mathbb{P}^4}) & \longrightarrow & T_{[X]}\mathbb{P}(V) & \longrightarrow & H^1(X, T_X) & \longrightarrow & \mathbb{C} & \to 0 \\ & \cong & & \cong & & \cong & & & \\ & \mathbb{C}^{15} & & \mathbb{C}^{34} & & \mathbb{C}^{20} & & & \end{array}$$

Kodaira: "This seemed very strange at that time".

<u>Def</u> (i) The smooth family $\pi: X \to B$ is complete at $0 \in B$ if for any smooth family $X' \to B'$ and $0' \in B'$ with (given isom.) $X'_{0'} \cong X_0$, there exist $0' \in B'' \subset B'$ and mphism $f: B'' \to B$ s.t.

$X'|_{B''} \to B''$ is the pullback of $X \to B$ along f:

$$\begin{tikzcd} X' \arrow[d] & \supset & X'|_{B''} \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & X \arrow[d] \\ B' & \supset & B'' \arrow[r, "f"] & B \end{tikzcd}$$

(ii) π is moreover (locally) universal at 0 if f is unique, i.e. for any two choices (B'', f) and $(\tilde{B}'', \tilde{f})$ there exists B''' s.t. $f|_{B'''} = \tilde{f}|_{B'''}$.

(iii) π is versal if $(df)_{0'}$ is unique.

<u>Def</u>

Thm (Kodaira–Spencer) Let $X \to B$ smooth family.

If KS is surjective at $0 \in B$, then it is complete at $0 \in B$.

$(\pi : X \to B, 0, \mathrm{id} : X_0 \to X_0)$

Cor If $X \to B$ is complete, let $0 \in B' \subset B$ submfld s.t.

$$KS : T_0 B' \longrightarrow H^1(X_0, T_{X_0})$$

isomphism.

Then $X_{B'} = \pi^{-1}(B') \xrightarrow{\pi'} B'$ is versal.

Proof: By thm, $\pi' : X_{B'} \longrightarrow B'$ is complete.

If we have fiber diagram,

$$\begin{array}{ccc}
\tilde{X} & \longrightarrow & X_{B'} \\
\downarrow & & \downarrow \\
\tilde{B} & \xrightarrow{f} & B' \\
\cup & & \cup \\
\tilde{0} & \longrightarrow & 0
\end{array}$$

we have

$$\begin{array}{ccc}
T_{\tilde{B},0} & & \\
 & \searrow^{KS_0} & \\
(df)_0 \downarrow & \circlearrowleft & H^1(X_0, T_{X_0}) \\
 & \nearrow^{\sim}_{KS_{B'}} & \\
T_{B',0} & &
\end{array}$$

$\Rightarrow (df)_0$ unique $\square$

$\Rightarrow$ We can always construct a versal family out of a complete one.

Theorem of existence (Kodaira - Spencer)

Let Y be a compact complex mfld with $H^2(Y, T_Y) = 0$.
Then there is a deformation family

$$(\pi : X \to B,\ 0 \in B,\ \varphi : X_0 \xrightarrow{\sim} Y)$$

where $KS(0) : T_{B,0} \longrightarrow H^1(X_0, T_{X_0})$ is an isomorphism

Moreover:

(a) If $h^1(X_t, T_{X_t})$ is constant for all $t \in B$, then π is versal for all of its fibers

(b) If $H^0(Y, T_Y) = 0$, then π is universal.

Rmks. (a) We write $\mathrm{Def}(Y) := B$ and call

$$\left(\begin{array}{c} X \\ \downarrow \\ \mathrm{Def}(Y) \end{array},\ 0 \in \mathrm{Def}(Y),\ \varphi : X_0 \xrightarrow{\sim} Y \right)$$

a (uni)versal deformation family of Y.

Warning: $X \to \mathrm{Def}(Y)$ is not unique, but given two universal def. families $\mathrm{Def}(Y)$, $\widetilde{\mathrm{Def}}(Y)$ of Y they are isomorphic on an open subset ~~of~~ :

Precisely given $\left(\begin{array}{c} X \\ \downarrow \\ \mathrm{Def}(Y) \end{array}, 0, \varphi \right)$, $\left(\begin{array}{c} \tilde{X} \\ \downarrow \\ \widetilde{\mathrm{Def}}(Y) \end{array}, \tilde{0}, \tilde{\varphi} \right)$ there ~~is~~ are

opens $U \subset \mathrm{Def}(Y)$ and $V \subset \widetilde{\mathrm{Def}}(Y)$ s.t.

$$\left(\begin{array}{c} X|_U \\ \downarrow \\ U \end{array}, 0, \varphi \right) \overset{(f,F)}{\cong} \left(\begin{array}{c} \tilde{X} \\ \downarrow \\ \widetilde{\mathrm{Def}}(Y) \end{array}, \tilde{0}, \tilde{\varphi} \right) \quad \text{with } f \text{ unique.}$$

~~Applying~~

(b) A versal family exists for any compact mfld Y if we allow B to be singular.

One still has

$$T_0 \mathrm{Def}(Y) \cong H^1(Y, T_Y).$$

(c) Condition $H^2(Y, T_Y)$ is not necessary for B to be smooth

Famous example: • Calabi-Yau manifolds.

• degree d hypersurfaces in $\mathbb{P}^3$ for $d \geq 6$.

(d) (Later) In situation of remark (a) ~~assume~~ let $H^2(Y, T_Y) = H^0(Y, T_Y) = 0$ but assume $g \in \mathrm{Aut}(Y)$. Consider

$$\left(\begin{array}{c} X \\ \downarrow \\ \mathrm{Def}(Y), \end{array} 0, \varphi \right), \quad \left(\begin{array}{c} X \\ \downarrow \\ \mathrm{Def}(Y), \end{array} 0, g \circ \varphi \right)$$

~~$\Rightarrow \exists U, U' \subset \mathrm{Def}(Y)$ open, $(f, F): (X_U \to U, 0, \varphi) \xrightarrow{\sim} (X_{U'} \to U', 0, g\circ\varphi)$~~

After shrinking $\mathrm{Def}(Y)$, there exist

$$(f, F): \left(\begin{array}{c} X \\ \downarrow \\ \mathrm{Def}(Y) \end{array}, 0, \varphi \right) \xrightarrow{\sim} \left(\begin{array}{c} X \\ \downarrow \\ \mathrm{Def}(Y) \end{array}, 0, g \circ \varphi \right)$$

$$(\Leftrightarrow) \quad \begin{array}{ccc} X & \xrightarrow{F} & X \\ \downarrow & & \downarrow \\ \mathrm{Def}(Y) & \xrightarrow{f} & \mathrm{Def}(Y) \end{array} \quad \text{and } g \circ \varphi \circ F|_{X_0} = \varphi$$

$$\Rightarrow F|_{X_0} = \varphi^{-1} g \varphi.$$

$t \in \mathrm{Def}(Y)$:

$F_t: X_t \to X_{f(t)}$ isomorphism. Upshot $\mathrm{Aut}(Y)$ acts on $\mathrm{Def}(Y)$.

$\rightsquigarrow$ Fibers over $\{t, f(t)\}$ isomorphic.

Existence:

Thm Let Y ~~cpt~~ compact complex mfld s.t. $H^2(Y, T_Y) = 0$.
Then there is a ~~smooth~~ deformation family

$$\pi = (\pi : X \longrightarrow B,\quad 0 \in B,\ \varphi : X_0 \xrightarrow{\sim} Y)$$

where $KS(0) : T_{B,0} \longrightarrow H^1(X_0, T_{X_0})$ is an isomorphism.

(In particular π is versal at 0 and $Def(Y) := B$ is smooth)

Moreover:

(a) If $h^1(X_t, T_{X_t})$ is constant for all $t \in B$, then ~~the family~~ π is versal for all of its fibers

(b) If $H^0(Y, T_Y) = 0$, then π is universal at 0.

~~(c) $Def(Y)$ is not unique, but given two $Def(Y)$, $\widetilde{Def}(Y)$, by versality, $\exists\, U \subset Def(Y)$ open, $0 \in U$, s.t. $f : U \to \widetilde{Def}(Y)$ whose $(df)_0$ invertible~~ the fibers $\mathcal{Y} \to Def(Y)$ are isom. $\Rightarrow$ so ~~Kuranishi's~~ unique on an open subset $0 \in U$.

Rmks to (2 pages later)

(1) A versal family exists for any X if we allow the base B to be singular.

(B can be constructed as fiber of 0 of the Kuranishi map

$$\begin{aligned} H^1(X, T_X) &\longrightarrow H^2(X, T_X) \\ \theta &\longmapsto [\theta, \theta] \end{aligned}$$

)

(2) Condition $H^2(X, T_X) = 0$ is not necessary for B to be smooth.

Ex: Calabi-Yau manifolds have smooth deformation spaces even if $H^2(X, T_X) \cong H^{n-2}(X, \Omega_X^{n-1}) \neq 0$ in genl

Ex 2: degree d hypersurfaces in $\mathbb{P}^3$ for $d \geq 6$.

Examples:

(1) Complex tori:

$$M = B \times \mathbb{C}^g / \mathbb{Z}^{2g}$$

π is versal, but several (many) fibers of π can be isomorphic

$$\downarrow \pi$$

$$B = \{ \omega \in M_{g\times g}(\mathbb{C}) \mid \det(\mathrm{Im}(\omega)) > 0 \}$$

$$M_\omega = \mathbb{C}^g / \Lambda_\omega, \quad \Lambda_\omega = \mathrm{Span}(e_1, \dots, e_g, \omega_1, \dots, \omega_g).$$

$$M_\omega \cong M_{\omega'} \iff \omega' = (a\omega + b)(c\omega + d)^{-1}$$

for $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in M_{2g\times 2g}(\mathbb{Z})$.

with $\det = 1$

$$\Gamma = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in M_{2g\times 2g}(\mathbb{Z}) \;\middle|\; \det\begin{pmatrix} a & b \\ c & d \end{pmatrix} = 1 \right\}.$$

$\Gamma \curvearrowright B$ not properly discontinuous

$\Rightarrow$ B/Γ not complex mfld, not Hausdorff.

$\left(\forall s \in B \; \exists s \in U \text{ s.t. } \exists t \in U \text{ with } \Gamma \cdot t \cap U \text{ infinite} \right.$

$\left. \Rightarrow gU \cap U \neq \emptyset \;\; \forall \text{ nbhds } U \text{ of } s \right)$

In fact, $\forall s \in B \; \forall U$ open there are infinitely many points $t_1, t_2, \dots \in U$ s.t. $M_{t_1} \cong M_{t_i} \; \forall i$

Properly discontinuous: $G \curvearrowright X$ p.d. if

- $\forall x \in X \; \exists U_x \subset X$ s.t. $gU_x \cap U_x = \emptyset$ for $g \neq 1$.
- $\forall x, y \in X$ s.t. $x \notin Gy$, $\exists x \in U_x$, $y \in U_y$ s.t. $U_x \cap gU_y = \emptyset \;\; \forall g$.

(2)

$$\underset{x_0,x_1,x_2}{\mathbb{P}^2} \times \underset{y_0,y_1}{\mathbb{P}^1} \times \mathbb{C}_t \supset S = V(x_1 y_0^2 - x_0 y_1^2 - t x_2 y_0 y_1)$$

Consider $\pi = pr_3 : S \longrightarrow \mathbb{C}_t$.

For $t \neq 0$: $S_t \cong \mathbb{P}^1 \times \mathbb{P}^1$, $H^1(S_t, T_{S_t}) = 0$,

(*) For $t = 0$: $S_0 = \mathbb{P}(\mathcal{O}(2) \oplus \mathcal{O})$, $H^1(S_0, T_{S_0}) = \mathbb{C}$, ~~KS~~
$= \mathbb{F}_2$

$$\begin{array}{c} S_0 \\ \big\downarrow p \\ \mathbb{P}^1 \end{array} \qquad p_* T_p = \mathcal{O}(-2) \oplus \mathcal{O} \oplus \mathcal{O}(2)$$

One can check:

$$KS(t) = \begin{cases} \text{isom.} & \text{for } t = 0 \\ 0 & \text{for } t \neq 0 \end{cases}$$

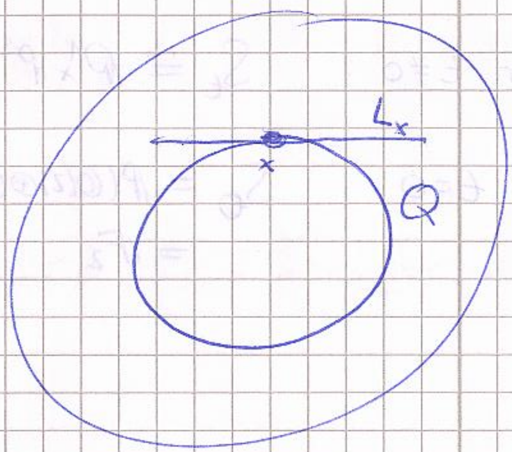
$\Rightarrow \pi$ versal at 0, but not elsewhere.

(*) Consider projection onto $\mathbb{P}^2$. : $\tilde{p} : S_t \longrightarrow \mathbb{P}^2$

For $t \neq 0$, projection is double cover branched along a smooth conic.

Any such is isomorphic to $\mathbb{P}^1 \times \mathbb{P}^1$:

the two rulings are the preimages of the tangents to the conic $Q \cong \mathbb{P}^1$



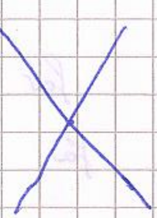
Since $\mathbb{P}^1$ simply connected, we the local system of

$\{L_x', L_x''\} \longrightarrow \{x\}$

is trivial, so we get two sets of $\mathbb{P}^1$'s of lines.

$\tilde{p}^{-1}(L_x) = L_x' \cup L_x''$

$L_x' \cdot L_x''$



For $t=0$, the conic degenerates to S_0 is the small resolution of the sing. double cover.

$\left(\tilde{p} : S_0 \longrightarrow \mathbb{P}^2 \text{ contracts a } \mathbb{P}^1 \text{ to } [0,0,1] \in \mathbb{P}^2 \right)$

Example (3)

$$\begin{array}{ccc} E_\tau & \subset & \mathbb{H}\times\mathbb{C}/\mathbb{Z}^2 \\ \downarrow & & \downarrow \\ \tau & \in & \mathbb{H} \end{array} \qquad (\tau, z) \sim (\tau, z + m\tau + n).$$

$$\mathrm{Aut}(E_i)/_{\text{translations}} = \langle i \rangle \quad \curvearrowleft \text{ multiplication by } i$$

Let $\quad f(\tau) = -1/\tau$

$\qquad F(\tau, z) = \left(-1/\tau, +\frac{z}{\tau}\right).$

Then (f, F) defines map

$$\left(\begin{array}{c} \mathbb{H}\times\mathbb{C}/\mathbb{Z}^2 \\ \downarrow \\ \mathbb{H} \end{array}, 0, \mathrm{id} \right) \longrightarrow \left(\begin{array}{c} \mathbb{H}\times\mathbb{C}/\mathbb{Z}^2 \\ \downarrow \\ \mathbb{H} \end{array}, 0, \iota\circ\mathrm{id} \right)$$

In fact, $SL_2(\mathbb{Z})$ acts on $\mathbb{H}\times\mathbb{C}/\mathbb{Z}^2$ by

$$\begin{cases} \tau \longmapsto \dfrac{a\tau+b}{c\tau+d} \\ z \longmapsto \dfrac{z}{c\tau+d} \end{cases}$$