Lecture 3

Last time

S alg K3 surface $/\mathbb{C}$: $H^1(S, \mathcal{O}_S) = 0$, $\Lambda^2 \Omega_S \cong \mathcal{O}_S$.

- $\chi_{top}(S) = 24$
- Hodge diamond

$$\begin{array}{ccccc} & & 1 & & \\ & 0 & & 0 & \\ 1 & & 20 & & 1 \\ & 0 & & 0 & \\ & & 1 & & \end{array}$$

- Singular Cohomology : $H^1(S, \mathbb{Z}) = 0$
  $H^2(S, \mathbb{Z})$ no torsion.
  $H^3(S, \mathbb{Z}) = 0$ $(\ast)$

$(\ast)$ **Lemma**: Let $\tilde{S} \xrightarrow{f} S$ be connected (finite) étale cover, then $f = \mathrm{id}$.

Proof Noether's formula: $\chi(S, \mathcal{O}_S) = \frac{1}{12}(K_S \cdot K_S + \chi_{top}(S))$
S sm. proj surface:

Here $\chi_{top}(\tilde{S}, \mathcal{O}_{\tilde{S}}) = \deg(f) \cdot \chi(S, \mathcal{O}_S)$

$\Lambda^2 \Omega_{\tilde{S}} \cong \mathcal{O}_{\tilde{S}}$.

$\Rightarrow$ $\chi(\tilde{S}, \mathcal{O}_{\tilde{S}}) = h^0(\mathcal{O}_{\tilde{S}}) - h^1(\mathcal{O}_{\tilde{S}}) + h^2(\mathcal{O}_{\tilde{S}}) = 2 - h^1(\mathcal{O}_{\tilde{S}})$

$\chi(\tilde{S}, \mathcal{O}_{\tilde{S}}) = \frac{1}{12} \chi_{top}(\tilde{S}) = \frac{1}{12} \deg(f) \chi_{top}(S) = 2 \deg(f)$

$\Rightarrow \deg(f) = 1$ $\square$

Cor $H^3(S, \mathbb{Z}) \underset{PD}{\cong} H_1(S, \mathbb{Z}) = 0$: Pf. We have that $H_1(S, \mathbb{Z})$ torsion, because $b_1 = 0$.
$(\pi_1^{alg}(S) = 0)$

(*) use that any $\mathbb{Z}$-module splits as direct sum of $\mathbb{Z}$'s & $\mathbb{Z}/n\mathbb{Z}$'s.

If there is $\alpha \neq 0$ in $H_1(S,\mathbb{Z})$ non trivial torsion,
then there exist a quotient(*)

$$\pi_1(S,x) \twoheadrightarrow H_1(S,\mathbb{Z}) \longrightarrow \mathbb{Z}/d\mathbb{Z} \longrightarrow 0$$

Which defines a degree d étale cover: Contradiction.

Poincaré duality:

$$H^2(S,\mathbb{Z}) \times H^2(S,\mathbb{Z}) \longrightarrow \mathbb{Z}.$$

unimodular pairing.

Q: What is the lattice $H^2(S,\mathbb{Z})$?

§Lattices

<u>Def</u> A lattice is a free $\mathbb{Z}$-module L together with a symmetric integral bilinear map

$$(-,-): L \times L \longrightarrow \mathbb{Z}.$$

We usually assume that L is non-degenerate, i.e.

$$\begin{aligned} L &\longrightarrow L^* \\ x &\longmapsto (x,-) \end{aligned}$$

is injective.

<u>Def:</u> L is unimodular, if $L \longrightarrow L^*$ isomorphism.
is even if $(x,x) \in 2\mathbb{Z}$ for all $x \in L$.
is odd if not even.
is of signature (k,l) if intersection matrix of $(-,-)$ wrt integral basis of L has k positive, l negative eigenvalues over $\mathbb{R}$.

(Intersection matrix $A = (e_i, e_j)$ for basis $\{e_i\}$)

is positive definite, if of signature $(m,0)$ ~~for~~.
is indefinite if " (m,n), ~~for~~ $m,n > 0$.

Classification of unimodular lattices.

__Thm__ Let L odd, unimodular, indefinite, ~~PK~~ of signature (m,n).
Then
$$L \cong (1)^{\oplus m} \oplus (-1)^{\oplus n}$$

$$\left((1)^{\oplus m} \oplus (-1)^{\oplus n} = \text{Lattice } \mathbb{Z}^{m+n} \text{ with intersection matrix } \begin{pmatrix} 1 & & & & & & \\ & 1 & & & & & \\ & & \ddots & & & & \\ & & & 1 & & & \\ & & & & -1 & & \\ & & & & & \ddots & \\ & & & & & & -1 \end{pmatrix} \right)$$

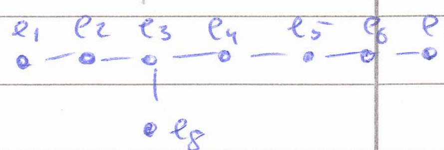
__Thm 2__ There exist even, unimodular lattices of signature (m,n) only if $m-n \equiv 0 \ (8)$.
If $m,n>0$, then ~~this~~ such lattice is unique.

What are examples?

(1) $U \cong \mathbb{Z}^2$ with intersection form $\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ of signature $(1,1)$.

"hyperbolic lattice".

(2) $E_8 := \mathbb{Z}^{\oplus 8}$ with intersection matrix



$$\begin{pmatrix} 2 & -1 & & & & & & \\ & \ddots & -1 & & & & & \\ & & \ddots & & & & -1 & \\ & & & \ddots & & & & \\ & & & & & -1 & & \\ & & & & -1 & \ddots & & \\ & -1 & & & & & 2 & \end{pmatrix}$$ unimodular, of signature $(8,0)$

Cor If L even, unimodular, of signature (m,n) with $m,n>0$,
Then

$$L \cong E_8^{\oplus \frac{m-n}{8}} \oplus U^{\oplus n}$$

$$L \cong E_8(-1)^{\frac{n-m}{8}} \oplus U^{\oplus m}.$$

Q: What is $U(-1) = ?$ Ex: $U(-1) \cong U$.

$\left(8\left(\frac{m-n}{8}\right)\right) + n\,(1,1)$

Q: How to see that E_8 is unimodular?

Prop Let L non-deg lattice with intersection matrix A (w.r.t some basis).
Then

$$|L^*/L| = |\det(A)|.$$

Ref: Filter, Appendix A.1.

Rmk: Positive definite unimodular lattices:

Back to K3 surfaces:

Consider $H^2(S,\mathbb{Z}) =$ rank 22 lattice

Hirzebruch Signature Thm: let (σ_+,σ_-) signature of $H^2(S,\mathbb{Z})$. Then

$$\sigma_+ - \sigma_- = \frac{1}{3}\int_S \left(c_1(S)^2 - 2c_2(S)\right)$$

$$= \frac{2}{3}(-24) = -16$$

$$\Rightarrow \sigma_+ = 3, \quad \sigma_- = 19.$$

Wu's formula $c_1(S) \equiv 0 \ (2) \quad \Rightarrow \quad (-,-)$ even on $H^2(S,\mathbb{Z})$

Poincaré duality: $H^2(S,\mathbb{Z})$ unimodular

Cor: $H^2(S,\mathbb{Z}) \cong E_8(-1)^{\oplus 2} \oplus U^{\oplus 3}$.

§ Kummer K3 surfaces

A abelian surface

$\tau : A \to A$ multiplication by -1, involution.

$\mathrm{Fix}(\tau) = \{x \in A \mid \tau(x) = x\} \cong A[2]$ 16 points.

$\tau(x) = x \Leftrightarrow -x = x \Rightarrow 2x = 0$

$\tilde{A} = Bl_{A[2]} A$

$S := \tilde{A}/\langle \tilde{\tau} \rangle$

exist by blowup property

$$\begin{tikzcd} \tilde{A} \arrow[r] \arrow[d] \arrow[loop left, "\tilde{\tau}"] & S \arrow[d] \\ A \arrow[r] \arrow[loop left, "\tau"] & A/\langle \tau \rangle \end{tikzcd}$$

Rmk This is not a fibre diagram:

x, y local coords on A, with $\tau(x) = -x$, $\tau(y) = -y$.

x^2, xy, y^2 local coords on $A/\langle \tau \rangle$

y, U local coords on (one chart of) $Bl_{A[2]}(A)$ $\left(\begin{matrix} xv = yu \\ \Rightarrow x = yU, \ U = \frac{u}{v} \end{matrix} \right)$

$t = y^2, U$ local coords on S,

Local coordinate ring of $A \times_{A/\langle \tau \rangle} S = \mathbb{C}[x,y] \otimes_{\mathbb{C}[x^2, xy, y^2]} \mathbb{C}[~~y~~ t, U]$

$$= \mathbb{C}[x, y, t, u] \Big/ \begin{pmatrix} t = y^2 \\ x^2 = tU^2 \\ xy = tU \end{pmatrix}$$

$$= \mathbb{C}^3_{x,y,u} \Big/ \begin{pmatrix} x^2 = y^2 U^2 \\ y \cdot (\cancel{x}\, x - yU) \end{pmatrix}$$

not smooth //

non reduced structure at exceptional divisor $y = 0$.

Prop S is a K3 surface.

Proof

Claim 1: S is smooth.

$$\begin{tikzcd} \tilde{A} \arrow[r, "f"] \arrow[d, "\pi"] & S \\ A & \end{tikzcd}$$

Step 1: let $E_i \subset \tilde{A}$ exceptional divisors.

f étale away from E_i

$\Rightarrow$ Only need to check at points of $f(E_i)$.

Step 2: let $\alpha \in A[2]$ and $tr_\alpha : A \to A$, $x \mapsto x+\alpha$ translation by α.

We have that tr_α & τ commute:

$$tr_\alpha(\tau(x)) = tr_\alpha(-x) = -x+\alpha \overset{-\alpha=\alpha}{=} -(x+\alpha) = \tau(tr_\alpha(x))$$

$\Rightarrow$ Only need to consider exceptional divisor over $O_A \in A$.

Step 3 Write $A = \mathbb{C}^2_{x,y}/\Lambda$

$\Rightarrow$ Have local coords x, y around O s.t. $\begin{cases} \tau(x) = -x \\ \tau(y) = -y \end{cases}$

$Bl_0(\mathbb{C}^2) = V(xv - yu) \subset \mathbb{C}^2_{x,y} \times \mathbb{P}^1_{[u,v]}$

$x = y\frac{u}{v}$ on $D(v)$

$= yU$, $U = \frac{u}{v}$, $U = \frac{x}{y}$

$\Rightarrow$ ~~On $D(v) \subset \tilde{A}$ have local~~

Local coords y, U on $\tilde{A}$ (around one chart).

$\tilde{\tau}(y) = -y$

$\tilde{\tau}(U) = \tilde{\tau}\left(\frac{x}{y}\right) = U.$

$\Rightarrow$ Local coords y^2, U on S.

(The local rings of S are the $\mathbb{Z}_2$-invariant part of the local ring of $\tilde{A}$)

$\Rightarrow S$ smooth.

<u>Claim 2: $\Lambda^2 \Omega_S \cong \mathcal{O}_S$</u>

Let $\alpha \in H^0(A, \Lambda^2 \Omega_A)$ constant non-zero 2-form

$\tau|_{H^0(\Omega_A)} = -id$ and $\alpha \in \Lambda^2 H^0(\Omega_A)$

$\Rightarrow \tau^*(\alpha) = \alpha$.

Alternative α = descend of $dx \wedge dy$ via $\mathbb{C}^2_{x,y} \longrightarrow A = \mathbb{C}^2_{x,y}/\Lambda$.

$\tau^*(\alpha) = \alpha \quad \Rightarrow \quad \tilde{\tau}^*(\pi^*(\alpha)) = \pi^*(\alpha)$.

$\Rightarrow \exists \sigma \in H^0(S, \Lambda^2 \Omega_S)$ s.t. $f^*(\sigma) = \pi^*(\alpha)$.

In local coords, $\alpha = dx \wedge dy$.

$x = yU$

$\pi^*(\alpha) = y\, dU \wedge dy = \frac{1}{2} dU \wedge d(y^2)$

$= f^*\left(\frac{1}{2} dU \wedge dt\right) \quad f\ t = y^2$

$\Rightarrow \sigma$ non-vanishing

$\Rightarrow \Lambda^2 \Omega_S \cong \mathcal{O}_S$.

Claim 3: $H^1(S,\mathcal{O}_S)=0$:

We check $H^0(S,\Omega_S) \cong H^1(S,\mathcal{O}_S)^* = 0$.

We have:

$$H^0(S,\Omega_S) \xhookrightarrow{f^*} H^0(\tilde{A},\Omega_{\tilde{A}})^{\mathbb{Z}_2} \Longrightarrow H^0(A,\Omega_A)^{\mathbb{Z}_2} = 0.$$

(Use that $0 \to \pi^*\Omega_A \to \Omega_{\tilde{A}} \to \bigoplus_{i=1}^{16} i_*\Omega_{E_i} \to 0$)

$\square$

Rmk Observe that S contains 16 distinct rational curves

$$f(E_1), \dots, f(E_{16}).$$

This characterizes Kummer surfaces among K3 surfaces (Huybrechts, Rmk 14.3.19)

§ The classical Kummer quartic.

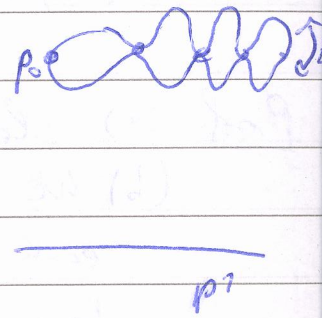
C smooth proj curve of genus 2.
$A = J(C)$.
$S = \mathrm{Kum}(A)$

$$\begin{tikzcd} & \hat{A} \arrow[r, "f"] \arrow[d] & S \\ C \arrow[ur] \arrow[r, hook] & A \arrow[loop below, "\tau"] & \end{tikzcd}$$

Let $p_0 \in C$ Weierstrass point.

$$C \hookrightarrow A$$
$$p \longmapsto \mathcal{O}_C(p - p_0)$$

$$\begin{tikzcd} C \arrow[d, "|\omega_C|"] \\ \mathbb{P}^1 \end{tikzcd}$$



Lemma $\tau(C) = C$, and $\tau|_C = \iota$.
(τ induces the hyperelliptic involution on C)

Proof Need to show
$$\tau(\mathcal{O}_C(p - p_0)) = \mathcal{O}(q - p_0) \quad \text{for some } q \in C.$$
$$\Longleftrightarrow \quad \mathcal{O}(p_0 - p) = \mathcal{O}(q - p_0)$$
$$\Longleftrightarrow \quad \mathcal{O}(2p_0) = \mathcal{O}(p + q)$$

$\Rightarrow \exists!\, q$ with this property, namely $q = \iota(p)$. $\square$

Cor: $C/\langle\iota\rangle \cong \mathbb{P}^1 \xhookrightarrow{f} S$

In fact, for every $\alpha \in A[2]$, let $C_\alpha = C + \alpha$.
$$\tau(C_\alpha) = \tau(C + \alpha)$$
$$= \tau(C) - \alpha = \tau(C) + \alpha = C_\alpha.$$

$\Rightarrow$ Get $F_\alpha = f(C_\alpha) \cong \mathbb{P}^1 \subset S$

Two sets of 16 ~~disjoint~~ rational curves:

$$\begin{cases} F_\alpha := f(C_\alpha) \\ E_\alpha = f(E_i) \end{cases} \qquad \alpha \in A[2]$$

~~Prop~~ <u>Lemma</u> (a) The E_α are disjoint

(b) The F_α are disjoint

(c) Each E_α meets 6 of the F_β's.

(d) Each F_α meets 6 of the E_β's.

<u>Proof</u> (a) clear

(b) We claim that C_α and C_β for $\alpha \neq \beta$ meet each other transversely in 2-torsion points, so become disjoint in $\tilde{A}$.

Can assume $\alpha = 0$.

2-torsion points are of the form $\begin{cases} \mathcal{O}(p_i - p_0), & p_i \in C \text{ Weierstraß} \\ \mathcal{O}(p_i + p_j - 2p_0), & p_i, p_j \text{ distinct Weierstraß}, \neq p_0. \end{cases}$

If $\beta = \mathcal{O}(p_i - p_0)$, and $x \in C \cap C_\beta$, then

$$x = \mathcal{O}(p - p_0) = \mathcal{O}(q - p_0) \otimes \mathcal{O}(p_i - p_0)$$

$\Rightarrow$ ~~perp~~ Two solutions $(p,q) \in \{(p_0, p_i), (p_i, p_0)\}$

If $\beta = \mathcal{O}(p_i + p_j - 2p_0)$ then $\mathcal{O}(p - p_0) = \mathcal{O}(q - p_0) \otimes \mathcal{O}(p_i + p_j - 2p_0)$ has solutions $(p,q) \in \{(p_i, p_j), (p_j, p_i)\}$. //(b)