Pandharipande–Thomas theory of $X = \mathrm{Tot}_S(\mathcal{L})$

S smooth proj surface.

$\mathcal{L} \in \mathrm{Pic}(S)$

$$X = \mathrm{Tot}(\mathcal{L}) = \mathrm{Spec}(\mathrm{Sym}^{\bullet}(\mathcal{L}^{-1}))$$
$$= \mathrm{Spec}_S(\underbrace{\bigoplus_{i \geq 0} \mathcal{L}^{-i}}_{\text{graded}})$$

$\Rightarrow \mathbb{C}^* \curvearrowright X$ by fiberwise scaling.

$$H_2(X, \mathbb{Z}) \cong H_2(S, \mathbb{Z}) \quad , \; i: S \hookrightarrow X \text{ zero section}$$
$$i_*(\beta) \longleftarrow\!\!| \; \beta$$

$$\int_{[P_n(X,\beta)]^{vir}} \prod \tau_{k_i}(\gamma_i) := \int_{[P_n(X,\beta)^{\mathbb{C}^*}]} \left(\prod \tau_{k_i}(\gamma_i)\Big|_{\text{Fix Com}} \right) \cdot \frac{1}{e(N^{vir})}$$

Q: How to evaluate?

Last time (Maulik–Kool?):

$$P_n(X,\beta)^{\mathbb{C}^*} = \bigsqcup_{\substack{r \geq 0 \\ \sum_{i=0}^{r} (n_i - (r-i)N + 1 - g_i) = n \\ \sum_{i=0}^{r} (i+1)\beta_i = \beta}} S^{[n_0, \dots, n_r, 0]}_{\beta_0, \dots, \beta_r}$$

$$S^{[n_0, \dots, n_r]}_{\beta_1, \dots, \beta_r} = \{ (D_1, \dots, D_r, Z_0, \dots, Z_r) \in \mathrm{Hilb}_{\beta_1} \times \dots \times \mathrm{Hilb}_{\beta_r} \times S^{[n_0]} \times \dots \times S^{[n_r]} \mid I_{Z_i}(-D_i) \subset I_{Z_{i-1}} \}$$

What is ~~[~~ fixed invt class $[P_n(X,\beta)^{\mathbb{C}^*}]^{vir}$?

virtual normal bundle $\frac{1}{e(N^{vir})}$

deformation

under this isomorphism ?

Easiest case today : $r = 0$, no nesting.

$$\cancel{P_n(X,\beta)^{\mathbb{C}^*} = P_n(S,\beta) \sqcup (\text{nested terms})}$$

$$\cancel{\cong}$$

$$\mathrm{Hilb}$$

Last time:

$$\cancel{P_n(S,\beta) \cong}$$

$$Z_0 \subset D_1$$

$$S^{[n_0,\beta]}_{\beta_0} = \{ (D_1, Z_0, \phi) \mid \mathcal{O}_S(-D_1) \subset I_{Z_0} \}$$

$$= \mathrm{Hilb}^{n+g-1}(\mathcal{C}/\mathrm{Hilb}_\beta)$$

$\beta_0 = \beta$

$n_0 + 1 - g = n$

Directly: $P_n(X,\beta)^{\mathbb{C}^*} = P_n(S,\beta) \sqcup$ nested stuff.

$$\cong$$

$$\mathrm{Hilb}^{n+g-1}(\mathcal{C}/\mathrm{Hilb}_\beta)$$

P.–

Fixed Obstruction theory:

$$\begin{cases} I^\circ = [\mathcal{O}_X \longrightarrow F] \\ E^\circ = \left(R\mathcal{H}om(I, I)_0[1]\right)^\vee \\ T^{vir} = R\mathcal{H}om(I, E)_0[1]. \end{cases}$$

Need to ~~[illegible]~~ understand $(E^\circ)^{fix}|_{P_n(X,\beta)^{\mathbb{C}^*}}$.

Idea for $P_n(S,\beta)$:

$$\begin{tikzcd} {[}\ \mathcal{L}^{-1} \otimes \mathfrak{t}^{-1} \otimes \mathcal{O}_X \arrow[r] \arrow[d] & 0\ {]} \arrow[d] & \\ {[}\ \mathcal{O}_X = \bigoplus_{i \geq 0} \mathcal{L}^{-i} \otimes \mathfrak{t}^{-i} \arrow[r] \arrow[d] & i_*(F)\ {]} \arrow[d] & = I^\circ \\ {[}\ i_*\mathcal{O}_S \arrow[r] & i_*(F)\ {]} & = i_* I_S. \end{tikzcd}$$

$\Rightarrow$ $\cancel{\mathcal{L}}\ \pi^*(\mathcal{L}^\vee) \otimes \mathfrak{t}^{-1} \longrightarrow I^\circ \longrightarrow i_*(I_S)$.

also: $I^\circ \longrightarrow \mathcal{O}_X \longrightarrow i_* F$.

~~Apply (2) to (1)~~ Apply (1) to (2) termwise:

$$\begin{tikzcd}
R\mathcal{H}om(\pi^*\mathcal{L}^\vee\otimes t^{-1}, I^\bullet) \arrow[r] & \mathcal{H}om(\pi^*\mathcal{L}^\vee\otimes t^{-1}, \mathcal{O}_X) \arrow[r] & R\mathcal{H}om(\pi^*\mathcal{L}^\vee\otimes t^{-1}, i_*F) \\
R\mathcal{H}om(I^\bullet, I^\bullet) \arrow[r] \arrow[u] & R\mathcal{H}om(I^\bullet, \mathcal{O}_X) \arrow[r] \arrow[u] & R\mathcal{H}om(I^\bullet, i_*F) \arrow[u] \\
R\mathcal{H}om(i_*I_S, I^\bullet) \arrow[r] \arrow[u] & R\mathcal{H}om(i_*I_S, \mathcal{O}_X) \arrow[r] \arrow[u] & R\mathcal{H}om(i_*\mathcal{I}_S, i_*F) \arrow[u]
\end{tikzcd}$$

Take fixed pt:

$$R\mathcal{H}om(\pi^*\mathcal{L}^\vee\otimes t^{-1}, \mathcal{O}_X)^{fix} = \cancel{\mathcal{H}om}(RH^\bullet(X, \pi^*\mathcal{L})\otimes t)^{fix}$$

$$= \left(R\Gamma(S, \bigoplus \mathcal{L}^{-i} t^{-i}\otimes\mathcal{L})\otimes t\right)^{fix}$$

$$= R\Gamma(S, \mathcal{O}_S) = \cancel{\ }.$$

$$R\mathcal{H}om(\pi^*\mathcal{L}^\vee\otimes t^{-1}, i_*F) = R\mathcal{H}om(\pi^*\mathcal{L}^\vee, i_*F)\otimes t$$

$$= R\Gamma(S, i_*F\otimes\mathcal{L})\otimes t$$

fixed pt $= 0$.

$$R\mathcal{H}om(i_*I_S, \mathcal{O}_X) = R\mathcal{H}om(\mathcal{O}_X, i_*I_S\otimes\omega_X)^\vee$$

$$= R\Gamma(S, I_S\otimes\omega_S\otimes\mathcal{L}^\vee)^\vee\otimes(t^{-1})^\vee$$ fixed pt $=0$.

$\Omega_X = ?$ $\quad 0 \to \pi^*\mathcal{L}\otimes t \to T_X \to \pi^*T_S \to 0$

$$0 \to \pi^*\Omega_S \to \Omega_X \to t^{-1}\otimes\pi^*\mathcal{L}^\vee \to 0$$

$$\omega_X = \omega_S\otimes\mathcal{L}^\vee\otimes t^{-1}$$

$$R\mathcal{H}om_X(i_*I_S, i_*F)^{fr} \cong R\mathcal{H}om_S(i^*i_*I_S, F)^{fr} \cong R\mathcal{H}om_S(I_S, i_*F)$$

$$\underbrace{I_S \otimes N^\vee_{S/X}[1]}_{I_S \otimes \mathcal{L}^\vee \otimes \mathcal{E}^{-1}[1]} \longrightarrow i^*i_*I_S \to I_S.$$

[Fact (Huybrechts book, 11.4) $i: Y \hookrightarrow X$ smooth hyp. surf., $\mathcal{F} \in D^b(Y)$.

$$\mathcal{F} \otimes N^\vee_{Y/X}[1] \longrightarrow j^*j_*\mathcal{F} \longrightarrow \mathcal{F} \longrightarrow \cdot[1]$$]

Upshot:

$$R\mathcal{H}om_S(I_S, F)[-1] \longrightarrow R\mathcal{H}om_X(I^\cdot, I^\cdot)^{fr}_{P_n(S,\beta)} \longrightarrow R\Gamma(S, \mathcal{O}_S)$$

Fixed perfect obs. thy:

$$\tilde{E}^\bullet = R\mathcal{H}om_S(I_S, F)^\vee \longrightarrow \mathbb{L}_{P_n(S,\beta)}.$$

Problem Find embedding: $P_n(S,\beta) \xhookrightarrow{j}$ smooth

+ description $j_*[P_n(S,\beta)]^{vir}$

Assumption: $H^2(S, \mathcal{O}_S) = 0$.

$$P_{n+1-g}(S,\beta) \xrightarrow{\cong} \mathrm{Hilb}^n(\mathcal{C}/H_\beta^{\mathrm{Hilb}})$$

Embeddings:

$$\mathrm{Hilb}^n(\mathcal{C}/H_\beta) \hookrightarrow S^{[n]} \times \mathrm{Hilb}_\beta \xhookrightarrow{\quad} \cancel{S^{[n]} \times \mathrm{Hilb}_{\beta+LA]}}$$

$$\hookrightarrow S^{[n]} \times \mathrm{Hilb}_\beta.$$

§1

Lemma

(a) Hilb_β carries a natural perfect obstruction theory $F^\bullet \longrightarrow \mathbb{L}_{\mathrm{Hilb}_\beta}$ with

$$(F^\bullet)^\vee = R\pi_*(\mathcal{O}_\mathcal{C}(\mathcal{C}))$$

$$\begin{array}{ccc} \mathcal{C} \subset & \mathrm{Hilb}_\beta \times S \\ & \downarrow \pi \\ & \mathrm{Hilb}_\beta \end{array}$$

(b) If $H^1(S,L) = 0$ for all line bundles L with $c_1(L) = \beta$, then Hilb_β is smooth.

Proof: (b)

$C \subset S$, $\quad T_{[C]} \mathrm{Hilb}_\beta = \mathrm{Hom}(C, N_{C/S})$

$= H^0(C, \mathcal{O}_C(C))$.

$$0 \longrightarrow \mathcal{O}_S \longrightarrow \mathcal{O}_S(C) \longrightarrow \mathcal{O}_C(C) \longrightarrow 0$$

$$\Rightarrow 0 \longrightarrow H^0(S,\mathcal{O}_S) \longrightarrow H^0(S, \mathcal{O}_S(C)) \longrightarrow H^0(\mathcal{O}_C(C))$$

$$\longrightarrow H^1(S,\mathcal{O}_S) \longrightarrow \underbrace{H^1(S,\mathcal{O}_S(C))} \longrightarrow H^1(C, \mathcal{O}_C(C))$$

$$\longrightarrow \cancel{H^2(S,\mathcal{O}_S)} \longrightarrow \quad " \quad \longrightarrow 0$$

~~Part 2: How to calculate the expected [illegible]~~

By assumption, $H^1(S, \mathcal{O}(C)) = 0$, so $\dim T_{[C]} \mathrm{Hilb}_\beta$ constant.

$$\begin{array}{ccc} \mathrm{Hilb}_\beta & & \mathcal{L}_{univ} \\ \downarrow & & \\ \mathrm{Pic}_\beta & \xleftarrow{\pi} & \mathrm{Pic}_\beta(S) \times S \end{array}$$

By assumption: $R^1\pi_* \mathcal{L}_{univ} = R^2\pi_* \mathcal{L}_{univ} = 0$.

$\Rightarrow \pi_* \mathcal{L}_{univ}$ vector bundle.

$$\Rightarrow \quad \mathbb{P}(\pi_* \mathcal{L}_{univ}) \longrightarrow \mathrm{Hilb}_\beta \qquad \text{set-theoretic bij.}$$
$$(L, s) \longmapsto [V(s)] \qquad \text{isom. on tgt space.}$$

$\Rightarrow$ <u>isom</u>. $\square$

(a) • Dend Hilbert scheme.

• Obs: ~~Hilb~~ let $A \subset S$ ~~[illegible]~~ sufficiently ample s.t. $H^1(S, \mathcal{L}) = 0$ for all $c_1(\mathcal{L}) = \beta + [A]$.

$$\mathrm{Hilb}_\beta \longrightarrow \mathrm{Hilb}_{\beta + [A]} = \mathrm{Hilb}_\gamma, \quad \gamma = \beta + A.$$
$$C \longrightarrow C + A.$$

Hilb_β cut out from $\mathrm{Hilb}_{\beta + A}$ by vector bundle.

$$\cancel{\mathcal{D}} \qquad \cancel{\mathcal{O}(\mathcal{D})} \; H^0(\mathcal{O}(\mathcal{D}|_A)) \qquad (*)$$

$$\mathcal{D} \subset \mathrm{Hilb}_\gamma \times S \qquad \mathcal{O} \xrightarrow{s_\mathcal{D}} \mathcal{O}(\mathcal{D}) \longrightarrow \mathcal{O}(\mathcal{D})|_{\mathrm{Hilb}_\gamma \times A} \quad \Big| \pi$$

$$\downarrow$$

$$\mathrm{Hilb}_\gamma \qquad \Rightarrow \mathrm{Hilb}_\beta = V\left(\mathcal{O} \longrightarrow \pi_*\left(\mathcal{O}(\mathcal{D})|_{\mathrm{Hilb}_\gamma \times A}\right)\right). \quad \square$$

§2: Embedding $\mathrm{Hilb}^n(\mathcal{C}/\mathrm{Hilb}_\beta) \hookrightarrow S^{[n]} \times \mathrm{Hilb}_\beta$

$(Z \subset C) \mapsto (Z, C)$

<u>Idea</u>: $Z \subset C \iff$ Section $s \in H^0(S, \mathcal{O}(C))$ vanishes on Z.

$\iff s|_Z = 0$ in $H^0(Z, \mathcal{O}_Z(C))$

$$\mathcal{C} \subset S^{[n]} \times \mathrm{Hilb}_\beta \times S \supset \mathcal{Z}$$

pulled back from $\mathrm{Hilb}_\beta \times S$

pulled back from $S^{[n]} \times S$.

$$\downarrow$$

$$S^{[n]} \times \mathrm{Hilb}_\beta$$

$$0 \to \mathcal{O} \xrightarrow{s_{\mathcal{C}}} \mathcal{O}(\mathcal{C}) \longrightarrow \mathcal{O}(\mathcal{C})|_{\mathcal{Z}} \to 0$$

univ. section

$\bar{s}_{\mathcal{C}}$

$\Rightarrow \mathrm{Hilb}^n(\mathcal{C}/S_\beta) = $ Zero locus of $\pi_*(\bar{s}_{\mathcal{C}}) : \mathcal{O} \longrightarrow \pi_*(\mathcal{O}(\mathcal{C})|_{\mathcal{Z}})$

$= \mathcal{O}(\mathcal{C})^{[n]}$

$\Rightarrow$ Relative perfect obstruction theory:

$$\mathcal{E}^\bullet \longrightarrow \mathbb{L}_{\mathrm{Hilb}^n(\mathcal{C}/S_\beta)/\mathrm{Hilb}_\beta}, \quad (\mathcal{E}^\bullet)^\vee = [T_{S^{[n]}} \longrightarrow \mathcal{O}(\mathcal{C})^{[n]}]$$

given by

$$\begin{array}{ccc} \mathcal{E}^\bullet = [(\mathcal{O}(\mathcal{C})^{[n]})^* & \longrightarrow & \Omega_{S^{[n]}}] \\ \downarrow & & \downarrow \\ \mathbb{L}_{\mathrm{Hilb}^n(\mathcal{C}/\mathrm{Hilb}_\beta)/\mathrm{Hilb}_\beta} = [I/I^2 & \longrightarrow & \Omega_{S^{[n]}}|] \end{array}$$

Thm (Kool–Thomas – Pandharipande) ~~Let $[P_n(S,\beta)]^{vir}$~~

The pushforward of virtual class associated to fixed p.o.t.

$$E^\bullet = R\mathcal{H}om(I_S, F)^\vee \longrightarrow \mathbb{L}_{P_n(S,\beta)}$$

is

$$j_*[P_n(S,\beta)]^{vir} = j_*\Big(c_{top}\big(\mathcal{O}(C)^{[n]}\big) \cap [S^{[n]}] \times [H_\beta]^{vir}\Big)$$

$$= c_{top}\big(\mathcal{O}(\mathcal{D}-A)^{[n]}\big) \cdot c_{top}\big(\pi_*(\mathcal{O}(\mathcal{D})|_A)\big)$$

$$\in A_*\big(S^{[n]} \times H_{\beta+[A]}\big)$$

<u>Main step</u>: Compare $(E^\bullet)^\vee = R\mathcal{H}om(I_S, F)$

with $(F^\bullet)^\vee = R\pi_*(\mathcal{O}_C(C)) = T^{vir}_{Hilb_\beta}$

and $(\mathcal{E}^\bullet)^\vee = [T_{S^{[n]}} \longrightarrow \mathcal{O}(C)^{[n]}] = T^{vir}_{Hilb^n(C/S_\beta)/H_\beta}$

<u>Prop</u>: We have a distinguished triangle

$$(\mathcal{E}^\bullet)^\vee \longrightarrow (E^\bullet)^\vee \longrightarrow (F^\bullet)^\vee$$

$$\Big(\overset{\shortparallel}{} \quad T^{vir}_{Hilb^n(C/H_\beta)/H_\beta} \longrightarrow T^{vir}_{P_n(S,\beta)} \longrightarrow T^{vir}_{Hilb_\beta} \Big)$$

<u>Rmk</u>: In fact we have

~~$(\mathcal{E}^\bullet)^\vee \longrightarrow (E^\bullet)^\vee$~~

$$\begin{tikzcd}
F^\bullet \arrow[r] \arrow[d] & E^\bullet \arrow[r] \arrow[d] & \mathcal{E}^\bullet \arrow[d] \\
\mathbb{L}_{Hilb_\beta} \arrow[r] & \mathbb{L}_{P_n(S,\beta)} \arrow[r] & \mathbb{L}_{Hilb^n(C/Hilb_\beta)/Hilb_\beta}
\end{tikzcd}$$

We need two computations: $\quad I^\bullet = [\mathcal{O}_S \longrightarrow F]$

$$(i_* F)^\vee = R\mathcal{H}om(i_* F, \mathcal{O}_S)$$

$$= R\mathcal{H}om(i_* F, \omega_S) \otimes \omega_S^\vee$$

$$= i_* R\mathcal{H}om(F, \omega_C[-1]) \otimes \omega_S^\vee \quad , \omega_C = \omega_S(C)|_C .$$

$$= i_* R\mathcal{H}om_C(F, \mathcal{O}_C) \otimes \mathcal{O}(C)[-1]$$

$$= i_*(I_Z)(C)[-1]$$

Note: $0 \longrightarrow \mathcal{O}_C \longrightarrow F \longrightarrow Q \longrightarrow 0 \qquad / \mathcal{H}om(-, \mathcal{O}_C)$

$$0 \to F^* \longrightarrow \mathcal{O}_C \longrightarrow \mathcal{E}xt^1(Q, F) \longrightarrow 0$$

$I_{Z \subset C} =: I_Z$

$$\boxed{\mathcal{E}xt^i(F, \mathcal{O}_C) = 0 \quad i > 0.}$$

Lemma

$$(I^\bullet)^\vee = \mathcal{J}_Z \otimes \mathcal{O}(C).$$

Proof

$$\begin{tikzcd}
\mathcal{O}(-C) \arrow[r] \arrow[d] & \mathcal{O}_S \arrow[r] \arrow[d, equal] & i_* \mathcal{O}_C \arrow[d] \\
I^\bullet \arrow[r] & \mathcal{O}_S \arrow[r] & F
\end{tikzcd} \qquad \Big| \; (\;)^\vee$$

$$\begin{tikzcd}
\cancel{i_*(\mathcal{O}_C)(C)[-1]}\, 0 \arrow[r] & \mathcal{O}_S \arrow[r] & \mathcal{O}(C) \arrow[r] & i_*(\mathcal{O}_C(C)) \arrow[r] & 0 \\
\cancel{i_*(I_Z)(C)[-1]}\, 0 \arrow[r] & \mathcal{O}_S \arrow[r] \arrow[u, equal] & (I^\bullet)^\vee \arrow[r] \arrow[u, hook] & i_*(I_Z)(C) \arrow[r] \arrow[u, hook] & 0
\end{tikzcd}$$

Snake lemma:

$$0 \to (I^\bullet)^\vee \longrightarrow \mathcal{O}(C) \longrightarrow \mathcal{O}_Z(C) \longrightarrow 0.$$

$$\Rightarrow (I^\bullet)^\vee = \bigoplus \mathcal{J}_Z \otimes \mathcal{O}(C).$$

where $\mathcal{J}_Z \subset \mathcal{O}_S$ ideal shf of $Z \subset S$. $\square$

Proof of Prop:

$$(E^\bullet)^\vee = R\mathcal{H}om(I_\bullet, F)$$

$$= R\mathcal{H}om(F^\vee, I_\bullet^\vee)$$

$$= R\mathcal{H}om\big((i_* I_Z) \otimes \mathcal{O}(C)[-1], \mathcal{J}_Z \otimes \mathcal{O}(C)\big)$$

$$= R\mathcal{H}om_S(i_* I_Z, \mathcal{J}_Z)[1]$$

Two exact sequences:

$$0 \longrightarrow \mathcal{J}_Z \longrightarrow \mathcal{O}_S \longrightarrow \mathcal{O}_Z \longrightarrow 0.$$

$$0 \longrightarrow i_* I_Z \longrightarrow i_* \mathcal{O}_C \longrightarrow \mathcal{O}_Z \longrightarrow 0.$$

$$\begin{tikzcd}
R\mathcal{H}om(i_*\mathcal{O}_C, i_*\mathcal{O}_Z)[-1] \arrow[r] \arrow[d] & R\mathcal{H}om(i_*\mathcal{O}_C, \mathcal{J}_Z) \arrow[r] \arrow[d] & R\mathcal{H}om(i_*\mathcal{O}_C, \mathcal{O}_S) \arrow[d] \\
R\mathcal{H}om(i_* I_Z, \mathcal{O}_Z)[-1] \arrow[r] \arrow[d] & R\mathcal{H}om(i_* I_Z, \mathcal{J}_Z) \arrow[r] \arrow[d] & R\mathcal{H}om(i_* I_Z, \mathcal{O}_S) \arrow[d] \\
R\mathcal{H}om(i_*\mathcal{O}_Z, i_*\mathcal{O}_Z) \arrow[r] & R\mathcal{H}om(i_*\mathcal{O}_Z, \mathcal{J}_Z)[1] \arrow[r] & R\mathcal{H}om(i_*\mathcal{O}_Z, \mathcal{O}_S)[1]
\end{tikzcd}$$

- $R\mathcal{H}om(i_*\mathcal{O}_C, \mathcal{O}_S) = ?$

$0 \to \mathcal{O}_S(-C) \longrightarrow \mathcal{O}_S \longrightarrow \mathcal{O}_C \longrightarrow 0 \quad | \quad \mathcal{H}om(-, \mathcal{O}_S)$

$\Rightarrow 0 \to \mathcal{O}_S \longrightarrow \mathcal{O}_S(C) \longrightarrow \mathcal{E}xt^1(\mathcal{O}_C, \mathcal{O}_S) \longrightarrow 0$

$$\Rightarrow \begin{cases} \mathcal{E}xt^2(\mathcal{O}_C, \mathcal{O}_S) = 0 \\ \mathcal{E}xt^1(\mathcal{O}_C, \mathcal{O}_S) = \mathcal{O}_C(C). \end{cases}$$

- $R\mathcal{H}om(i_*\mathcal{O}_Z, \mathcal{O}_S) = \mathcal{E}xt^2(\mathcal{O}_Z, \mathcal{O}_S)[-2]$,

and we have iso

$$\mathcal{E}xt^2(\mathcal{O}_Z, \mathcal{O}_S) \longrightarrow \mathcal{E}xt^2(\mathcal{O}_Z, \mathcal{O}_Z).$$

$\mathcal{O}_S \to \mathcal{O}_Z$

(by Serre duality: $\mathcal{H}om(\mathcal{O}_Z, \mathcal{O}_Z) \longrightarrow \mathcal{H}om(\mathcal{O}_S, \mathcal{O}_Z)$ iso)

- $R\mathcal{H}om(i_*\mathcal{O}_C, i_*\mathcal{O}_Z)$:

$0 \to \mathcal{O}_S(-C) \longrightarrow \mathcal{O}_S \longrightarrow \mathcal{O}_C \longrightarrow 0 \quad | \mathcal{H}om(-, i_*\mathcal{O}_Z)$

$\mathcal{H}om(i_*\mathcal{O}_C, i_*\mathcal{O}_Z) \xrightarrow{\sim} \mathcal{H}om(\mathcal{O}_S, i_*\mathcal{O}_Z) \longrightarrow \mathcal{H}om(\mathcal{O}_S(-C), i_*\mathcal{O}_Z) \longrightarrow \mathcal{E}xt^1(\mathcal{O}_C, i_*\mathcal{O}_Z) \to 0$

$\rightsquigarrow \mathcal{E}xt^2(\mathcal{O}_C, i_*\mathcal{O}_Z) = 0.$

$$\Rightarrow \begin{cases} \mathcal{H}om(i_*\mathcal{O}_C, i_*\mathcal{O}_Z) = \mathcal{H}om(i_*\mathcal{O}_Z, i_*\mathcal{O}_Z) \\ \mathcal{E}xt^1(i_*\mathcal{O}_C, i_*\mathcal{O}_Z) = i_*\mathcal{O}_Z(C) \\ \mathcal{E}xt^2(i_*\mathcal{O}_C, i_*\mathcal{O}_Z) = 0. \end{cases}$$

$$\mathrm{Cone}\begin{pmatrix} \mathcal{E}xt^1(i_*\mathcal{O}_Z, i_*\mathcal{O}_Z) \\ \downarrow \\ i_*\mathcal{O}_Z(C) \end{pmatrix} \longrightarrow R\mathcal{H}om(i_*I_Z, J_Z)[1] \longrightarrow i_*\mathcal{O}_C(C) \qquad / R\Gamma(S,-)$$

- $R\Gamma(\mathcal{E}xt^1(i_*\mathcal{O}_Z, i_*\mathcal{O}_Z)) = \mathrm{Ext}^1_S(i_*\mathcal{O}_Z, i_*\mathcal{O}_Z)$

$\mathcal{E}xt^1$ 0-dim,

$H^1(\mathcal{H}om(i_*\mathcal{O}_Z, i_*\mathcal{O}_Z)) = 0$

$$= \mathrm{Hom}(\mathcal{J}_Z, \mathcal{O}_Z) = T_{S^{[n]},Z}.$$

$0 \to \mathcal{J}_Z \to \mathcal{O}_S \to \mathcal{O}_Z \to 0 \quad / \mathrm{Hom}(-, \mathcal{O}_Z)$

$0 \to \mathrm{Hom}(\mathcal{J}_Z, \mathcal{O}_Z) \longrightarrow \mathrm{Ext}^1(\mathcal{O}_Z, \mathcal{O}_Z) \to 0$

$$\Rightarrow \mathrm{Cone}\begin{pmatrix} T_{S^{[n]},Z} \\ \downarrow \\ H^0(Z, \mathcal{O}_Z(C)) \end{pmatrix} \longrightarrow R\mathrm{Hom}_S(i_*I_Z, J_Z)[1] \longrightarrow R\Gamma(C, \mathcal{O}_C(C))$$

$\square$

(*) $R\pi_*(\mathcal{O}_C(C))^\vee \longrightarrow K_{\mathrm{Hilb}_\beta}$

comes from looking at

$$\begin{array}{c} \mathrm{Hilb}_\beta \times S \supset C \\ \big\downarrow \pi \\ \mathrm{Hilb}_\beta \end{array}$$

$$\mathbb{L}_{C/\mathrm{Hilb}_\beta \times S}[-1] \cong \mathcal{O}_C(-C) \longrightarrow \mathbb{L}_{\mathrm{Hilb}_\beta} \oplus \mathbb{L}_S \longrightarrow \pi^* \mathbb{L}_{\mathrm{Hilb}}$$

⇝ dualizing, adjunction, gain $\mathbb{T}_{\mathrm{Hilb}_\beta} \longrightarrow R\pi_*(\mathcal{O}_C(C))$.

Relationship with presentation of H_β as ~~ze~~ zero locus of section in H_β:

Over H_β we have:

$$0 \to \mathcal{O}_C(-A) \longrightarrow \mathcal{O}_D \longrightarrow \mathcal{O}_A \to 0 \quad / \otimes \mathcal{O}(D),$$

$$0 \to \mathcal{O}_C(C) \longrightarrow \mathcal{O}_D(D) \longrightarrow \mathcal{O}(D)|_A \to 0 \quad / \pi_*$$

$$R\pi_*(\mathcal{O}_C(C)) = [\underbrace{\pi_* \mathcal{O}_D(D)}_{\cong\, T_{\mathrm{Hilb}_\delta}} \longrightarrow \underbrace{\pi_*(\mathcal{O}(D)|_A)}_{\text{vector bundle.}}]$$

///