8 PT theory of Tot(L): part 1

Definition 8.1. Let X be any smooth (quasi-)projective variety over $\mathbb{C}$. A *stable pair* on X is a morphism $s\colon \mathcal{O}_X \to F$ of coherent sheaves satisfying the following conditions:

1. F is pure of dimension 1 i.e. every nontrival subsheaf $0 \subsetneq G \subseteq F$ has support of dimension 1.
2. Coker(s) has support of dimension 0.
3. Supp(F) is proper

for any given $n \in \mathbb{Z}$ and $\beta \in H_2(X, \mathbb{Z})$ we obtain a fine (quasi-)projective moduli space of stable pairs which on closed points looks like:

$$P_n(X,\beta) = \{\mathcal{O}_X \xrightarrow{s} F \mid s \text{ stable pair}, \chi(F) = n, [F] = \beta\}/\sim$$

where two stable pairs get identified if they admit an isomorphism under which the two sections correspond.

Remark 8.2. 1. From talks no. 2 and 3 we know that $P_n(X,\beta)$ admits a perfect obstruction theory if X is projective and $\dim X \leq 3$. Hence we obtain a virtual cycle

$$[P_n(X,\beta)]^{vir} \in H_{2vdim}(P_n(X,\beta))$$

with $vdim = \int_\beta c_1(X)$. Similar to Gromov-Witten invariants, this lets us define *Pandharipande-Thomas invariants* - see the next talk for more. If X is only quasi-projective, we may not be able to define such invariants. But if X admits an action by a torus T so that the fixed locus $P_n(X,\beta)^T$ is compact, we may define for any $\gamma \in H_T^*(P_n(X,\beta))$:

$$\langle\gamma\rangle_{n,\beta}^{PT} \coloneqq \sum_{\substack{X_i \subset P_n(X,\beta)^T \\ X_i \text{ conn comp}}} \int_{[X_i]^{vir}} \frac{\gamma\mid_{X_i}}{e^T(N_{X_i})}$$

where the virtual normal bundle N_{X_i} and the virtual cycles $[X_i]^{vir}$ are defined exactly as in the proof of the virtual localization formula. If $P_n(X,\beta)$ is already compact, then this gives us the right invariant while extending the definition to the non-compact case.

2. Since stable pairs have 1-dimensional support, we should see PT theory as a way to count curves in X - just like Gromov-Witten theory! Indeed, for any stable pair $\mathcal{O}_X \to F$, we see that F is almost the structure sheaf of a curve - up to the finite cokernel.
If X is a smooth curve, it is easy to see that a stable pair is the same datum as an effective divisor on X.

Now consider the following example: Let S be a smooth projective surface over $\mathbb{C}$ and $\mathcal{L} \in \mathrm{Pic}(S)$ a line bundle on it. Further let X be the the total space of $\mathcal{L}$ equipped with the canonical $T = \mathbb{C}^\times$-action that scales the fiberes over S.
In this talk, we will try to identify the fixed loci of $P_n(X,\beta)$ and embedd them into smooth spaces over which we have some control. The next talk will compute the virtual cycles and the virtual normal bundles of these loci and give an algorithm for computing the PT-invariants on X.
But before we do this, we first have to answer the question: What do stable pairs on a surface look like?

Proposition 8.3. For any $n \in \mathbb{Z}$ and $\beta \in H_2(S,\beta)$, we have an isomorphism of schemes:

$$P_n(S,\beta) = \mathrm{Hilb}^{n+g-1}(\mathcal{C}/\mathrm{Hilb}^\beta(S))$$

where $\mathcal{C} \to \mathrm{Hilb}^\beta(S)$ is the universal subcurve of S of class β and g is the arithmetic genus of any such curve, i.e.:

$$2g-2 = \int_S (K_S+\beta).\beta$$

Proof. The proof can be found in Appendix B of the paper *stable pairs and BPS invariants* by Pandharipande and Thomas. We only sketch the argument:
The first step is to show

$$P_n(C) = \mathrm{Hilb}^{n+g-1}(C)$$

for any Gorenstein curve C. Indeed, any stable pair $\mathcal{O}_C \to F$ of support C can be dualized to yield an injection $F^\vee \hookrightarrow \mathcal{O}_C$ and $V(F^\vee)$ is a 0-dim subscheme. Conversely, for any 0-dimensional $Z \subset X$ defined by the ideal $I_Z \subset \mathcal{O}_C$ one can show that $\mathcal{O}_C \to I_Z^\vee$ is a stable pair. It is somewhat nontrivial that the Euler characteristics actually match up here.
Then we note that any subcurve $C \subset S$ is always Gorenstein and the above proof goes through in the same way for the universal family of curves. □

Now we go back to X. Take $\beta \in H_2(S)$ and $n \in \mathbb{Z}$:

Theorem 8.4.

$$P_n(X,\beta)^T = \coprod_{\substack{r\geq 0 \\ \sum_{i=0}^r (n_i-(r-i)N-g+1)=n \\ \sum_{i=0}^r (i+1)\beta_i=\beta}} S^{[n_0,\dots,n_r,0]}_{\beta_0,\dots,\beta_r}$$

where the g_i are such that $2g_i - 2 = \int_S (K_S+\beta).\beta$ and $N = (\beta.c_1(\mathcal{L}))$ and the nested Hilbert scheme:

$$S^{[n_0,\dots,n_r]}_{\beta_1,\dots,\beta_r} = \{(D_1,\dots,D_r,Z_0,\dots,Z_r) \in \prod_i S^{[n_i]} \times \prod_j \mathrm{Hilb}^\beta_i(S) \mid I_{Z_i}(-D_i) \subset I_{Z_{i-1}} \forall i\}$$

Proof. For any stable pair $[\mathcal{O}_S \xrightarrow{s} F] \in P_n(X, \beta)$, being in the fixed locus is the same as F and s being T-equivariant. Since the projection $X \to S$ is affine, we can push the stable pair forward without any loss of information. Since the projection is T-equivariant with the trivial action on S, we can decompose both sheaves into their weight spaces:

$$\mathcal{O}_X = \bigoplus_{i \geq 0} \mathcal{L}^{-i}\mathfrak{t}^{-i} \xrightarrow{s=\bigoplus_i s_i} \bigoplus_{i \geq 0} F_i \mathfrak{t}^{-i}$$

where $\mathfrak{t}$ is the canonical 1-dim representation of T. The $\mathcal{O}_X$-module structure of F gives us maps

$$\phi_i \colon F_i \otimes \mathcal{L}^{-1} \to F_{i+1}$$

so that the diagram:

$$\begin{array}{ccc} \mathcal{L}^{-i} \otimes \mathcal{L}^{-1} & = & \mathcal{L}^{-i-1} \\ \downarrow{\scriptstyle s_i \otimes 1} & & \downarrow{\scriptstyle s_{i+1}} \\ F_i \otimes \mathcal{L}^{-1} & \xrightarrow{\phi_i} & F_{i+1} \end{array}$$

commutes. Furthermore the stability of F tells us that each $\mathcal{O}_S \to F_i \otimes \mathcal{L}^i$ is a stable pair and since the support of F is proper, we have $F_i = 0$ for all but finitely many i. Setting $G_i = F_i \otimes \mathcal{L}^i$, we now see:

$$P_n(X, \beta)^T = \coprod_{\substack{r \\ \sum_{i=0}^r (n_i - iN) = n \\ \sum_{i=0}^r \beta_i = \beta}} \Big\{ (G_0, ..., G_r, \mathcal{O}_S \to G_0, G_0 \to G_1, ..., G_{r-1} \to G_r) \\ \mid \forall i \colon \mathcal{O}_S \to G_0 \to ... \to G_i \in P_{n_i}(S, \beta_i) \Big\}$$

Let's further break down the right hand side:
The first two sheaves give us the following diagram:

$$\begin{array}{ccc} \mathcal{O}_S & & \\ \downarrow & \searrow & \\ G_0 & \longrightarrow & G_1 \end{array}$$

Since the support of a stable pair is the kernel of its section we see that $C_1 \subset C_0$ for $C_i = \mathrm{Supp}(G_i)$. This yields:

$$\begin{array}{ccc} \mathcal{O}_{C_0} & \twoheadrightarrow & \mathcal{O}_{C_1} \\ \downarrow & & \downarrow \\ G_0 & \longrightarrow & G_1 \end{array}$$

Further, applying $\mathcal{H}om_{\mathcal{O}_{C_0}}(..., \mathcal{O}_{C_0})$ gives:

$$\begin{array}{ccc}
\mathcal{O}_{C_0} & \hookrightarrow & \mathcal{H}om_{\mathcal{O}_{C_0}}(i_*\mathcal{O}_{C_1}, \mathcal{O}_{C_0}) = i_*\mathcal{H}om_{\mathcal{O}_{C_1}}(\mathcal{O}_{C_1}, i^!\mathcal{O}_{C_0}) = i_*\mathcal{O}_{C_1}(-D) \\
\uparrow & & \uparrow \\
G_0^\vee & \hookleftarrow & \mathcal{H}om_{\mathcal{O}_{C_0}}(i_*G_1, \mathcal{O}_{C_0}) = i_*\mathcal{H}om_{\mathcal{O}_{C_1}}(G_1, i^!\mathcal{O}_{C_0}) = i_*(G_1^\vee \otimes \mathcal{O}_{C_1}(-D))
\end{array}$$

where the bottom arrow is injective because the other ones are and the equalities use Grothendieck-Verdier duality for the inclusion $i\colon C_1 \hookrightarrow C_0$ with

$$i^!\mathcal{O}_{C_0} = i^*\mathcal{O}_{C_0} \otimes \omega_{C_1} \otimes i^*\omega_{C_0}^{-1} = \omega_S(C_1) \mid_{C_1} \otimes \omega_S^{-1}(-C_0) \mid_{C_1} = \mathcal{O}_{C_1}(-D)$$

and $D = C_0 - C_1$. Now let I_i be the preimage of $G_i^\vee$ under $\mathcal{O}_S \to \mathcal{O}_{C_i}$ i.e. $V_S(I_i) = V_{C_i}(G_i^\vee)$. The injection on the right of the preceding diagram now witnesses $I_0 \supset I_1(-D)$ and we obtain a point in a certain nested Hilbert scheme by also doing this for the other G_i. Since C_0 contains all the supports, we have to add the zero at the end of the superscript. This process is reversible and also works in families (trust me!) and so we are done. □

There's one thing that we have still left to do: embedd the fixed loci into nice smooth spaces!
Since I'm lazy, I'll just assume $h^1(\mathcal{O}_S) = 0$ in which case we can embedd

$$S^{[n_0,\dots,n_r]}_{\beta_0,\dots,\beta_r} \hookrightarrow \prod_i S^{[n_i]} \times \prod_i \mathbb{P}(H^0(\mathcal{O}_S(\beta_i)))$$