Remarks on last time:

(1) X sm. proj. var.

$h^{p,0} = \dim H^0(X, \Omega_X^p)$

$P_n = \dim H^0(X, nK_X)$ Plurigenera

are birational invariants.

(Hartshorne: Ex II.8.8).

(2) $h^{p,q}$ are not birational invariant, and also not diffeomorphism invariant.

<u>Ex</u> $X_1 = X(70, 16, 16, 14, 7, 6) \subset \mathbb{P}^9$, $h^{3,0} = 51838824307721$

$X_2 = X(56, 49, 8, 6, 5, 4, 4) \subset \mathbb{P}^{10}$, $h^{3,0} = 535\ldots$

– diffeomorphic (Wang)

arXiv: 2004.07142

<u>Fact</u> A rational linear combination of Hodge & Chern numbers of X is a diffeomorphism invariant iff (after adding a multiple of $\chi_p - \int \mathrm{Td}_p$.) it is a linear combination of Betti & Pontryagin classes

(Kotschick – Schreieder, The Hodge ring of Kähler manifolds)

(3) Hirzebruch – Signature Thm:

General formula of signature on $H^{2n}(X, \mathbb{R})$

in terms of Pontryagin numbers of X (Chern numbers)

Lecture 4: K3 surfaces.

Last time:

A abelian surface.

$(-1) = \overset{0}{\tau}$ involution

$\mathrm{Fix}(\tau) = A[2]$

$S = \tilde{A}/\langle\tilde{\tau}\rangle$.

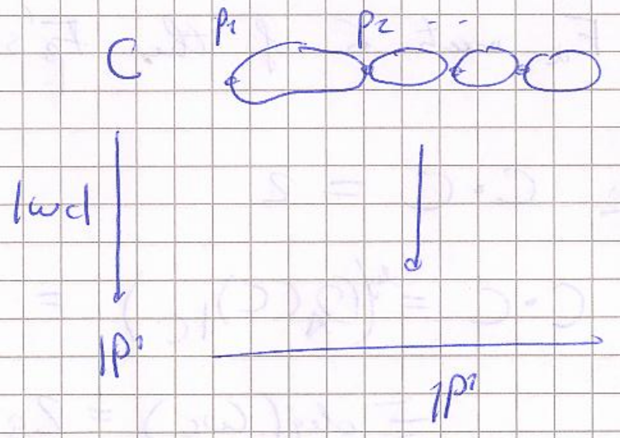
Kummer K3 surface.

$$\begin{array}{ccccc} \mathbb{P}^1 \cong E_\alpha \subset & \overset{\mathrm{Bl}_{A[2]}(A)}{\tilde{A}} & \xrightarrow{\quad f \quad} & S \\ \tilde{\tau} \circlearrowright & \downarrow & & \downarrow \\ \alpha \in & A & \longrightarrow & A/\pm 1 \\ & \circlearrowright \tau & & \end{array}$$

$\{E_\alpha\}_{\alpha \in A[2]}$ set of 16 disjoint $\mathbb{P}^1$'s curves in S.

Special case $A = \mathrm{Jac}(C)$ Jacobian of genus 2 curve.

$$C \hookrightarrow A$$
$$p \longmapsto \mathcal{O}(p - p_1)$$



For $\alpha \in A[2]$:

$\tau(C_\alpha) = C_\alpha$

$\tau|_{C_\alpha} =$ Covering involution of $C \to \mathbb{P}^1$

6 Weierstrass points $p_1, \dots, p_6$.

$\left(C_a = C + a \text{ for } a \in A \right)$.

$$\begin{array}{ccc} & & \tilde{A} \\ & \nearrow & \downarrow \\ C_\alpha & \subset & A \end{array}$$

$F_\alpha := C_\alpha/\langle\tilde{\tau}\rangle \cong \mathbb{P}^1$

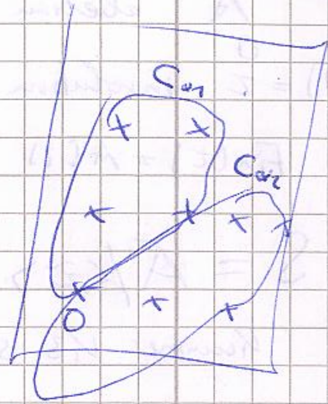
$\cap$

S 16 $\mathbb{P}^1$'s.

Rmk:

- Each $g=2$ curve has 6 WPs.

$\Rightarrow$ Each E_α meets 6 of the E_β's.



- Conversely, how many C_α pass through $O_A \in A$?

$$O \in C_\alpha = C + \alpha$$

$$\Leftrightarrow \quad \mathcal{O} = \mathcal{O}(p - p_0) \otimes \mathcal{L}_\alpha$$

$$\Leftrightarrow \quad \mathcal{L}_\alpha = \mathcal{O}(p_0 - p)$$

$$: \mathcal{L}_\alpha^\vee \cong \mathcal{L}_\alpha \Rightarrow \mathcal{O}(2p_0) \cong \mathcal{O}(2p) \Rightarrow p \in \{p_1, \dots, p_6\}$$

$$\Leftrightarrow \quad \mathcal{L}_\alpha = \mathcal{O}(p_i - p_0) \qquad 6 \text{ solutions}$$

$\Rightarrow$ Each E_α meets 6 of the F_β's.

- Lemma: $C \cdot C = 2$

Proof
$$C \cdot C = \deg(\mathcal{O}_A(C)|_C) = \deg(\omega_A(C)|_C)$$
$$= \deg(\omega_C) = 2g - 2 = 2 \qquad \square$$

Thus C_α and C_β ($\alpha \neq \beta$) meet in ≤ 1 or 2 points.

[Exercise: C_α, C_β meet in precisely 2 points which are fixed points.

$\Rightarrow \tilde{C}_\alpha \subset \hat{A}$ are disjoint

$\Rightarrow F_\alpha = f(\tilde{C}_\alpha)$ are disjoint.

$\square$

Classical Geometry

Let $\Theta := [C] \in \mathrm{Pic}(A)$ Theta divisor.

$\mathbb{P}^3 \cong |2\Theta|$ Linear system.

$$\chi(\mathcal{O}(2\Theta)) = \frac{1}{2}(2\Theta)^2 = 4.$$

$$\begin{cases} H^2(\mathcal{O}(2\Theta)) = H^0(\mathcal{O}(-2\Theta)) = 0. \\ H^1(\mathcal{O}(2\Theta)) = 0 \end{cases}$$

(Either directly, since $h^0(\mathcal{O}(\Theta)) = \chi(\mathcal{O}(\Theta)) = 1$, so $h^1(\mathcal{O}(\Theta))$ or use Kodaira vanishing thm: $H^i(X, \mathcal{L} \otimes \omega_X) = 0$ for $i > 0$, $\mathcal{L}$ ample)

Fact: $\vartheta : A \longrightarrow \hat{A} = \mathrm{Pic}^0(A)$ is a group ~~hom~~ isomorphism. ~~which is~~

$a \longmapsto \Theta_a - \Theta$

Consider:

$$\begin{tikzcd} & A/\pm 1 \arrow[dr, hook, "\tilde{f}"] & \\ f : A \arrow[ur] \arrow[rr] & & \mathbb{P}(H^0(\mathcal{O}(2\Theta))) \end{tikzcd}$$

$a \longmapsto \Theta_a + \Theta_{-a}$.

If $\Theta_a + \Theta_{-a} = \Theta_b + \Theta_{-b}$

$\Rightarrow \Theta_a = \Theta_b$ or $\Theta_a = -\Theta_b$

$\Rightarrow \vartheta(a) = \vartheta(\pm b)$

$\Rightarrow a = \pm b$. $\Rightarrow \tilde{f}$ injective on $\mathbb{C}$-points.

In fact, closed embedding (it separates tangent vectors).

Exercise:

$$f^* \mathcal{O}(1) \cong \mathcal{O}(2\Theta),$$

hence f can be identified with map coming from linear system $|2\Theta|$.

Proof :

$$A \times A \xrightarrow{m_-} A$$
$$(x,y) \longmapsto x-y$$

$$A \times A \xrightarrow{m_+} A$$
$$(x,y) \longmapsto x+y$$

$$m_-^{-1}(\Theta) = \{(x,y) \mid x-y \in \Theta\}$$

$$\Rightarrow m_-^{-1}(\Theta)_y = \Theta_y \;,\; m_+^{-1}(\Theta) = \Theta_{-y}$$

$$\Rightarrow \tilde{\Theta} = m_+^{-1}(\Theta) + m_-^{-1}(\Theta) \subset A \times A$$
$$\downarrow pr_2$$
$$A$$

family of divisors in $|2\Theta|$ parametrized by A.

$$\Rightarrow \exists f : A \longrightarrow \mathbb{P}(H^0(\mathcal{O}(2\Theta))) \text{ s.t. } \tilde{\Theta} = (f \times id)^{-1}(Z)$$

universal subscheme

cut out by $\mathcal{O}(2\Theta) \boxtimes \mathcal{O}(1)$.

$$Z \subset A \times \mathbb{P}(\mathcal{O}(2\Theta))$$

$$\Rightarrow (f \times id)^*(\mathcal{O}(2\Theta) \boxtimes \mathcal{O}(1)) \otimes p_1^* \mathcal{O}(2\Theta)^\vee = pr_2^* f^* \mathcal{O}(1)$$
$$\cong$$
$$pr_2^*(\mathcal{L})$$

To evaluate $\mathcal{L}$ restrict to $x \times A$ and observe $\tilde{\Theta}|_{x \times A} = C_x + C_x$

~~What~~ are

$$S_0 := f(A/\pm 1) \subset \mathbb{P}^3$$

- S_0 has 16 nodes (images of $A[2]$).
- $\deg(S_0) = S_0 \cdot H^2$

$$= \frac{1}{2}(f_*[A]) \cdot H^2$$

$$= \frac{1}{2}(2\Theta)^2 = 4.$$

$\Rightarrow S_0$ quartic

- $\deg(f(C_\alpha)) = \frac{1}{2} C_\alpha \cdot 2C = 2$

$f(C_\alpha) = C_\alpha/\mathbb{Z}_2$

$\Rightarrow$ 16 conics $f(C_\alpha)$

In fact, $2C_\alpha \in |\mathcal{O}(2\Theta)|$

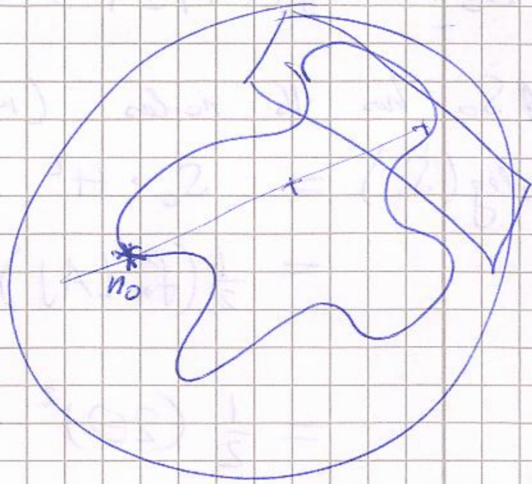
$\Rightarrow$ There are 16 Hypplan $H_\alpha \subset \mathbb{P}^3$ s.t. $H_\alpha \cap S_0 = 2f(C_\alpha)$

$\Rightarrow$ (S_0 16 Hypoplans tangt to a conic).

Projection

- $n_0 \in S_0$ node.

$\mathbb{P}^2 \cong P \subset \mathbb{P}^3$ plane away from n_0



$$\begin{tikzcd}
n_0 \in S_0 \arrow[rr, dashed, "p"] & & P \\
& x \longmapsto (l_{n_0 x} \cap P) & \\
E_{n_0} \subset Bl_{n_0}(S_0) \arrow[uu, bend left] \arrow[uurr, bend right, "\tilde{p}"'] & &
\end{tikzcd}$$

projection from n_0

$$\tilde{p}(E_{n_0}) = \mathbb{P}(\text{Normal cone of } n_0)$$
$$\cong \text{Conic} \subset \mathbb{P}^2.$$

For $y \in P \setminus \tilde{p}(E_{n_0})$, fiber of $\tilde{p}^{-1}(y) = (l_{n_0 y} \cap S) \setminus \{n_0\}$

length $= 4 - 2 = 2.$

$\Rightarrow \tilde{p}$ double cover.

- Branch locus?

Let H_α be hyperplanes s.t. $H_\alpha \cap S_0 = 2 f(C_\alpha) = 2 F_\alpha$

$$\cancel{\tilde{p}_*(H_\alpha \cap S)} \quad \tilde{p}_* H_\alpha = \begin{cases} \mathbb{P}^2 & \text{if } n_0 \notin H_\alpha \\ l_i \text{ line} & \text{if } n_0 \in H_\alpha. \end{cases}$$

$$\Rightarrow \begin{cases} \tilde{p}(F_\alpha) = l_i & \text{if } n_0 \in \alpha. \\ \& \quad \tilde{p}^{-1}(l_i) = H_i \cap S_0 = 2 F_\alpha \end{cases}$$

$\Rightarrow \{F_\alpha\}_{\alpha \in A(S)}$ s.t. $n_0 \in F_\alpha$ is ramif. divisor.

$\Longrightarrow \tilde{p}$ double cover of $\mathbb{P}^2$ branched along

$$l_1 \cup \dots \cup l_6 \quad , \quad l_i = \tilde{p}_*(\hat{H}_i).$$

- The lines $l_1, \dots, l_6$ meet in

$$5 + 4 + 3 + 2 + 1 = 15$$

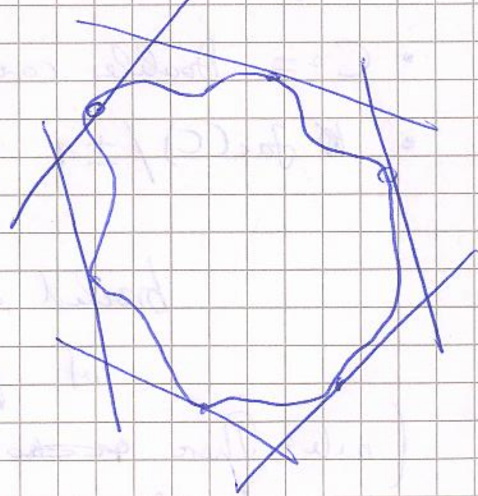
points.

$\Rightarrow$ Branched cover locally near intersection:

$$t^2 = x_J$$

$\Rightarrow$ these give the 15 nodes of $\tilde{S}_0 = Bl_0(S_0)$.

- $\tilde{p}(E_0) \subset \mathbb{P}^2$ which meets l_i tangentially (in a single point).
  (since F_α configuration above:
  we have $E_{n_0} \cdot F_\alpha = 1$)



The double cover of conic $\tilde{p}(E_0)$
branched along the 6 points
of tangency is C.

Exercise: Conversely, any quartic surface $S \subset \mathbb{P}^3$ with 16 ODPs is ~~of the~~ of the above form.

Proof:

- Project S from a node to $\mathbb{P}^2$
- Branch divisor is a degree 6 curve with 15 nodes

$C \subset \mathbb{P}^2$, $\deg = 6$, 15 nodes.

$$0 \to \mathcal{O}_C \to \mathcal{O}_{\tilde{C}} \to \bigoplus_{i=1}^{15} \mathcal{O}_{x_i} \to 0 \qquad g = \frac{(d-1)(d-2)}{2} = 10$$

$$\Rightarrow \sum_{i=1}^{N} (1-g_i) = \chi(\mathcal{O}_{\tilde{C}}) = \chi(\mathcal{O}_C) + 15$$
$$= 1 - 10 + 15 = 6.$$

$\Rightarrow N = 6,\ g_i = 0.$

$\Rightarrow C$ union of 6 lines.

- Image of E_{n_0} is a conic Q tangent to the 6 lines.
- $C \cong$ Double cover of Q along $Q \cap l_i$.
- $Jac(C)/\pm 1$ quartic which is also double cover of $\mathbb{P}^2$ branched along the 6 tangents to Q at point $p_i = Q \cap l_i$

(note: There ~~exists~~ is a unique conic, and the 6 points on Q determine the lines l_i).

Classification of algebraic Surfaces

S sm. proj surface.

Basic invariants:

$$\text{Irregularity} \quad q = h^1(\mathcal{O}_S)$$

$$\text{Geom. genus} \quad p_g = h^2(\mathcal{O}_S) = h^0(\mathcal{O}(K))$$

$$\text{"Plurigenera"} \quad P_m(S) = h^0(\mathcal{O}(mK))$$

Def (Kodaira dimension). birational invariant

X sm. proj variety / compact complex mfld.

$\kappa(X) :=$ maximal dimension of images $\phi_{mK_X}(X)$

map associated to $|mK_X|$

$=$ Asymptotic behavior of $P_m(X) = \dim H^0(\mathcal{O}(mK_X))$, i.e.

$$P_m(X) = m^{\kappa(X)} + \dots$$

$= \operatorname{tr.deg}\left(\bigoplus_{m\geq 0} H^0(\mathcal{O}(mK_X))\right)$.

Convention: If $P_m = 0 \ \forall m$, $\kappa(X) := -\infty$.

Class. of Curves: $\kappa(C) \in \{-\infty, 0, 1\}$

$\mathbb{P}^1$ ell. curves $g \geq 2$.

Surfaces: $\kappa(S) \in \{-\infty, 0, 1, 2\}$.

Def A surface S is minimal if it does not contain a divisor $E \cong \mathbb{P}^1 \subset S$ with $E \cdot E = -1$.

Equivalently: Every birational morphism $S \to S'$ is an isomorphism.

$\hookrightarrow$ This follows since a birational morphism of surfaces is the composition of blowups at points, so at each step we contract some (-1) curve E.

Thm

(a) Every surface S has a birational morphism to a minimal one S_{min}.

(b) If S is not ruled (i.e. birational to $C \times \mathbb{P}^1$ for some curve C), then S_{min} is unique.

$\Rightarrow$ Can restrict to minimal surfaces.

Proof of $(*)$: Beauville Thm V.19:

$$\begin{tikzcd} \tilde S \arrow[d, dashed, "\varepsilon_n"'] \arrow[ddr, "f"] & \\ \cdot \arrow[d, "\varepsilon_1"'] & \\ S' & S \end{tikzcd}$$

$E \subset \tilde S$ exceptional divisor of ε_n

$C = f(E)$ curve.

$K_S \cdot C \le -2 \Rightarrow C \cdot C \ge 0$ (genus formula).

$\Rightarrow P_n = \dim H^0(S, nK_S) = 0 \ \forall n$

$\Rightarrow Alb(S)$ has $\dim \le 1$

(otherwise pullback of 2-form gives a 2-form).

$\Rightarrow S \to Alb(S)$ with fiber $\mathbb{P}^1 = C$.

Classification of Surfaces

$\kappa(S) = -\infty$: Rational surfaces , Ruled surfaces

minimal: $\mathbb{P}^2$, $\mathbb{F}(n)$ $n \neq 1$.

minimal: $\mathbb{P}_C(E)$, $\begin{array}{c} E \\ \downarrow \\ C \end{array}$ rank 2 vector bundle on curve C $g(C) \geq 1$.

$\mathbb{F}(n) = \mathbb{P}(\mathcal{O} \oplus \mathcal{O}(n))$
Hirzebruch surface.

$\kappa(S) = 0$:

Let S minimal with $\kappa(S) = 0$.

Then we have one of the following:

(1) $p_g = 0$, $q = 0$: Then $2K_S = 0$ and S is an Enriques surface.

(2) $p_g = 0$, $q = 1$: S Bielliptic

(3) $p_g = 1$, $q = 0$: S K3 surface.

(4) $p_g = 1$, $q = 2$: S Abelian surface.

Elau

Enriques Surfaces:

Y $\quad p_g = 0 \Rightarrow H^0(Y, \omega_Y) = 0 \Rightarrow \omega_Y \neq \mathcal{O}_Y$.

$\quad 2K_Y = 0 \Rightarrow \omega_Y^{\otimes 2} \cong \mathcal{O}_Y$.

$X :=$ Double cover of Y wrt ω_Y.

$$\begin{array}{c} X \\ \Big\downarrow \pi \text{ étale} \\ Y \end{array}$$

$$\begin{cases} \pi^* \omega_Y = \mathcal{O}(\text{Ramification div}) = \mathcal{O}. & \Rightarrow p_g(X) = 1 \\ \chi_{top}(X) = 2 \cdot \chi_{top}(Y) \end{cases}$$

$\Rightarrow \chi(\mathcal{O}_X) = 2 \cdot \chi(\mathcal{O}_Y) = 2(1 - q + p_g) = 2.$

$\Rightarrow \Lambda^2 \Omega_X \cong \mathcal{O}_X,\ H^1(\mathcal{O}_X) = 0$

$\Rightarrow X$ K3 surface.

Conversely, if X K3 surface with $\tau \in \mathrm{Aut}(X)$ fixpoint free involution, then $Y = X/\langle \tau \rangle$ Enriques.

(Exercise).

[$H^0(Y, \Omega_Y) \subset H^0(X, \Omega_X) = 0. \Rightarrow q = 0.$

$\chi(\mathcal{O}_Y) = \frac{1}{2}\chi(\mathcal{O}_X) = 1 \Rightarrow p_g = 0$]

Example

(1) $\tau \circlearrowright S$ branched along $D = (4,4)$ divisor.

$$\begin{array}{c} S \\ \big\downarrow {\scriptstyle 2:1} \\ \tau \circlearrowright \mathbb{P}^1 \times \mathbb{P}^1 \end{array} \qquad \tau : ([x_0, x_1], [y_0, y_1]) \longmapsto ([x_0, -x_1], [y_0, -y_1])$$

Assume (i) D does not contain ~~preserve~~ fix points of τ

(ii) D is τ invariant (and smooth).

$\Longrightarrow$ $S/\langle\tau\rangle$ Enriques.

(2) $S = \mathrm{Km}(E \times F)$

$a \in (E \times F)[2] \rightsquigarrow t_a$

$\sigma = id_E \times (-id_F)$

$\sigma \circ t_a \circlearrowright \mathrm{Km}(E \times F)$ fixed pt free

Bielliptic surface

Def A surface S is bielliptic if

$$S = (E \times F)/G$$

where
- E, F elliptic curves
- $G \circlearrowright E$ faithfully by translations
- $G \circlearrowright F$ " s.t. $F/G \cong \mathbb{P}^1$.

There are 7 cases: $G \circlearrowright E$ by translation $\rightsquigarrow$ $G \subset \mathbb{Z}/a\mathbb{Z} \oplus \mathbb{Z}/b\mathbb{Z}$

e.g. $G = \mathbb{Z}/2$ $G \circlearrowright E$ by translation by 2-torsion pt.

$G \circlearrowright F$ by (-1).

$K(S) = 1$: S minimal $\Rightarrow$ S elliptic surface with base curve of $g \geq 2$.

$K(S) = 2$: General type (Wilderness)