Hilbert schemes and applications

Seminar, Universität Bonn, Sommer 2020

Organizer: Georg Oberdieck

Friday 2-4, Room 1.008

The *Hilbert scheme* of a projective variety X is the scheme that parametrizes all of its closed subschemes. The (closed) points of the Hilbert scheme are in 1-to-1 correspondence with the closed subschemes if X. A *family of closed subscheme* parametrized by a Noetherian scheme T is a closed subscheme of the product $X \times T$ which is flat over T.

By a fundamental result of Grothendieck [5] the Hilbert scheme exists and (if we fix the Hilbert polynomial) is a projective scheme over the ground field. In other words each component of the Hilbert scheme can be described as the zero locus of a finite collection of polynomials in projective space. This makes the Hilbert scheme an object which is unique to algebraic geometry. Parallel constructions in other fields of geometry usually yield an object which are too large to study. For example, the 'space' of all smooth submanifolds of a given compact smooth manifold is infinite-dimensional.

The goal of this seminar is to understand the proof of existence of the Hilbert scheme, explore its properties and demonstrate some of its many applications.

The most basic example of the Hilbert scheme is the Grassmannian which parametrizes all linear subspaces of projective space. As the Grassmannians the Hilbert schemes for general projective varieties have often rich and beautiful geometry with lots of applications. We will see some of these. For example the Hilbert scheme of zero-dimensional subschemes of length n on a surface is a smooth projective variety of dimension $2n$. One can construct geometrically actions of Kac-Moody Lie algebras (infinite-dimensional Lie algebras) on their cohomology. If the surface is symplectic then the Hilbert scheme of points on it is also symplectic. This yields one of the few examples of a Hyperkähler variety. More generally the topological invariants of Hilbert schemes are a very interesting object to study. For example for singular plane curves the topological Euler characteristic of their Hilbert schemes have relations to knot invariants. For curves in threefolds the Euler characteristics are related to Donaldson–Thomas theory and physics.

But contrary to the above examples the geometry of the Hilbert scheme can also behave quite badly in general. In fact it can be as bad as possible: By a result of Vakil the Hilbert schemes of curves in projective space satisfy Murphy's law, i.e. every singularity type of finite type over $\mathbb{Z}$ can appear on that moduli space.

The first half of the seminar will concern the existence of the Hilbert scheme and requires background from algebraic geometry about flatness, sheaf cohomology, the Hilbert polynomial and the Grassmannians. The second half is geared towards examples; a general algebraic geometry background is assumed.

The schedule for the talks is as follows:

Talk 1. April 24: Hilbert functors (Patrick Seifner) In the first part of the talk we introduce the Hilbert and Quot scheme functors $\mathfrak{H}ilb(X)$ and $\mathfrak{Q}uot(X)$ where X is a projective variety. We also recall the Hilbert polynomial and that it is locally constant in flat families. From this we conclude that there is a decomposition of the Hilbert and Quot scheme functors as a coproduct according to Hilbert polynomial,

$$\mathfrak{H}ilb(X) = \bigsqcup_P \mathfrak{H}ilb_P(X).$$

Our goal in the next lectures is to show that these functors are representable. Explain that the projective space and the Grassmannian are examples of Hilbert schemes and Quot schemes (we assume the universal property is known). The reference for this section is Section 3 of [17]; for the definition of the quot scheme functor see also Section 9. A further reference is [2, Section 5]. In the second part follow the discussion about m-regularity in Section 4 of [17] (Defn. 4.1, Prop 4.2 and 4.3). If there is time left, discuss Remark 4.5, i.e. that the m-regularity bound can be extremely large even in reasonable cases. See also [10, Lecture 14].

Talk 2. May 8: Flattening stratification (Thibaud van den Hove) Discuss [10, Lecture 8] (Further references [17, 2]).

Talk 3. May 15: Proof of Existence (Bence Hevesi) Detailed discussion of Section 8 of [17]. Explain how this generalizes to the existence of the Quot scheme. Further possible topics: Grothendieck's original proof (see [2]). Existence of scheme of morphisms $X \to Y$ [2, Thm 5.23].

A great reference for the proof of existence is also Lecture 15 in [10] (Mumford is interested there in the case of curves on surfaces but also discusses the general case).

Talk 4. May 22: A counter-example to the integral Hodge conjecture (Jonas Baltes)
Follow Section 2 of [16], see also Kollár in [18, page 134].

Talk 5. May 29: Local structure (Solomiya Mizyuk)
The main goal of the talk is to describe the tangent space of the Hilbert scheme: The tangent space to a point $[Z] \in \mathsf{Hilb}(X)$ is

$$T_{[Z]}\mathsf{Hilb}(X) = \mathrm{Hom}_{\mathcal{O}_X}(\mathcal{I}_Z, \mathcal{O}_Z).$$

Explain how this generalizes the description of the tangent space to the Grassmannian. Reference is [7, Thm 3.2]. Discuss the example of Hilbert scheme of n points of a d-dimensional smooth variety. Show it is smooth if $d = 1, 2$ but singular if $d \geq 3$ and $n \geq 3$. Prove the smoothness for $d = 2$ along [7, Thm.3.3]. For a non-smooth point in the Hilbert scheme take a power of the maximal ideal in $\mathbb{C}^d$, see [9, Cor.18.30].

In the second half of the talk explain the statement of Murphy's law about Hilbert schemes: Every singularity type can appear in the Hilbert scheme of curves in projective space, see [19]. Then present Mumford's example of a curve in $\mathbb{P}^3$ where the corresponding point in the Hilbert scheme is non-reduced [11, Part II]. (You can choose freely in what depth to present this example.)

Talk 6. June 5: Hilbert scheme of points of surfaces (Mingyu Ni) Recall from the last lecture that the Hilbert scheme of points of curves and surfaces is smooth, while it becomes singular for higher-dimensional varieties. For a curve it is precisely the symmetric product, which is smooth, while in the surface case it is a desingularization of the symmetric product, [7, Sec.3]. The Hilbert scheme of 2 points on a surface S is the quotient $Bl_\Delta(S \times S)/\mathbb{Z}_2$.

The goal of this talk is to prove Göttsche's formula that states that if S is a smooth surface (over $\mathbb{C}$) then the generating series of its topological Euler characteristic is given by

$$\sum_{n=0}^{\infty} e(\mathsf{Hilb}_n(S))q^n = \left(\prod_{n\geq 1} \frac{1}{1-q^n}\right)^{e(S)}.$$

Start with the case $S = \mathbb{C}^2$ and use the $\mathbb{C}^*$-action to compute the left hand side in terms of the number of partitions. This is elementary combinatorics. For the general case we reduce to the local case by a cut and paste-arguments. More generally, state the formula for the Betti numbers and Hodge numbers of the Hilbert schemes. An elementary proof (using the Hilbert-Chow morphism) can be found in [3]. You can choose freely in what depth the results of that paper are covered.

Talk 7. June 12: The cohomology of the Hilbert scheme of points of a surface and the Heisenberg algebra (Liao Wang)

The goal of this talk is to present the structure of the cohomology of the Hilbert scheme of points $\mathsf{Hilb}_n(S)$ (also denoted by $S^{[n]}$) of a smooth projective surface. The rough idea is to consider all the cohomologies simultaneously,

$$H^*(\mathsf{Hilb}\, S) = \bigoplus_{n\geq 0} H^*(\mathsf{Hilb}_n S, \mathbb{Q})$$

and then use geometric correspondences to construct an action of the Heisenberg algebra which makes $H^*(\mathsf{Hilb}\, S)$ an irreducible module. Concretely, follow the paper [12] resp. the book [13] of Nakajima. First define the Nakajima correspondences $S^{[n,n+i]} \subset S^{[n]} \times S^{[n+i]} \times S$ and the operators $\mathfrak{q}_i(\alpha)$. Then follow Nakajimas work to prove the commutation relations. Finally using Göttsches formula conclude that the representation is irreducible.

This talk creates an interesting link between the geometry of Hilbert schemes of points and representation theory.

Talk 8. June 19: Hilbert scheme of plane curve singularities and knot invariants (Maximilian Schimpf) Let C be an reduced complex curve embedded in a smooth surface. In analogy with Göttsche's formula we can form the generating series

$$P_C(t) = \sum_{n\geq 0} e(\mathsf{Hilb}_n(C))q^n.$$

If C is a smooth curve, then the right hand side is well known by MacDonald's formula. For C singular, the right hand side is related to the HOMFLY polynomial of the knots associated to the singularities of the curve. The goal of the talk is to prove a structure result for the series $P_C(t)$ and explain the statement of the above relation.

Concretely, begin by defining $P_C(t)$ and compute it in the case of a smooth curve, a curve with a node, and a cusp (Example 3,4 and 5 in [14] and the first page where it is explained how to reduce to the local situation.

Beware: There is a paper with a very similar title about the HOMFLY homology). Then state the structure result of Proposition 12 of [14] and give its proof which can be found after Theorem 13. This will take most of the class. In the remainder state and explain Conjecture 1 of [14]. This gives a topological interpretation of the numbers $n_{g,h}$. If time permits also state Conjecture 2.

Remark. Conjecture 1 and 2 in [14] was eventually proven in [8]. These conjectures are the starting point of a much more general story that relates categorified knot invariants to sheaves on the Hilbert scheme of points on the plane, see the work of Negut, Oblomkov and many others.

Talk 9. June 26: Göttsche Conjecture (Mauro Varesco)

In this talk we discuss Göttsche's formula for the number of δ-nodal curves in a linear system of dimension δ on an algebraic surface. The formula was conjectured in [4] and a short proof using the Hilbert scheme geometry was found in [6]. First explain the formula in [4] and then relate it to the Hilbert schemes following sections 2 and 3 of [6]. The final part of the proof (Section 4) is considered in the next section.

Talk 10. July 3: Tautological integrals over the Hilbert scheme (Denis Nesterov)

The goal of this talk is to prove a universality results for integrals of tautological classes over the Hilbert scheme of points of a smooth surface following the paper of Ellingsrud, Göttsche, Lehn [1], in particular Section 3. We apply the result to prove Göttsche's conjecture in [6, Section 4].

Talk 11. July 10: Curves on threefolds: Virtual classes and Donaldson–Thomas theory (Georg Oberdieck)

The Hilbert scheme of points on a surface is smooth and its geometry can be controlled in many instances. This had useful applications to curves in linear systems on surfaces (this is not surprising since a curve on a surface is just a divisor which can be twisted away). In this talk we go beyond surfaces and consider Hilbert schemes of curves on threefolds. It turns out that in this case the Hilbert scheme can be quite singular (Talk 4) but still in a controlled way. It is what is called *virtually smooth*.

Concretely, we discuss deformation and obstruction spaces of the Hilbert scheme, and from this data construct a perfect obstruction theory and a virtual class. This leads to the definition of Donaldson-Thomas invariants. An introductory reference is [15].

Remark. Further topics that may be discussed:

1. Construction of moduli space of sheaves from Quot schemes (after the book of Huybrechts and Lehn)

2. The Hilbert scheme of points of a K3 surface as a hyper-Kähler variety (after Beauville)

3. Hilbert schemes and Quiver varieties (following an article of Kuznetsov)

4. The Chow motive of the Hilbert scheme of points of a surface (after an article of de Cataldo and Migliorini)

5. Abstract Hilbert schemes, i.e. Hilbert schemes of objects in general abelian categories (satisfying some properties), following a paper by Artin and Zhang.

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