Lecture 5
Classification of algebraic surfaces

S sm. proj surface.
Basic invariants:

$$\text{Irregularity } q = h^1(\mathcal{O}_S)$$
$$\text{Geom. genus } p_g = h^2(\mathcal{O}_S) = h^0(\mathcal{O}(K))$$
$$\text{"Plurigenera" } P_m(S) = h^0(\mathcal{O}(mK))$$

Def (Kodaira dimension) birational invariant

X sm. proj variety / compact complex mfld.

$\kappa(X)$ = maximal dimension of images $\phi_{mK_X}(X)$

map associated to $|mK_X|$

... = Asymptotic behavior of $P_m(X) = \dim H^0(\mathcal{O}(mK_X))$, i.e.
$$P_m(X) = m^{\kappa(X)} + \ldots$$

$$= \text{tr.deg}\left(\bigoplus_{m \geq 0} H^0(\mathcal{O}(mK_X))\right).$$

Convention: If $P_m = 0 \ \forall m$, $\kappa(X) := -\infty$.

Class. of Curves: $\kappa(C) \in \{-\infty, 0, 1\}$

$\mathbb{P}^1$ ell. curves $g \geq 2$.

Surfaces: $\kappa(S) \in \{-\infty, 0, 1, 2\}$.

Def A surface S is minimal if it does not contain a divisor $E \cong \mathbb{P}^1 \subset S$ with $E \cdot E = -1$.

Equivalently: Every birational morphism $S \to S'$ is an isomorphism.

↳ This follows since a birational morphism of surfaces is the composition of blowups at points, so at each step we contract some (-1) curve E.

Thm

(a) Every surface S has a birational morphism to a minimal one S_{min}.

(b) If S is not ruled (i.e. birational to $C \times \mathbb{P}^1$ for some curve C), then S_{min} is unique.

$\Rightarrow$ Can restrict to minimal surfaces.

Proof of (*): Beauville Thm V.19:

$$\begin{tikzcd} & \tilde{S} \arrow[dl, dotted, "\varepsilon_n"'] \arrow[ddr, "f"] & \\ \cdot \arrow[d, "\varepsilon_1"'] & & \\ S' & & S \end{tikzcd}$$

$E \subset \tilde{S}$ exceptional divisor of ε_n

$C = f(E)$ curve.

$K_S \cdot C \leq -2 \Rightarrow C \cdot C \geq 0$ (genus formula).

$\Rightarrow P_n = \dim H^0(S, nK_S) = 0 \ \forall n$

$\Rightarrow Alb(S)$ has dim ≤ 1

(otherwise pullback of 2-form gives a 2-form).

$\Rightarrow S \to Alb(S)$ with fiber $\mathbb{P}^1 = C$.

Classification of Surfaces

$\kappa(S) = -\infty$: Rational surfaces, Ruled surfaces.

minimal: $\mathbb{P}^2$, $\mathbb{F}(n)$, $n \neq 1$.

minimal: $\mathbb{P}_C(E)$, $\begin{array}{c} E \\ \downarrow \\ C \end{array}$ rank 2 vector bundle on curve C, $g(C) \geq 1$.

$\mathbb{F}(n) = \mathbb{P}(\mathcal{O} \oplus \mathcal{O}(n))$
Hirzebruch surface

$\kappa(S) = 0$:

Let S minimal with $\kappa(S) = 0$.
Then we have one of the following:

(1) $p_g = 0$, $q = 0$: Then $2K_S = 0$ and S is an Enriques surface.
(2) $p_g = 0$, $q = 1$: S Bielliptic
(3) $p_g = 1$, $q = 0$: S K3 surface.
(4) $p_g = 1$, $q = 2$: S Abelian surface.

Ex.

Enriques Surfaces:

Y $p_g = 0 \Rightarrow H^0(Y, \omega_Y) = 0 \Rightarrow \omega_Y \neq \mathcal{O}_Y$.

$2K_Y = 0 \Rightarrow \omega_Y^{\otimes 2} = \mathcal{O}_Y$.

$X :=$ Double cover of Y wrt ω_Y.

$$\begin{array}{c} X \\ \big\downarrow \pi \text{ etale} \\ Y \end{array}$$

$$\begin{cases} \pi^* \omega_Y = \mathcal{O}(\text{Ramification div}) = \mathcal{O}. \quad \Rightarrow p_g(X) = 1 \\ \chi_{top}(X) = 2 \cdot \chi_{top}(Y) \end{cases}$$

$\Rightarrow \chi(\mathcal{O}_X) = 2 \cdot \chi(\mathcal{O}_Y) = 2(1 - q + p_g) = 2.$

$\Rightarrow \Lambda^2 \Omega_X \cong \mathcal{O}_X, \quad H^1(\mathcal{O}_X) = 0$

$\Rightarrow X$ K3 surface.

Conversely, if X K3 surface with $\tau \in \mathrm{Aut}(X)$ fixpoint free involution, then $Y = X/\langle \tau \rangle$ Enriques.

(Exercise).

$\ulcorner$ $H^0(Y, \Omega_Y) \subset H^0(X, \Omega_X) = 0. \Rightarrow q = 0.$

$\chi(\mathcal{O}_Y) = \frac{1}{2}\chi(\mathcal{O}_X) = 1 \quad \Rightarrow p_g = 0$ $\lrcorner$

Example

(1) S branched along $D = (4,4)$ divisor.

$$\begin{array}{c} \tilde{\tau} \circlearrowleft S \\ \downarrow 2:1 \\ \tau \circlearrowleft \mathbb{P}^1 \times \mathbb{P}^1 \end{array}$$

$\tau : ([x_0, x_1], [y_0, y_1]) \longmapsto ([x_0, -x_1], [y_0, -y_1])$

Assume (i) D does not ~~preserve~~ contain fix points of τ

(ii) D is τ invariant (and smooth)

$\Longrightarrow$ $S/\langle \tilde{\tau} \rangle$ Enriques.

(2) $S = Km(E \times F)$

$a \in (E \times F)[2] \rightsquigarrow t_a$

$\sigma = id_E \times (-id_F)$

$\sigma \circ t_a \circlearrowleft Km(E \times F)$ fixed point free /

Bielliptic surface

Def A surface S is bielliptic if

$$S = (E \times F)/G$$

where
- E, F elliptic curves
- $G \circlearrowleft E$ faithfully by translations
- $G \circlearrowleft F$ " s.t. $F/G \cong \mathbb{P}^1$.

There are 7 cases: $G \circlearrowleft E$ by translation $\rightsquigarrow G \subset \mathbb{Z}/a\mathbb{Z} \oplus \mathbb{Z}/b\mathbb{Z}$

e.g. $G = \mathbb{Z}/2$ $G \circlearrowleft E$ by translation by 2-torsion pt.

$G \circlearrowleft F$ by (-1). //

$\kappa(S) = 1$: S minimal $\Rightarrow$ S elliptic surface with base curve of $g \geq 2$.

$\kappa(S) = 2$: General type (Wilderness)

§ Deformations of complex manifolds

Def A smooth family of compact complex manifolds consists of

(a) X, B ~~...~~ compact, complex connected mflds

(b) $\pi : X \longrightarrow B$ proper, smooth morphism.

proper = preimage of compact subset is compact.

smooth = Jacobian has full rank at each point.

$$\begin{tikzcd} X \arrow[d] & \supset & X_0 \arrow[d] \\ B & \ni & 0 \end{tikzcd}$$

We say $X_t = \pi^{-1}(t)$ is a deformation of X_0.

Ex 1 (Hypersurfaces in $\mathbb{P}^n$)

$V = H^0(\mathbb{P}^n, \mathcal{O}(d))$

$f_0, \dots, f_N \in V$ basis.

$$\underset{y_0 \dots y_N}{\mathbb{P}(V)} \times \underset{x_0 \dots x_n}{\mathbb{P}^n} \supset Z = \left\{ [y_0, \dots, y_N, x_0, \dots, x_n] \;\middle|\; \sum y_i f_i(x) = 0 \right\}.$$

universal hypersurface.

$U \subset \mathbb{P}(V)$ open locus of smooth hypersurfaces

$Z_U = \pi^{-1}(U)$

$\rightsquigarrow \pi : Z_U \longrightarrow U$ smooth family

$$\begin{tikzcd} Z_U \arrow[d] & \subset & Z \arrow[d, "\pi"] \arrow[r, hook] & \mathbb{P}(V) \times \mathbb{P}^n \\ U & \subset & \mathbb{P}(V) & \end{tikzcd}$$

Ex 2 (Complex torus)

Elliptic curve $E_\tau = \mathbb{C}/\mathbb{Z}+\tau\mathbb{Z}$, $\tau \in \mathbb{C}$
$Im(\tau) > 0$

$\tau \in \mathbb{H} = \{ z \in \mathbb{C} \mid Im(z) > 0 \}$

Family version:

$$\begin{array}{ccc} E_\tau & \subset & X = \mathbb{H}\times\mathbb{C}/\mathbb{Z}^2 \\ \downarrow & & \downarrow \pi \\ \tau & \in & \mathbb{H} \end{array} \qquad , \quad (\tau, z) \sim (\tau, z + m\tau + n)$$

Complex torus:

$$T_\omega = \mathbb{C}^g / \Lambda_\omega \ , \quad \Lambda_\omega = \mathrm{Span}(\underbrace{e_1, \dots, e_g}_{\text{std basis}}, \omega_1, \dots, \omega_g)$$

s.t. $e_1, \dots, e_g, \omega_1, \dots, \omega_g$ linearly independent over $\mathbb{R}$

$\iff Im(\omega_1, \dots, \omega_g)$ non-deg.

$\iff \det(\quad) \neq 0$

$$B := \{ W = (\omega_1, \dots, \omega_g) \in M_{g\times g}(\mathbb{C}) \mid \det(Im(\omega)) > 0 \}$$

$$\begin{array}{ccc} T_\omega & \subset & B \times \mathbb{C}^g / \mathbb{Z}^{2g} \\ \downarrow & & \downarrow \\ \omega & \in & B \end{array} \qquad , \quad (\omega, z) \sim (\omega, z + \sum m_i e_i + \sum n_i \omega_i)$$

Problem: Understand all deformations of a given compact mfld X_0.

Topologically:

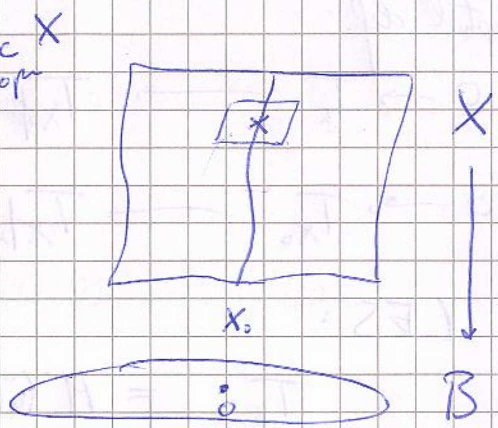
Thm (Ehresmann): Any smooth family $\pi: X \to B$ is differentially locally trivial:

i.e. $\forall b \in B \; \exists_{b \in} U_b \subset B$ s.t. $\pi^{-1}(U_b) \cong U_b \times X_b$.

Kodaira - Spencer map.

$\forall x \in X_0 \; \exists_{0 \in} V_x \subset B$ ~~stad~~ $x \in U_x \subset X_0$, $x_0 \in \mathcal{U}_x \subset_{op} X$ s.t.

~~$\pi^{-1}(V_x) \cong V_x \times U_x$~~ s.t. $\mathcal{U}_x \cong U_x \times V_x$



~~Let~~ X_0 compact, so there exist

$x_1, \dots, x_n \in X$ s.t. U_{x_i} cover X_0.

Replace V_x by $V = V_{x_1} \cap \dots \cap V_{x_n}$.

$$\Rightarrow \underbrace{\pi^{-1}(V)}_{\cup \mathcal{U}_i} \cong \bigcup V \times U_i \quad , \; U_i = U_{x_i}$$

$$= \coprod_{i=1}^{n} \underset{t}{V} \times \underset{z_i}{U_i} \Big/ (t, z_i) \sim (t, \varphi_{ij}(t, z_j))$$

$$\Rightarrow X_t = \coprod U_i / z_i \sim \varphi_{ij}(t, z_j) \quad \text{dif. det. by chch.}$$

Cocycle condition: $\varphi_{ik}(t, z_k) = \varphi_{ij}(t, \varphi_{jk}(t, z_k)) \quad \Big| \frac{\partial}{\partial t_b}\Big|_{t=0}$

$$\frac{\partial \varphi_{ik}(t, z_k)}{\partial t_b} = \frac{\partial \varphi_{ij}}{\partial t_b}(t, z_j) + \underbrace{(d\varphi_{ij})}_{\text{in } z_j\text{-var}} \cdot \frac{\partial \varphi_{jk}}{\partial t_b}(t, z_k)$$

Define $\quad \theta^b_{ij} := \frac{\partial \varphi_{ij}}{\partial t_b} \in H^0(U_i \cap U_j, T_{U_i \cap U_j})$.

$$\Rightarrow \cancel{\theta^b_{ik}} \cancel{\theta_{ij}} \quad \theta^b_{ij}\Big|_{U_{ijk}} - \theta^b_{ik}\Big|_{U_{ijk}} + \theta^b_{jk}\Big|_{U_i \cap U_j \cap U_k} = 0 \in H^0(U_{ijk}, T)$$

$\Rightarrow$ Cocycle: $\Theta \in \check{H}^1(\mathcal{U}, T) = H^1(X_0, T_{X_0})$

$$KS: T_0 B \longrightarrow H^1(X_0, T_{X_0}) \quad \text{Kodaira–Spencer map.}$$
$$\sum \lambda_b \frac{\partial}{\partial t_b} \longmapsto \left(\sum \lambda_b \Theta_{ij}^b\right)_{ij}$$

Conceptual def:

$$0 \to T_\pi \longrightarrow T_X \longrightarrow \pi^* T_B \longrightarrow 0 \quad /_{|X_0}.$$
$$0 \to T_{X_0} \longrightarrow T_X|_{X_0} \longrightarrow T_{B,0} \otimes \mathcal{O} \longrightarrow 0$$

LES:

$$T_{B,0} = H^0(T_{B,0} \otimes \mathcal{O}) \xrightarrow{\delta \,=\, KS} H^1(X_0, T_{X_0})$$

Intuition: KS classifies deformations of X_0 to first order.
(Given Θ_{ij}^b we can set $\varphi_{ij}(t) = \varphi_{ij}^{X_0} + \sum t_b \Theta_{ij}^b + O(t^2)$.)

<u>Rmk</u>: Not every $\Theta \in H^1(X_0, T_{X_0})$ need to ~~too~~ be in the image of KS for a smooth family:
Having a deformation implies
$$[\Theta, \Theta] = 0 \in H^2(X_0, T_{X_0}).$$
Ex: $\mathbb{P}^1 \times A$, A abelian sfc: there are Θ with $[\Theta, \Theta] \neq 0$.
Here $[-,-] : T_X \times T_X \longrightarrow T_X$ Lie bracket defined by
$$[X, Y] = \sum (X \cdot Y_i - Y \cdot X_i) \frac{\partial}{\partial x_i} \quad \text{for } X = \sum X_i \frac{\partial}{\partial x_i}, \; Y = \sum Y_i \frac{\partial}{\partial x_i}.$$
Certainly, $[X, X] = 0$ for vector fields, but we also consider interactions $[\Theta_{ij}, \Theta_{jk}] \in H^0(U_{ijk}, T_{U_{ijk}})$ which can be non-zero

Examples

(1) $V = H^0(\mathbb{P}^n, \mathcal{O}(d))$

$$\begin{tikzcd} X \arrow[d] & \subset & Z_U \arrow[d, "\pi"] \\ 0 \in & U & \subset \mathbb{P}(V) \end{tikzcd}$$

$$\begin{tikzcd} 0 \arrow[r] & T_X \arrow[r] & T_{\mathbb{P}^n}|_X \arrow[r] & N_{X/\mathbb{P}^n} \arrow[r] & 0 \end{tikzcd}$$

$$N_{X/\mathbb{P}^n} = \mathcal{O}_{\mathbb{P}^n}(d)|_X$$

$$\Rightarrow \quad 0 \longrightarrow H^0(X, T_X) \longrightarrow H^0(T_{\mathbb{P}^n}|_X) \longrightarrow H^0(N_{X/\mathbb{P}^n})$$

$$\xrightarrow{\delta} H^1(X, T_X) \longrightarrow H^1(T_{\mathbb{P}^n}|_X) \longrightarrow 0$$

(*) Kodaira vanishing

$$(*) \quad N_{X/\mathbb{P}^n} = \mathcal{O}(d)|_X = \mathcal{O}(-n-1+d)|_X \otimes \mathcal{O}(n+1)$$
$$= \omega_X(n+1)$$
$$\Rightarrow H^i(\omega_X \otimes \mathcal{L}) = 0 \text{ for } i > 0,\ \mathcal{L} \text{ ample}.$$

Note:

$$0 \to \mathcal{O} \longrightarrow \mathcal{O}(d) \longrightarrow \mathcal{O}(d)|_X \to 0$$

$$\Rightarrow H^0(\mathcal{O}(d)|_X) = H^0(\mathcal{O}(d))/\mathbb{C}$$
$$\cong T_{[X]}\mathbb{P}(V)$$

Under this identification

$$\delta : T_{[X]}\mathbb{P}(V) \cong H^0(X, N_{X/\mathbb{P}^n}) \longrightarrow H^1(X, T_X)$$

is Kodaira Spencer map.

Assume $n \geq 3$, $d \geq 3$.

$$H^1(T_{\mathbb{P}^n}|_X) = \begin{cases} 0 & \text{if } (n,d) \neq (3,4) \\ \mathbb{C} & \text{if } (n,d) = (3,4) \end{cases}$$

$$H^0(T_{\mathbb{P}^n}|_X) = H^0(T_{\mathbb{P}^n})$$

$$= H^0(\mathcal{O}(1))^{\oplus(n+1)}/\mathbb{C} \cong T_{id} PGL(n+1)$$

$$0 \to \mathcal{O} \to \mathcal{O}(1)^{\oplus n+1} \longrightarrow T_{\mathbb{P}^n} \to 0$$

$H^0(X, T_X) = 0$ (*Nakano vanishing*)

$H^q(M, \Omega^p \otimes \mathcal{L}) = 0$ if $p+q > \dim(M)$, $\mathcal{L}$ ample.

$$\Rightarrow \quad 0 \to \underbrace{T_{id} PGL(n+1)}_{\text{proj. transf of } X} \longrightarrow \underbrace{T_{[X]}\mathbb{P}(V)}_{\text{inf def of } X \subset \mathbb{P}^n} \longrightarrow \underbrace{H^1(X, T_X)}_{\text{inf def of } X} \to 0$$

(2)

$$\begin{array}{c} B \times \mathbb{C}^g/\mathbb{Z}^{2g} \\ \Big\downarrow \pi \\ B = \{\omega \in M_{g\times g}(\mathbb{C}) \mid \det(\operatorname{Im}(\omega)) > 0\} \end{array}$$

$\dim(B) = g^2$.

$\cancel{h^1(T_{torus})}$ $h^1(\pi^{-1}(b), T_{\pi^{-1}(b)}) = g\, h^1(\mathcal{O}_A) = g^2$.

$T_A \cong \mathcal{O}_A^{\oplus g}$

in fact KS is isomorphism

$\xleftarrow{\text{Ex}(3)}$

$$\begin{array}{l} \mathbb{P}(\mathcal{E}) \\ \big\downarrow \pi \\ X \end{array} \qquad \pi_* \mathcal{O}(1) = \mathcal{E}^\vee$$

$$0 \longrightarrow p^*(\mathcal{E})(1) \longrightarrow T_\pi \longrightarrow 0 \quad / \pi_*$$

$$0 \longrightarrow \mathcal{E} \otimes \mathcal{E}^\vee \longrightarrow \pi_* T_\pi \longrightarrow 0$$

$$0 \longrightarrow T_\pi \longrightarrow T_{\mathbb{P}(\mathcal{E})} \longrightarrow \pi^*(T_X) \longrightarrow 0 \quad / \pi_*$$

$$\frac{\mathcal{E} \otimes \mathcal{E}^\vee}{\mathcal{O}} \longrightarrow R\pi_* T_{\mathbb{P}(\mathcal{E})} \longrightarrow T_X$$

$\Rightarrow$ $H^0(\ \text{"}\ ) \longrightarrow H^0(\mathbb{P} T_{\mathbb{P}(\mathcal{E})}) \longrightarrow H^0(X, T_X)$

$\longrightarrow H^1\left(\frac{\mathcal{E} \otimes \mathcal{E}^\vee}{\mathcal{O}}\right) \longrightarrow H^1(T_{\mathbb{P}(\mathcal{E})}) \longrightarrow H^1(X, T_X)$

$\longrightarrow H^2\left(\frac{\mathcal{E} \otimes \mathcal{E}^\vee}{\mathcal{O}}\right) \longrightarrow \ldots$

$\cancel{H^1\left(\frac{\mathcal{E} \otimes \mathcal{E}^\vee}{\mathcal{O}}\right)}$

$$H^1(\mathcal{O}) \longrightarrow \mathrm{Ext}^1(\mathcal{E}, \mathcal{E}) \longrightarrow H^1\left(\frac{\mathcal{E} \otimes \mathcal{E}^\vee}{\mathcal{O}}\right)$$

$$\longrightarrow H^2(\mathcal{O}) \longrightarrow \mathrm{Ext}^2(\mathcal{E}, \mathcal{E}) \longrightarrow H^2\left(\frac{\mathcal{E} \otimes \mathcal{E}^\vee}{\mathcal{O}}\right)$$

tr   tr

identified with
$\mathrm{Ext}^1(\mathcal{E}, \mathcal{E})_0$.

~~Ex~~

$$M = \mathbb{C}^g / \Lambda_\omega \quad , \quad \Lambda_\omega = \mathrm{Span}(e_1, \dots, e_g, \omega_1, \dots, \omega_g)$$

linearly indep. over $\mathbb{R}$.

η^2 positive

$\mathcal{U} = \mathbb{C}^g$ $\quad B \subset \mathbb{C}^{g^2}$ open subset corr. to $(\omega_1, \dots, \omega_g)$

"$\{\omega \mid \det(\mathrm{Im}(\omega)) \neq 0\}$ defining complex tori

$(e_1, \dots, e_g, \omega_1, \dots, \omega_g)$ lin. ind. over $\mathbb{R}$ $\iff \det(\mathrm{Im}(\omega_1, \dots, \omega_g)) \neq 0$

$$\mathcal{M} = \bigcup_\omega \times \mathbb{C}^g / (\omega, z) \sim (\omega, z + \lambda) \quad \lambda \in \Lambda_\omega$$

$\downarrow$ smooth family

$\mathcal{U} \quad \mathbb{C}^{g^2}$-pd.

Computation shows KS is an isomorphism.

$$H^1(M, T_M) = H^1(M, \mathcal{O}^{\oplus g}) = \mathbb{C}^{g^2}$$

$$T_\omega \mathcal{U} \longrightarrow H^1(M_\omega, \Theta T_{M_\omega})$$