Lecture 2

§ Branched covers

X variety/scheme, $\mathcal{L} \in Pic(X)$, $s \in H^0(X, \mathcal{L}^{\otimes d})$, $s \not\equiv 0$

$$\begin{array}{l} L := Tot(\mathcal{L}) \\ \Big\downarrow \pi \quad = \underline{Spec}(\mathcal{O}_X \oplus \mathcal{L}^\vee \oplus (\mathcal{L}^\vee)^2 \oplus \dots) \\ X \end{array}$$

$t \in H^0(L, \pi^*\mathcal{L})$ canonical section,

Geometrically: t is sending x to $(x, \pi(x)) \in L_{\pi(x)}$.
$\iff t : L \longrightarrow \pi^*(L) = L \times_X L$ is the diagonal.

Algebraically: $Coh(L) \simeq$ Coherent $\mathcal{A}$-algebras, $\mathcal{A} = \mathcal{O} \oplus \mathcal{L}^\vee \oplus (\mathcal{L}^\vee)^2 \oplus \dots$
$\mathcal{F} \longmapsto \pi_*(\mathcal{F})$.

$$(\mathcal{O}_L \longrightarrow \pi^*\mathcal{L}) \overset{\pi}{\longmapsto} \begin{pmatrix} \mathcal{O}_X \oplus \mathcal{L}^\vee \oplus (\mathcal{L}^\vee)^2 \oplus \dots \\ \searrow^{\simeq} \quad \searrow^{\simeq} \quad \searrow \quad \dots \\ \mathcal{L} \oplus \mathcal{O}_X \oplus \mathcal{L}^\vee \oplus \dots \end{pmatrix}$$

Rmk $(t=0)$ is the zero section of $L \to X$.

~~Def~~ The ~~branched~~ cover of X wrt $\mathcal{L}$ branched along $D=(s=0)$.
is

$$\begin{array}{c}\tilde{X} = V(t^d - \pi^*(s)) \\ \downarrow \pi \\ X\end{array} \qquad \left(\begin{array}{l} t^d \in \Gamma(L, \pi\mathcal{L}^{\otimes d}) \\ s \in \end{array}\right)$$

~~Lemma~~

~~Assume that X and D are smooth.~~

Lemma Assume X is smooth.

(a) ~~If X and~~ D is smooth, then $\tilde{X}$ is smooth.

(b) $\omega_{\tilde{X}} = \pi^*(\omega_X \otimes \mathcal{L}^{\otimes(d-1)})$

(c) ~~If X~~, D smooth, Let $D_0 = (\pi^{-1}(D))_{red}$. ~~Also~~ Then $\mathcal{O}(D_0) = \pi^*\mathcal{L}$.

((d) If X projective connected and $L^k \neq 0$ for $k = 1, \dots, d-1$, then $\tilde{X}$ is connected.)

Proof

(a) Let $x_1, \dots, x_n$ local coords of X s.t. $D = (x_1 = 0)$.

$\Rightarrow t, x_1, \dots, x_n$ loc. coords for L.

$\Rightarrow \tilde{X} = V(x_1 - t^d)$ has local coords $t, x_2, \dots, x_n$.

(b) $\omega_{\tilde{X}} = \omega_L(\tilde{X})_{|\tilde{X}}$

$= \omega_L \otimes \pi^*(\mathcal{L}^d)_{|\tilde{X}}$

$= \pi^*(\omega_X \otimes \mathcal{L}^{d-1})$.

~~Claim~~: What is ω_L?

$$0 \longrightarrow \pi^*\Omega_X \longrightarrow \Omega_L \longrightarrow \omega_\pi \longrightarrow 0.$$

$$\omega_\pi \cong T_\pi^\vee \underset{(*)}{\cong} \pi^* L^\vee$$

$$\Rightarrow \omega_L = \det(\Omega_L) = \pi^*(\det(\Omega_X) \otimes L^\vee) = \pi^*(\omega_X \otimes L^\vee).$$

(*) Either use that $T_\pi = \pi^*(\mathcal{M})$ for some $\mathcal{M} \in Pic(X)$ since fiberwise trivial, and then

$$\mathcal{M} = T_\pi|_{V(t)} = N_{L_0/L} = \mathcal{O}(L_0)|_{L_0} \overset{\text{check}}{=} \pi^*\mathcal{L}|_{L_0} = \mathcal{L}$$

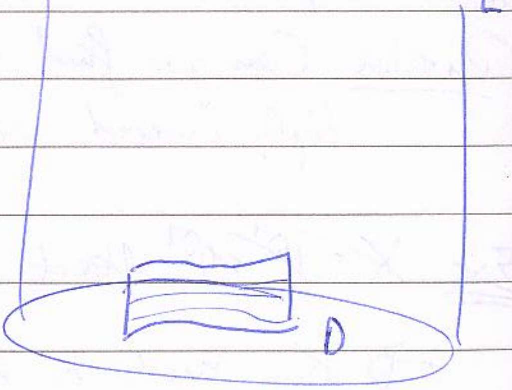
Or: $\omega_\pi = \Omega_{L/X} = \Omega_{A/\mathcal{O}_X} = L^* \otimes A$, using $A = \mathrm{Sym}[\mathcal{L}^\vee]$.)

(c). $\pi_{\tilde X}^{-1}(D) = \pi_L^{-1}(D) \cap \tilde X$

$= V(\pi^* s) \cap V(t^d - \cancel{\pi^* s})$

$\tilde X \subset L \qquad = V(\pi^* s - t^d) \cap V(t^d)$

$\pi_{\tilde X} \searrow \ \swarrow \pi_L \qquad = V(t^d|_{\tilde X})$ = section of $\pi_{\tilde X}^* \mathcal{L}^{\otimes d}$



If D reduced, $\pi^{-1}(D)_{red} = V(\pi^* s) \cap V(t) = V(t) \cap \tilde X =$

$=$ section of $\pi^* \mathcal{L}|_{\tilde X}$. $\to$ so any section gives $\mathcal{L}^i \cong \mathcal{O}$.

(d) $\pi_* \mathcal{O}_{\tilde X} = \mathcal{O}_X \oplus \mathcal{L}^\vee \oplus \dots \oplus (\mathcal{L}^\vee)^{d-1}$.

$$H^0(\tilde X, \mathcal{O}_{\tilde X}) = H^0(X, \mathcal{O}_X) \oplus \underbrace{H^0(\mathcal{L}^\vee) \oplus \dots \oplus H^0((\mathcal{L}^\vee)^{d-1})}_{0 \text{ if } \mathcal{L}^i \neq \mathcal{O}}$$

either $\mathcal{L}^{\otimes d}$ effective or if $\mathcal{L}^{\otimes d} \cong \mathcal{O}$ then are torsion line bundles.

Example

$D \subset \mathbb{P}^2 = X$ sextic curve, $\mathcal{L} = \mathcal{O}(3)$, $d = 2$.

$S :=$ $\underbrace{\text{Branched cover of deg 2 of } \mathbb{P}^2}_{\text{Double cover}}$ branched at D.

$$\omega_S = \pi^*(\omega_{\mathbb{P}^2} \otimes \mathcal{L})$$
$$= \pi^*(\mathcal{O}(-3) \otimes \mathcal{O}(3)) = \mathcal{O}_S.$$

$$\pi_* \mathcal{O}_S = \mathcal{O}_{\mathbb{P}^2} \oplus \mathcal{O}_{\mathbb{P}^2}(-3)$$

$H^0(\mathcal{O}_S) = \mathbb{C} \Rightarrow S$ connected.

$H^1(\mathcal{O}_S) = 0$.

$H^2(\mathcal{O}_S) = \mathbb{C}$.

#(moduli) / ≠

Question: Can you find a K3 surface S that is triple covered over a smooth surface?

Ex ~~$X = \mathbb{P}^1 \times \mathbb{P}^1$~~ Need $\omega_X \otimes \mathcal{L}^2 \cong \mathcal{O}$.

$\Rightarrow K_X$ needs to be divisible by 2.

Can take $X = \mathbb{P}^1 \times \mathbb{P}^1$

$\mathcal{L} = \mathcal{O}(1,1)$.

$D =$ ~~gg~~ smooth zero locus of section of $\mathcal{O}(3,3)$.

Rmk $f: S \to S$ non symplectic automorphism of K3 surface of prime order p, then $p \in \{2, 3, 5, 7, 11, 13, 17, 19\}$. How does case $p = 5$ look like?

§ Intersection theory of surfaces:

Let S smooth proper (hence projective) surface over $\mathbb{C}$.

Three ways:

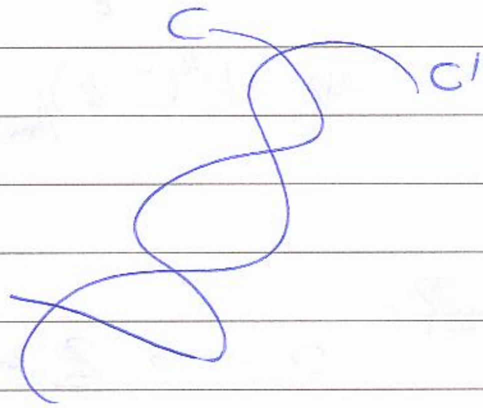
(1) Geometrically:

Let $C, C' \subset S$ distinct irreducible curves ($\Rightarrow C \cap C'$ finite set of pts)

For $x \in C \cap C'$,

$$m_x(C,C') := \dim_{\mathbb{C}}\left(\mathcal{O}_{S,x}/(f,g)\right)$$

where f, g local equations for C, C'.



$$C \cdot C' := \sum_{x \in C \cap C'} m_x(C,C').$$

?) Fulton style intersection thy.

Note: $C \cdot C' = h^0(\mathcal{O}_{C \cap C'}) = \chi(\mathcal{O}_{C\cap C'})$

$= \chi(\mathcal{O}_C) - \chi(\mathcal{O}_C(-C'))$.

$= \chi(\mathcal{O}_S) - \chi(\mathcal{O}_S(-C)) - \chi(\mathcal{O}_S(-C'))$
$+ \chi(\mathcal{O}_S(-C-C'))$.

$0 \to \mathcal{O}_C(-C') \longrightarrow \mathcal{O}_C \longrightarrow \mathcal{O}_{C\cap C'} \to 0$

$0 \to \mathcal{O}_S(-C) \longrightarrow \mathcal{O}_S \longrightarrow \mathcal{O}_C \to 0$

$0 \to \mathcal{O}_S(-C-C') \to \mathcal{O}_S(-C') \to \mathcal{O}_C(-C') \to 0$

(2) For $\mathcal{L}, \mathcal{L}' \in \mathrm{Pic}(S)$ define

$$\mathcal{L}\cdot\mathcal{L}' := \chi(\mathcal{O}_S) - \chi(\mathcal{L}^\vee) - \chi(\mathcal{L}'^\vee) + \chi(\mathcal{L}^\vee \otimes (\mathcal{L}')^\vee)$$

Prop This defines symmetric, bilinear form
$$\mathrm{Pic}(S) \times \mathrm{Pic}(S) \longrightarrow \mathbb{Z}.$$

(3) (Topologically)

$$(-,-): H^k(S,\mathbb{Z}) \times H^{4-k}(S,\mathbb{Z}) \xrightarrow{\cup} H^4(S,\mathbb{Z}) \xrightarrow{\int_S} \mathbb{Z}.$$

~~nondeg~~ unimodular ~~form~~ pairing (modulo torsion)

$$\Rightarrow H^k(S,\mathbb{Z})/_{\mathrm{tors.}} \longrightarrow \mathrm{Hom}(H^{4-k}(S,\mathbb{Z})/_{\mathrm{tors}}, \mathbb{Z}) \text{ iso}$$

Recall:
$$0 \longrightarrow \underline{\mathbb{Z}} \hookrightarrow \mathcal{O}_S \xrightarrow{\exp} \mathcal{O}_S^* \longrightarrow 0 \quad \text{exponential sequence.}$$

$$\mathrm{Pic}(S) = H^1(S, \mathcal{O}_S^*) \xrightarrow[\text{connecting homomorphism}]{c_1} H^2(S,\mathbb{Z})$$

Fact: c_1 is isometric: $\mathcal{L} \cdot \mathcal{L}' = \int_S c_1(\mathcal{L}) \cup c_1(\mathcal{L}')$.

(Alternative description of c_1 if $\mathcal{L} = \mathcal{O}_S(C)$ for C irreducible, smooth:
C defines $H^2(S) \longrightarrow H^2(C) \xrightarrow{\sim} \mathbb{Z}$
So elt in $H^2(S,\mathbb{Z})^\vee \cong H^2(S,\mathbb{Z})$ (up to torsion)
which is $c_1(\mathcal{L})$.)

Hirzebruch-Riemann-Roch thm:

$$\chi(S, \mathcal{F}) = \int_S ch(\mathcal{F})\, td(T_S)$$

coh. shef.

For lin bdle $\mathcal{L}$,

$$\chi(S, \mathcal{L}) = \int_S \left(1 + c_1(\mathcal{L}) + \frac{c_1(\mathcal{L})^2}{2}\right)\left(1 + \frac{1}{2}c_1(T_S) + \chi(\mathcal{O}_S)\right)$$

$$= \chi(\mathcal{O}_S) + \frac{1}{2}\left(c_1(\mathcal{L})^2 + c_1(T_S)\cdot c_1(\mathcal{L})\right)$$

$$= \chi(\mathcal{O}_S) + \frac{1}{2}(\mathcal{L}\cdot\mathcal{L} - \omega_S\cdot\mathcal{L}).$$

$$\begin{cases} td(T_S) = 1 + \frac{1}{2}c_1(T_S) + \frac{1}{12}\left(c_1(T_S)^2 + c_2(T_S)\right). \\ \text{Gauss-Bonnet: } \int_S c_2(T_S) = \chi_{top}(S) \end{cases}$$

$$\Rightarrow \quad \chi(S, \mathcal{O}_S) = \frac{1}{12}\left(c_1(S)^2 + \chi_{top}(S)\right). \qquad \text{(Noether's formula)}$$

Serre duality: locally free

$$H^k(S, \mathcal{F}) \cong H^{2-k}(\mathcal{F}^\vee \otimes \omega_S)^\vee.$$

Hodge decomposition X smooth proj var / $\mathbb{C}$
or compact Kähler manifold.

$$H^k(X, \mathbb{C}) = \bigoplus_{\substack{p+q=k \\ p,q \geq 0}} H^{p,q}(X)$$

$$\overline{H^{p,q}(X)} = H^{q,p}(X)$$

$$H^{p,q}(X) = H^q(X, \Omega^p).$$

§ Numerical invariants of a K3 surface.

S algebraic K3 / $\mathbb{C}$

- $H^1(S, \mathcal{O}_S) = 0$.
- $\Lambda^2 \Omega_S \cong \mathcal{O}_S$.

$$\begin{array}{ccccc} & & h^{0,0} & & \\ & h^{1,0} & & h^{0,1} & \\ h^{2,0} & & h^{1,1} & & h^{0,2} \\ & h^{2,1} & & h^{1,2} & \\ & & h^{2,2}. & & \end{array}$$

$H^0(S, \mathcal{O}_S) = \mathbb{C} \Rightarrow h^{0,0} = 1$

$H^1(S, \mathcal{O}_S) = 0 \qquad h^{1,0} = 0 = h^{0,1}$

they're just complex conjugates.

$$\begin{array}{ccccc} & & 1 & & \\ & 0 & & 0 & \\ 1 & & 20 & & 1 \\ & 0 & & 0 & \\ & & 1. & & \end{array}$$

$H^2(S, \mathcal{O}_S) \cong H^0(S, \omega_S)^\vee \qquad h^{2,0} = 1$
$= \mathbb{C}$

By Poincaré duality $H^3(S, \mathbb{Q}) = 0 \Rightarrow h^{2,1} = h^{1,2} = 0$
$H^4(S, \mathbb{C}) = \mathbb{C}$

Noether's formula: $2 = \chi(S, \mathcal{O}_S) = \frac{1}{12} \chi_{top}(S)$

$\Rightarrow \chi_{top}(S) = 24 = \sum h^{p,q} (-1)^{p+q}$
$= 4 + h^{1,1}(S)$

$\Rightarrow h^{1,1}(S) = 20$

Exponential sequence:

$$0 \longrightarrow \mathbb{Z} \longrightarrow \mathcal{O}_S \longrightarrow \mathcal{O}_S^* \longrightarrow 0.$$

$$0 \longrightarrow \mathbb{Z} \longrightarrow \mathbb{C} \longrightarrow \mathbb{C}^* \longrightarrow 0$$

$$0 \to H^1(S,\mathbb{Z}) \to \underset{\substack{\parallel \\ 0}}{H^1(S,\mathcal{O}_S)} \longrightarrow \mathrm{Pic}(S) \hookrightarrow H^2(S,\mathbb{Z}) \to H^2(S,\mathcal{O})$$

$$\Longrightarrow H^1(S,\mathbb{Z}) = 0.$$

Lemma $\mathrm{Pic}(S)$ has no torsion.

Proof Assume $\mathcal{L} \in \mathrm{Pic}(S)$ with $\mathcal{L}^{\otimes d} \cong \mathcal{O}_S$ and ~~$d \geq 1$ minimal with this property~~ $\mathcal{L}^{\otimes k} \not\cong \mathcal{O}$ for $0 < k < d$.

Let $\tilde{S}$ cover associated to $s : \mathcal{O}_S \longrightarrow \mathcal{L}^{\otimes d}$.

$$\begin{array}{c} \tilde{S} \\ \downarrow{\scriptstyle \pi} \\ S \end{array}$$

$\Rightarrow \pi : \tilde{S} \longrightarrow S$ étale cover.

~~$\Rightarrow \sigma_S : \mathcal{O}_S \longrightarrow \Lambda^2\omega_S \quad \pi^*\omega_S$~~

~~$\Rightarrow \pi^*\sigma_S$~~

$\Rightarrow \omega_{\tilde{S}} = \pi^*\omega_S \cong \mathcal{O}_{\tilde{S}}$.

$\Rightarrow H^0(\tilde{S}, \mathcal{O}_{\tilde{S}}) = \mathbb{C}$

$\quad H^2(\tilde{S}, \mathcal{O}_{\tilde{S}}) = H^0(\tilde{S}, \omega_{\tilde{S}})^\vee = \mathbb{C}$

$\Rightarrow \chi(\tilde{S}, \mathcal{O}_{\tilde{S}}) = 2 - h^1(\mathcal{O}_{\tilde{S}})$ $\qquad d = 1 \; \Uparrow$

$\parallel$

$\chi(S, \pi_*\mathcal{O}_{\tilde{S}}) = \chi(\mathcal{O}_S \oplus \mathcal{L}^\vee \oplus \dots \oplus (\mathcal{L}^\vee)^{d-1}) = d\,\chi(\mathcal{O}_S) = 2d.$ $\square$

Cor $H^2(S,\mathbb{Z})$ is torsion-free.

Proof ~~If $kx=0$ for $k\geq 1$, then~~ φ

$$0 \to \mathrm{Pic}(S) \xrightarrow{c_1} H^2(S,\mathbb{Z}) \xrightarrow{\varphi} H^2(S,\mathcal{O})$$

If $x\in H^2(S,\mathbb{Z})$ s.t. $kx=0$ for $k\geq 1$,

then $\varphi(kx)=0$ $\quad (H^2(S,\mathcal{O})=\mathbb{C}$ torsion free$)$.

$\Rightarrow x=c_1(\mathcal{L})$ with $\mathcal{L}^{\otimes k}\cong\mathcal{O} \overset{\text{Lemma}}{\Rightarrow} \mathcal{L}\cong\mathcal{O}$. $\square$

Lemma $H_1(S,\mathbb{Z})$ torsion-free~~, hence $H_1(S,\mathbb{Z})=0$~~.

Proof Argue similar to Lemma. If $H_1(S,\mathbb{Z})$ has torsion, then since it is a $\mathbb{Z}$-module (and so of the form $\mathbb{Z}^{\oplus r}\oplus\bigoplus_i \mathbb{Z}/p_i^{n_i}$) it has a torsion quotient

$$H_1(S,\mathbb{Z}) \twoheadrightarrow \mathbb{Z}/m\mathbb{Z}.$$

$\Rightarrow$ Get quotient $\pi_1(S)\to H_1(S,\mathbb{Z})\to\mathbb{Z}/m\mathbb{Z}$.

$\Rightarrow$ Cover $\tilde{S}\to S$ which is impossible by same argument. $\square$

Since $b_1(S)=0 \Rightarrow H_1(S,\mathbb{Z})=0 \overset{PD}{\Longrightarrow} H^3(S,\mathbb{Z})=0$.

[Poincaré duality: For any coefficient ring G,

$$H^i(S;G) \longrightarrow H_{n-i}(S;G)$$
$$\alpha \longmapsto \alpha\frown[S]$$

is an isomorphism.]

[UCT: $H^i(S,\mathbb{Z}) \underset{\text{non canonical}}{\cong} \mathrm{Hom}_{\mathbb{Z}}(H_i(S,\mathbb{Z}),\mathbb{Z})\oplus \mathrm{Ext}^1(H_{i-1}(S,\mathbb{Z}),\mathbb{Z})$.]

Intersection form on $H^2(S,\mathbb{Z})$:

$$H^2(S,\mathbb{Z}) \times H^2(S,\mathbb{Z}) \xrightarrow{\int_S (\,)\cup(-)} \mathbb{Z}$$

non-degenerate, unimodular

$$\left(\begin{array}{l} (x,y)=0 \ \forall y \in H^2 \\ \Rightarrow x=0 \end{array} \right)$$

$$\Updownarrow$$

$$H^2(S,\mathbb{Z}) \hookrightarrow H^2(S,\mathbb{Z})^*$$

$$H^2(S,\mathbb{Z}) \xrightarrow{\sim} H^2(S,\mathbb{Z})^*$$

isom.

What more can we say?

Hirzebruch-Signature thm: Let $\sigma = (\sigma_+, \sigma_-)$ signature of $(-,-)$.

$$\cancel{\text{index}} \quad \sigma_+ - \sigma_- = \int_S \frac{1}{3}\left(c_1(S)^2 - 2c_2(S)\right)$$

$$= -16$$

$$\Rightarrow \sigma_+ = 3 \quad , \sigma_- = 19.$$

__Wu's formula:__

$c_1(S) \equiv 0 \mod 2 \implies (-,-)$ even on $H^2(S,\mathbb{Z})$.

i.e. $(x,x) \in 2\mathbb{Z} \; \forall x$.

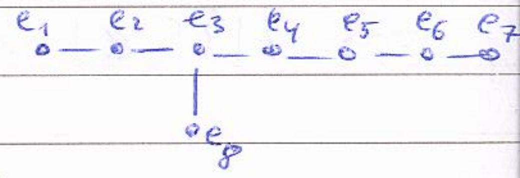
__Thm__ There exist a unimodular even lattice of signature (m,n) only if $m-n \equiv 0 \; (8)$.
It is unique, if $m>0$ and $n>0$ (indefinite case).

__Examples:__

$U = \mathbb{Z}^{\oplus 2}$ with intersection form $\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$.

unimodular, sign $(1,1)$

$E_8 = \mathbb{Z}^8$ with intersection form



$$\begin{array}{c c}
\begin{array}{cccccccc} & & & & & & & 8 \end{array} & \\
\begin{pmatrix}
2 & -1 & & & & & & \\
-1 & 2 & -1 & & & & & \\
 & -1 & 2 & -1 & & & & -1 \\
 & & -1 & 2 & -1 & & & \\
 & & & -1 & 2 & -1 & & \\
 & & & & -1 & 2 & -1 & \\
 & & & & & -1 & 2 & 0 \\
 & & -1 & & & & 0 & 2
\end{pmatrix} & \begin{array}{c} \\ \\ 3 \\ \\ \\ \\ \\ \\ \end{array} \\
\begin{array}{cccccccc} & & 3 & & & & & \end{array} &
\end{array}$$

unimodular, signature $(1,1)$.

__Use:__ If L non-degenerate lattice with intersection form A wrt some basis, then

$$|L^*/L| = |\det(A)|.$$

(this is an ~~ex~~example of Fulton, Appendix A.1. For a 1-dim'l regular ~~For a module M over~~ ring B, finitely gen modules M, N, and $\varphi: M \to N$ we have $-\ell_B(\mathrm{Ker}(\varphi)) + \ell_B(\mathrm{coker}(\varphi)) = \#(\text{zeros of } \det(\varphi)) = \ell_B(\text{ker} \det(\varphi)) - \ell_B(\mathrm{coker}(\det(\varphi)))$)

In thm.

If $m>n$, $L \cong U^{\oplus n} \oplus E_8^{\frac{m-n}{8}}$

$m<n$, $L \cong U^{\oplus m} \oplus E_8(-1)^{\frac{n-m}{8}}$.

$\lrcorner$

Rmk: If L not even, unimodular, signature (m,n), $m,n>0$, then $L \cong \langle 1 \rangle^{\oplus m} \oplus \langle -1 \rangle^{\oplus n}$

Cor Let S be an algebraic K3 surface / $\mathbb{C}$.

$\Rightarrow H^2(S,\mathbb{Z}) \cong U^{\oplus 3} \oplus E_8(-1)^{\oplus 2}$.

Exercise: Let $S \longrightarrow \mathbb{P}^2$ ~~branched~~ double cover branched along smooth sextic curve $C \subset S$. Compute $H^1(S, \Omega_S^1)$ directly.

Proof We have

$$0 \longrightarrow \pi^*\Omega_{\mathbb{P}^2} \longrightarrow \Omega_S \longrightarrow i_* N^*_{C_0/S} \longrightarrow 0$$

Generated by $dt/t\,dt$, and dt is generator of $N^*_{C_0/S}$. (normal direction $\frac{\partial}{\partial t}$).

We have $N^*_{C_0/S} = \mathcal{O}_S(C_0)|^*_{C_0}$

$$= \pi^*\mathcal{L}^\vee|_{C_0} = \mathcal{L}^*|_C.$$

We pushforward by π:

$$0 \longrightarrow \underset{\substack{\text{S/} \\ \Omega_{\mathbb{P}^2} \oplus \Omega_{\mathbb{P}^2}(-3)}}{\pi_*\pi^*\Omega_{\mathbb{P}^2}} \longrightarrow \pi_*\Omega_S \longrightarrow i_*\mathcal{L}^*|_C \longrightarrow 0$$

$$H^1(\mathcal{L}^*|_C) \cong H^0(C, \mathcal{L}|_C \otimes \omega_C)^\vee$$

$$\cong H^0(C, \mathcal{L}|_C \otimes \omega_{\mathbb{P}^2}(C)|_C)^\vee$$

$$= H^0(C, \mathcal{O}_C(C))^\vee$$

$$= H^0(C, N_{C/\mathbb{P}^2})^\vee.$$

Euler seq.

$$0 \longrightarrow \Omega_{\mathbb{P}^2} \longrightarrow \mathcal{O}(-1)^{\oplus 3} \longrightarrow \mathcal{O} \longrightarrow 0$$

$$\Rightarrow \quad H^1(\mathbb{P}^2, \Omega_{\mathbb{P}^2}) \cong \mathbb{C}, \quad H^2(\mathbb{P}^2, \Omega_{\mathbb{P}^2}) = 0.$$

$$\Rightarrow \quad H^2(\Omega_{\mathbb{P}^2}(-3)) = \mathrm{Ker}\left(H^2(\mathcal{O}(-4)^{\oplus 3}) \longrightarrow H^2(\mathbb{P}^2, \mathcal{O}(-3)) \right)$$

$$\cong \mathrm{Coker}\left(\mathbb{C} \longrightarrow H^0(\mathcal{O}_{\mathbb{P}^2}(1))^{\oplus 3} \right)^\vee$$

$$H^1(\Omega_{\mathbb{P}^2}(-3)) = 0.$$

Get

$$0 \longrightarrow H^1(\mathbb{P}^2, \Omega_{\mathbb{P}^2}) \longrightarrow H^1(S, \Omega_S) \longrightarrow H^0(C, N_{C/\mathbb{P}^2})^\vee$$

$$\longrightarrow \left(H^0(\mathcal{O}_{\mathbb{P}^2}(1))^{\oplus 3} / \mathbb{C} \right)^* \longrightarrow 0$$

Dualize, $H^1(S, \Omega_S) \cong H^1(S, T_S)^\vee$.

$$\Rightarrow \quad 0 \longrightarrow \overset{8}{H^0(\mathcal{O}_{\mathbb{P}^2}(1)^{\oplus 3}) / \mathbb{C}} \longrightarrow \overset{27}{H^0(C, N_{C/\mathbb{P}^2})} \longrightarrow \overset{20}{H^1(S, T_S)} \longrightarrow \overset{1}{\mathbb{C}} \to 0$$

tangent space to $PGL(3)$

inf. Deformations of $C \subset \mathbb{P}^2$

semireg. part.

4

$$\#(\text{moduli of } S \to \mathbb{P}^2) = h^0(\mathcal{O}_{\mathbb{P}^2}(6)) - 1 - 8$$

$$= \binom{6+2}{2} - 9$$

$$= \frac{8 \cdot 7}{2} - 9 = 28 - 9 = 19$$