Lecture 1: Examples & Plan

References: – Huybrechts, Lectures on K3 surfaces.
– Kondo, K3 surfaces
– Barth, Hulek, Peters, Van de Ven – Compact Complex Surfaces (Chapter on K3).
– Beauville – Complex Algebraic Surfaces.
– Beauville et al, Géométrie des surfaces K3: modules & périodes
↑ Asterisque Vol 126 (see Huybrechts book ref).
"easy" french.

Exam: Oral, 30.1.2023.

§1 Def & Examples ~~(K3 surfaces~~

~~Motto: There are plenty of K3 surfaces.~~

Def An algebraic K3 surface over a field k is a smooth proper ~~surface~~ variety $/k$ S of dim 2 s.t.

$$H^1(S, \mathcal{O}_S) = 0$$

$$\omega_S = \Lambda^2 \Omega_S^1 \cong \mathcal{O}_S$$

Rmk Variety = Separated, geometrically integral scheme of finite type$/k$

X smooth $\Leftrightarrow$ $\Omega_{X/k}$ locally free of rank $\dim(X)$

$\Leftrightarrow$ $X \otimes_k \bar{k}$ regular.

Rmk2 Any smooth proper surface is projective (Liu, Ch. 9.3)

$\Rightarrow$ Algebraic K3's are projective

$S \subset \mathbb{P}^N$

Rmk 3 Over $\mathbb{C}$, can consider analytification S_{an}.

GAGA: Alg geom. over S $\Longleftrightarrow$ Cmplx Geom. over S_{an}.
($Coh(X) \cong Coh(X_{an})$ for X proper)

Def A polarized K3 surface is a pair $(S, \mathcal{L})$ where
- S algebraic K3 surface
- $\mathcal{L} \in Pic(S)$ ample divisor, primitive in $Pic(S)$.

$d := \mathcal{L} \cdot \mathcal{L}$ degree of polarization.

§2. Examples /$\mathbb{C}$ Motto: There K3 surfaces are easy to construct
There are plenty of them.

(1) Quartic surface.

$S \subset \mathbb{P}^3$ smooth hypersurface of degree 4.

Check that S is a K3 surface.

(1) $0 \longrightarrow \mathcal{O}_{\mathbb{P}^3}(-4) \longrightarrow \mathcal{O}_{\mathbb{P}^3} \longrightarrow \mathcal{O}_S \to 0$

$$\Rightarrow \quad 0 \longrightarrow H^0(\mathbb{P}^3, \mathcal{O}_{\mathbb{P}^3}) \xrightarrow{\sim} H^0(S, \mathcal{O}_S)$$
$$\longrightarrow H^1(\mathcal{O}_{\mathbb{P}^3}(-4)) \longrightarrow \cancel{H^1(\mathbb{P}^3, \mathcal{O}_{\mathbb{P}^3})} \longrightarrow H^1(S, \mathcal{O}_S)$$
$$\longrightarrow H^2(\cancel{\quad}).$$

Vanishing: $H^i(\mathbb{P}^n, \mathcal{O}(m)) = 0$ for $0 < i < n$.

(2) Adjunction formula: $D \subset X$ Cartier divisor in X smooth
$$\omega_D = \omega_X(D)|_D.$$

$$\omega_S = \omega_{\mathbb{P}^3}(4)|_S$$
$$= \mathcal{O}(-4) \otimes \mathcal{O}(4)|_S$$
$$= \mathcal{O}_S. \qquad \square$$

$\mathcal{L} := i^*\mathcal{O}(1)$ polarization.

$$\mathcal{L}\cdot\mathcal{L} = i^*(H)^2$$
$$= H^2 \cdot [S]$$
$$= \deg_{\mathbb{P}^3}(S) = 4.$$

(2) $\mathbb{P}^4 \supset S = V(f,g)$ smooth complete intersection

(ii)

$$\omega_{\mathbb{P}^4} = \mathcal{O}(-5) \qquad D_f = V(f) \qquad \deg(f) = 2$$

$$D_g = V(g). \qquad \deg(g) = 3.$$

$$\omega_S = \omega_{\mathbb{P}^4}(D_f + D_g)|_S$$

$$= \mathcal{O}(-5 + \deg(f) + \deg(g))|_S$$

want $\deg(f) + \deg(g) = 5$

$$\not= \quad = \mathcal{O}_S.$$

$(1,4) \rightsquigarrow$ reduces to previous case

$(2,3)$

(i)

$$0 \longrightarrow \underset{\substack{\shortparallel \\ \mathcal{O}_{D_f}(-3)}}{\mathcal{O}_{D_f}(-\deg g)} \longrightarrow \mathcal{O}_{D_f} \longrightarrow \mathcal{O}_S \longrightarrow 0$$

$$0 \longrightarrow \mathcal{O}_{\mathbb{P}^4}(m-2) \longrightarrow \mathcal{O}_{\mathbb{P}^4}(m) \longrightarrow \mathcal{O}_{D_f}(m) \longrightarrow 0 \quad \Big| \otimes \mathcal{O}(m)$$

$$\Rightarrow \quad H^i(\mathcal{O}_{D_f}(m)) = 0 \quad \text{for } i = 1,2.$$

all m.

$$\Rightarrow \quad H^1(S, \mathcal{O}_S) = 0.$$

(iii) $\mathcal{L} = i^*\mathcal{O}(1)$

$$\mathcal{L} \cdot \mathcal{L} = \deg(S) = 2 \cdot 3 = \underline{6}.$$

(3) $\cancel{8}\mathbb{P}^5 \supset S = V(f_1, f_2, f_3)$, $\deg(f_i) = 2 \ \forall i$

K3 sufau of deg 8.

~~$\Rightarrow$ no~~

Rmk / Exercise: No other complete intersection in $\mathbb{P}^4$ K3s possible.

(4) $\mathbb{P}^2 \times \mathbb{P}^1 \supset S = V(f)$, $f \in H^0(\mathcal{O}(3,2))$

deg (3,2) hypersufau.

- $\omega_S = \omega_{\mathbb{P}^2\times\mathbb{P}^1}(S)|_S$. $\qquad \omega_{\mathbb{P}^2\times\mathbb{P}^1} = pr_1^*(\omega_{\mathbb{P}^2}) \otimes pr_2^*(\omega_{\mathbb{P}^1})$

$= \mathcal{O}(-3,-2)(S)|_S$.

$= \mathcal{O}_S$.

- $H^1(S, \mathcal{O}_S) = 0$

$\Rightarrow S$ a K3 surface.

Interesting:

$$\begin{tikzcd} S \arrow[r, hook] \arrow[dr, "\pi"'] & \mathbb{P}^2\times\mathbb{P}^1 \arrow[d, "\pi_2"] \\ & \mathbb{P}^1 \end{tikzcd} \qquad \begin{tikzcd} F_x \arrow[r] \arrow[d, "\pi_x"] & S \arrow[d] \\ x \arrow[r, hook] & \mathbb{P}^1 \end{tikzcd}$$

Fibers of π: For $x \in \mathbb{P}^1$, let $F_x := \pi^{-1}(x)$.

$\omega_{F_x} = \underset{\mathcal{O}_S}{\overset{\shortparallel}{\omega_S}}(F_x)|_{F_x}$ $\qquad \mathcal{O}(F_x) = \pi^*\mathcal{O}_{\mathbb{P}^1}(x)$

$= \pi_x^*(\mathcal{O}_{\mathbb{P}^1}(x)|_x) \cong \mathcal{O}_{F_x}$ $\qquad \Rightarrow F_x$ arithmetic genus 1 curve.

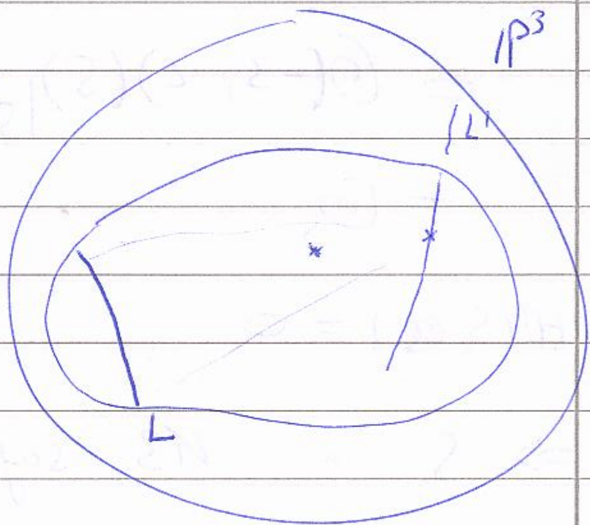
$\Rightarrow$ Any fibration of a K3 surface $S \to C$ is an elliptic fibration.

~~We can construct K3 surfaces~~
$\Rightarrow$ This yields other methods to construct K3's, e.g. by ~~their~~ using Weierstraß models

(5) (Another elliptic fibration).

Let $S \subset \mathbb{P}^3$ Quartic which contains a line $L \subset S$.

Let $L' \subset \mathbb{P}^3 \setminus L$ line



Projection to L': (x)

$$\begin{array}{ccc} \mathbb{P}^3 \setminus L & \longrightarrow & L' \cong \mathbb{P}^1 \\ \cup & \nearrow & \\ S \setminus L & & \end{array}$$

Exercise: $\begin{array}{ccc} S \setminus L & \longrightarrow & \mathbb{P}^1 \\ \cap & \nearrow \text{extends.} & \\ S & & \end{array}$ $\Rightarrow$ elliptic fibration $\begin{array}{c} S \\ \downarrow \\ \mathbb{P}^1 \end{array}$

Fibers $F_x = \underbrace{\left(\mathrm{Span}(L, x) \cap S - L\right)}_{\text{cubics in } \mathrm{Span}(L,x)}$.

(x) For point $p \in \mathbb{P}^3 \setminus L$ form plane $\mathrm{Span}(L, p)$ and send it to $\mathrm{Span}(L,p) \cap L'$.

(x) e.g. $L = (x_2 = x_3 = 0)$, $L' = (x_0 = x_1 = 0)$,

$$\begin{array}{ccc} \mathbb{P}^3 \setminus L & \longrightarrow & \mathbb{P}^1 \\ [x_0, \ldots, x_3] & \longmapsto & [x_2, x_3] \end{array}$$

[Proof Exercise:
$$\begin{tikzcd} S \arrow[r, phantom, "\subset"] & \mathbb{P}^3 \arrow[r, dashed] & \mathbb{P}^1 \\ Bl_L(S) = \tilde{S} \arrow[u, "\cong"] \arrow[r, phantom, "\subset"] & Bl_L(\mathbb{P}^3) \arrow[u] \arrow[ur] & \end{tikzcd}$$
]

Q: How many K3 surfaces do we get in this way?
moduli?

(1) Quartics:

$$\dim H^0(\mathbb{P}^3, \mathcal{O}(4)) = \binom{4+3}{3} = \binom{7}{3} = \frac{7\cdot 6\cdot 5}{6} = 35$$

$$\#\overset{(expected)}{moduli} = \dim \mathbb{P}H^0(\mathbb{P}^3, \mathcal{O}(4)) - \dim PGL(4)$$

$$= 34 - 15$$

$$= 19.$$

$\dim H^0(\mathbb{P}^n, \mathcal{O}(d)) = \binom{n+d}{d}$

(2) (2,3)-CI in $\mathbb{P}^4$:

$$h^0(\mathcal{O}_{\mathbb{P}^4}(2)) = \binom{6}{2} = 15, \quad h^0(\mathcal{O}_{\mathbb{P}^4}(3)) = \binom{4+3}{3} = 35$$

$$\#\text{moduli} = \dim \mathbb{P}H^0(\mathcal{O}(2)) + \dim \mathbb{P}H^0(\mathcal{O}(3)) - \dim PGL(5) - 5$$

$$= 14 + 34 - 24 - 5$$

$$= 24 - 5$$

$$= 19$$

(−5) Since $H^0(S, \mathcal{O}(3)) = \mathbb{C}^{29}$, and $H^0(\mathbb{P}^4, \mathcal{O}(3)) \twoheadrightarrow H^0(S, \mathcal{O}(3))$, Each (2,3) K3 surface $S \subset \mathbb{P}^4$ is contained in a 5-dim'l family of cubic hypersurfaces.

Proof: Either by Riemann-Roch or elementary:

$$0 \longrightarrow \mathcal{O}_{D_g}(1) \longrightarrow \mathcal{O}_{D_g}(3) \longrightarrow \mathcal{O}_S(3) \longrightarrow 0.$$

$$0 \longrightarrow \mathcal{O}_{\mathbb{P}^4} \longrightarrow \mathcal{O}_{\mathbb{P}^4}(3) \longrightarrow \mathcal{O}_{D_g}(3) \longrightarrow 0$$

$$0 \longrightarrow \mathcal{O}_{\mathbb{P}^4}(-2) \longrightarrow \mathcal{O}_{\mathbb{P}^4}(1) \longrightarrow \mathcal{O}_{D_g}(1) \longrightarrow 0.$$

$$\Rightarrow H^0(\mathcal{O}_S(3)) = H^0(\mathcal{O}_{D_g}(3)) / H^0(\mathcal{O}_{D_g}(1))$$

$$= \mathbb{C}^{35} - \mathbb{C}^5 - \mathbb{C}. = \mathbb{C}^{29}$$

(4) $\#\text{moduli} = \dim H^0(\mathbb{P}^2 \times \mathbb{P}^1, \mathcal{O}(3,2)) - 1 - \dim PGL(3) \times PGL(2)$

$$= 30 - 1 - (8+3) = 18.$$

$\mathrm{Pic}(\mathbb{P}^2 \times \mathbb{P}^1) \subset \mathrm{Pic}(S).$

§ Torelli theorems for K3 surfaces

(1) Classical case of curves.

C smooth proper curve of genus g / $\mathbb{C}$

$\gamma_1, \dots, \gamma_g, \delta_1, \dots, \delta_g \in H_1(C, \mathbb{Z})$ symplectic basis $\begin{pmatrix} \gamma_i \cdot \gamma_j = 0 \\ \gamma_i \cdot \delta_j = +1 \\ \delta_i \cdot \delta_j = 0 \end{pmatrix}$

$\omega_1, \dots, \omega_g$ basis of $H^0(C, \Omega_C^1)$.

Period matrix:

$$\Pi = \underbrace{\left(\int_{\gamma_i} \omega_j \;\middle|\; \int_{\delta_i} \omega_j \right)}_{2g} \Big\} g$$

Rieman Bilinear relations:

$$\int_C \omega \cdot \omega' = 0 \overset{(*)}{\Rightarrow} \sum_i \left(\int_{\gamma_i} \omega \int_{\delta_i} \omega' - \int_{\delta_i} \omega \int_{\gamma_i} \omega' \right) = 0$$

$$i \cdot \int_C \omega \cdot \bar{\omega} > 0 \Rightarrow i \cdot \sum_i \left(\int_{\gamma_i} \omega \int_{\delta_i} \bar{\omega} - \int_{\delta_i} \omega \int_{\gamma_i} \bar{\omega} \right)$$

$$= 2 \, \mathrm{Im} \left(\int_{\gamma_i} \bar{\omega} \cdot \int_{\delta_i} \omega \right) > 0.$$

(*) This follows from

$$\int_C \omega \cdot \omega' = \int_{\Delta_C \subset C \times C} pr_1^*(\omega) \cdot pr_2^*(\omega')$$

and the Kunneth decomposition

$$[\Delta_C] = \sum_i (\gamma_i \otimes \delta_i - \delta_i \otimes \gamma_i) + p \otimes 1 + 1 \otimes p.$$

(If $\gamma_1, \dots, \gamma_g$ not basis of $H^0(C, \Omega_C)$, $\exists\, \omega$ s.t. $\int_{\gamma_i} \omega = 0$)
$\Rightarrow$ contradicting positivity.

~~Normalize~~:
Choose basis $\omega_1, \dots, \omega_g$ s.t. $\int_{\gamma_i} \omega_j = \delta_{ij}$

$$\Pi = \left(\mathbb{1} \mid \tau \right)$$

(1) $\Rightarrow$ $\tau^t = \tau$
(2) $\Rightarrow$ $\mathrm{Im}(\tau) > 0$

$$\tau \in \mathbb{H}_g = \left\{ \tau \in M(g \times g, \mathbb{C}) \;\middle|\; \begin{matrix} \tau^t = \tau \\ \mathrm{Im}(\tau) > 0 \end{matrix} \right\}.$$ Siegel upper half plane.

Change of basis of γ_i, δ_i by $g \in Sp(2g, \mathbb{Z})$ $\Rightarrow$ $\tau \longmapsto g \cdot \tau = (A\tau + B)(C\tau + D)^{-1}$

$$g = \begin{pmatrix} A & B \\ C & D \end{pmatrix}.$$

$$\Rightarrow \quad \tau(C) \in \left[\mathbb{H}_g / Sp(2g, \mathbb{Z}) \right]$$

Thm (Torelli, 1914)

The period mapping

$$\tau : \mathcal{M}_g \longrightarrow \left[\mathbb{H}_g / Sp(2g, \mathbb{Z}) \right]$$

is injective.

(2) Case of K3 surfaces.

S K3 surface : $\Lambda^2\Omega_S \cong \mathcal{O}_S$, $H^1(S,\mathcal{O}_S)=0$

$$\Downarrow$$

$$H^0(\Lambda^2\Omega_S) = \mathbb{C}\sigma_S \ , \ \underbrace{\sigma_S : \mathcal{O}_S \xrightarrow{\sim} \Lambda^2\Omega_S}_{\text{symplectic}}.$$

$\Rightarrow$ $\sigma_S \in H^0(\Lambda^2\Omega_S) \subset H^2(S,\mathbb{C})$

$\Rightarrow$ $[\sigma_S] \in \mathbb{P}H^2(S,\mathbb{C})$ distinguished element.

Thm A (Torelli thm) ~~polar~~

Let S, S' be K3 surfaces $/\mathbb{C}$.
Let $\varphi: H^2(S,\mathbb{Z}) \longrightarrow H^2(S',\mathbb{Z})$ isometry (wrt. intersection pairing) s.t.

$$\varphi([\sigma_S]) = [\sigma_{S'}].$$

Then $S \cong S'$.
If moreover, $\varphi(H) = H'$ for ample classes $H \in Pic(S)$, $H' \in Pic(S')$
then $\exists$ unique $f: S' \to S$ s.t. $f^* = \varphi$.

Cor Let $f_1, f_2 \in Aut(X)$ s.t. $f_1^* = f_2^*$ $\Rightarrow$ $f_1 = f_2$.

Surjectivity of Period map

$\Lambda := E_8(-1)^{\oplus 2} \oplus U^{\oplus 3}$ rank 22 lattice.

$\ni v_d$ primitive, $v_d^2 = d$.

$\Lambda_d := v_d^{\perp}$.

$$\mathcal{D} = \left\{ x \in \mathbb{P}(\Lambda_d) \;\middle|\; \begin{array}{l} x \cdot x = 0 \\ x \cdot \bar{x} > 0 \end{array} \right\}.$$

$\circlearrowright$

$$\tilde{O}(\Lambda_d) = \{ g \in O(\Lambda_d) \mid g \text{ acts trivially on } \Lambda_d^*/\Lambda_d \}.$$

$\mathcal{M}_d$ = moduli space of polarized K3 surfaces of degree d.

$\downarrow \text{Per}$

$[\tilde{O}(\Lambda_d) \backslash \mathcal{D}]$ quotient stack.

__Thm__ The Period mapping

$$\mathcal{M}_d \longrightarrow [\tilde{O}(\Lambda_d) \backslash \mathcal{D}]$$

is an open embedding, with complement

$$\tilde{O}(\Lambda_d) \Big\backslash \left(\bigcup_{\substack{\alpha \in \Lambda \\ \alpha \cdot \alpha = -2}} \alpha^{\perp} \right).$$