Degeneracy loci, virtual cycles and Nested Hilb. schemes

Goal :- given $\sigma: E_0 \to E_1$ map of v.b., to construct a POT on the deepest degeneracy locus (where rk σ is min).

- applications : - POT on the Nested Hilb. schemes
  - Okounkov vanishing

Recall the stand constr of POT on a zero locus of a v.b.

§ zero loci

$$\begin{array}{l} E \quad \text{vect. bdle of rk } r \\ \downarrow \uparrow \sigma \in \Gamma(A, E) \\ Z = Z(\sigma) \overset{i}{\subset} A \quad \text{smooth} \\ \qquad\quad n = \dim A \end{array}$$

$I \subset \mathcal{O}_A$ ideal of Z

$\rightsquigarrow$ POT on Z :

$$\begin{array}{ccc} E^\bullet = [\overset{-1}{E^*|_Z} & \longrightarrow & \overset{0}{\Omega_A|_Z}] \\ \downarrow \qquad \downarrow & & \| \\ \mathbb{L}_Z = [\, I/I^2 & \longrightarrow & \Omega_A|_Z] \end{array}$$

iso on h^0

surj on h^{-1}

$\rightsquigarrow [Z]^{vir} \in A_{n-r}(Z)$

$i_*[Z]^{vir} = c_r(E) \cap [A]$

§ Degeneracy loci

A smooth, quasiproj cplex variety

$\sigma: E_0 \to E_1$ map of v.b. of rks r_0, r_1 resp.

$Z_k \subset A$ degeneracy locus where rk $\sigma \leq k$

$\| \{\Lambda^{k+1}\sigma = 0\}$

$r :=$ min rk of σ $\quad (Z_{r-1} = \emptyset)$

Put $Z := Z_r$

Fact1 $f: T \to A$ map of schemes. TFAE

(1) f factors through Z_r

(2) $\ker(f^*\sigma: f^*E_0 \to f^*E_1)$ is a rk $r_0 - r$ subbundle of f^*E_0

(3) $\ker(f^*\sigma)$ has a locally free subsheaf of rk $r_0 - r$.

((3) $\Rightarrow$ (1) : (3) $\Rightarrow$ rk$(f^*\sigma)$ on the generic pt is $\leq r \Rightarrow$ equal to r $\Rightarrow$ coker$(f^*\sigma)$ is a rk $(r_1 - r)$ sheaf

lower semicont of rk $\Rightarrow$ rk $f^*\sigma|_t \leq r \Rightarrow$ rk $f^*\sigma|_t = r$ $\forall t \in T$ closed $\Rightarrow$ coker $f^*\sigma$ loc. free $\Rightarrow$ ker $f^*\sigma$ is rk $(r_0 - r)$-subbdle $< f^*E_0$

(coker $f^*\sigma)|_t$ = coker $(f^*\sigma|_t)$ $\Rightarrow$ dim coker $f^*\sigma)|_t = r_1 - r$ $\forall t \in T$ closed)

so $f^*E_0/_{\ker f^*\sigma}$ loc free of rk r $\Rightarrow \Lambda^{r+1}(f^*E_0/_{\ker f^*\sigma}) = 0$. As $\underset{\Lambda^{r+1} f^*\sigma}{f^*\Lambda^{r+1}\sigma}$ factors through $\Lambda^{r+1}(f^*E_0/_{\ker f^*\sigma})$, it's zero.

$\Rightarrow f$ factors through $Z(\Lambda^{r+1}\sigma) = Z_r$ □

Cor As rk $\sigma|_Z = r$, $\ker(\sigma|_Z) = h^0(E_\bullet|_Z) =: h^0$ v.b. of rk $r_0 - r$

coker $\sigma|_Z = h^1(E_\bullet|_Z) =: h^1$ v.b. of rk $r_1 - r$

$$0 \to h^0 \to E_0|_Z \xrightarrow{\sigma|_Z} E_1|_Z \to h^1 \to 0$$

Thm 1 The degeneracy locus $Z = Z_r \overset{\iota}{\hookrightarrow} A$ inherits a POT

$$[(h^1)^* \otimes h^0 \to \Omega_A|_Z] \to \mathbb{L}_Z \qquad \rightsquigarrow [Z]^{vir} \in A_{n-k}(Z)$$

$n = \dim A$

$k = (r_0 - r)(r_1 - r) = rk((h^1)^* \otimes h^0)$

Fact 2 (Thom-Porteous formula)

$$\iota_*[Z]^{vir} = \Delta_{r_1-r}^{r_0-r}(c(E_1 - E_0)) \in A_{n-k}(A), \text{ where } \Delta_b^a(c) := \det(c_{b+j-i})_{1 \le i,j \le a}$$

Pf of Thm 1 idea: view Z as a zero locus

Consider $Gr := Gr(r_0 - r, E_0) \xrightarrow{q} A$ param. $(r_0 - r)$ dim subsp in fibers of $E_0 \to A$

$[V \hookrightarrow E|_a] \longmapsto a$

Gr represents the functor $\underline{Gr(r_0-r, E_0)}(Z) = \{ v \text{ vect. bdle on } Z \text{ of rk } r_0 - r\}$)

$\mathcal{U} \hookrightarrow q^* E_0$ universal v.b. on Gr : fiber over $[V \hookrightarrow E|_a] \in Gr$ is V

Consider the composition $\mathcal{U} \hookrightarrow q^* E_0 \to q^* E_1 \quad \rightsquigarrow \tilde{\sigma} \in \Gamma(\mathcal{U}^* \otimes q^* E_1)$

Claim $Z(\tilde{\sigma}) \subset Gr$ is iso to $Z \subset A$ under the restr. $\bar{q} := q|_{Z(\tilde{\sigma})} : Z(\tilde{\sigma}) \to A$

Pf. • on closed pts

$[V \hookrightarrow E_0|_x] \in Z(\tilde{\sigma}) \iff \begin{cases} V = \ker(\sigma|_x) \\ \dim V = r_0 - r \end{cases} \iff rk\, \sigma|_x = r \iff x \in Z$

• scheme theor : maps

have: $\ker \bar{q}^* \sigma \supset \mathcal{U}|_{Z(\tilde{\sigma})} \leftarrow$ loc. free sheaf of rk $r_0 - r$

$\Rightarrow$ by Fact 1, $\bar{q}$ factors through $Z \subset A$, so $\bar{q} : Z(\tilde{\sigma}) \to Z$

inverse map: $h^0 = \ker(\sigma|_Z)$ is loc. free of rk $r_0 - r$.

Consider $Z \to Gr$ its classif. map

$\searrow Z(\tilde{\sigma}) \subset Gr$

POT $Z \simeq Z(\tilde{\sigma}) \Rightarrow Z$ inherits POT of ~~the~~ a zero locus.

Note: $\bar{q}^* h^0 \simeq \mathcal{U}|_{Z(\tilde{\sigma})}$

$q^* \ker \sigma \simeq \mathcal{U}$

$$\begin{tikzcd}
(1) & \mathcal{U}|_{Z(\tilde{\sigma})} \otimes (h^1)^* \arrow[r, "(**)"] \arrow[d] & q^*\Omega_A|_{Z(\tilde{\sigma})} \arrow[d] \\
(2) & (\mathcal{U} \otimes (q^* E_1)^*)|_{Z(\tilde{\sigma})} \arrow[r] \arrow[d] \arrow[dr, phantom, "(*)"] & \Omega_{Gr}|_{Z(\tilde{\sigma})} \arrow[d] \\
& \mathcal{U}|_{Z(\tilde{\sigma})} \otimes (q^* E_0|_{Z(\tilde{\sigma})} / \mathcal{U})^*|_{Z(\tilde{\sigma})} \arrow[r, phantom, "\overset{*}{\simeq}"] & \Omega_{Gr/A}|_{Z(\tilde{\sigma})}
\end{tikzcd}$$

$*$ is due to $T_{Gr/A} \simeq \mathcal{H}om(\mathcal{U}, q^* E_0/\mathcal{U})$

Assume $(*)$ is commutative. Then have map $(**)$ on kernels.

By snake lemma, the two complexes (1) and (2) are quasi isom.

Note that $q|_{Z(\tilde{\sigma})}^* (1) = [h^0 \otimes (h^1)^* \to \Omega_A|_Z]$. As $q|_{Z(\tilde{\sigma})} : Z(\tilde{\sigma}) \xrightarrow{\sim} Z$, it is a POT on Z

ι_*

have: commut. diagr

$$\begin{tikzcd} Z(6) \arrow[r, hook, "j"] \arrow[d, equal] & Gr \arrow[d, "q"] \\ Z \arrow[r, "i"] & A \end{tikzcd}$$

$j_*[Z(6)]^{vir} = c_{v_1(v_0-v)}(\mathcal{U}^* \otimes q^*E_1) \in A_{\dim Gr - (v_0-v)v_1}(Gr)$

$\dim Gr = n + (v_0-v)v$, $= n - (v_1-v)(v_0-v)$

$i_*[Z]^{vir} = \Delta^{v_0-v}_{v_1-v}(c(E_1-E_0))$ by Thom-Porteous formula

$\Delta^a_b(c) := \det(c_{b+j-i})_{1\le i,j\le a}$

Cor If $v_0-v=1$, then $i_*[Z]^{vir} = c_{v_1-v_0+1}(E_1-E_0)\cap[Z]$

Thm 2 (Higher Thom-Porteous formula) for $v_0-v=1$ case

$$i_*(c_1((h^0)^*)^i \cap [Z]^{vir}) = c_{v_1-v_0+i+1}(E_1-E_0) \cap [Z] \quad \forall i\ge 0$$

Pf. $v_0-v=1 \Rightarrow Gr(v_0-v, A) = \mathbb{P}(E_0)$, $\mathcal{U} = \mathcal{O}_{\mathbb{P}(E_0)}(-1)$

$$i_*(c_1((h^0)^*)^i \cap [Z]^{vir}) = q_* j_*(c_1(\mathcal{O}_{\mathbb{P}(E_0)}(1))^i \cap [Z(6)]^{vir}) = q_*(c_1(\mathcal{O}_{\mathbb{P}(E_0)}(1))^i \cup c_{v_1}(\mathcal{O}(1)\otimes q^*E_1)\cap[Z(6)])$$

$$= q_*\left(c_1(\mathcal{O}(1))^i \cup \sum_{k=1}^{v_1} c_k(q^*E_1)\cup c_1(\mathcal{O}(1))^{v_1-k}\right)\cap[Z(6)]) =$$

$$= \sum_{k=1}^{v_1} q_*((c_1(\mathcal{O}(1))^{v_1+i-k} \cup q^*c_k(E_1))\cap[Z(6)]) = \sum_{k=1}^{v_1} q_*(c_1(\mathcal{O}(1))^{v_1+i-k} \cap (q^*(c_k(E_1)\cap[Z]))) =$$

$$= \sum_{k=1}^{v_1} s_{v_1-v_0+i-k+1}(E_0)\cap(c_k(E_1)\cap[Z]) = c_{v_1-v_0+i+1}(E_1-E_0)\cap[Z]$$

§ Nested Hilbert schemes on surfaces

Let S be a smooth cplx proj surface w/ $h^{0,1}(S) = h^{0,2}(S) = 0$

$S^{[n_1,n_2]} = \left\{ \begin{matrix} I_1 \subseteq I_2 \\ Z_1 \supseteq Z_2 \end{matrix} \quad len(\mathcal{O}_S/I_i) = n_i \right\}$ Nested Hilb. scheme

$n_1 \ge n_2 \in \mathbb{Z}_{\ge 0}$ represents functor: $B \longmapsto \{\mathcal{I}_1 \subseteq \mathcal{I}_2 \subset \mathcal{O}_{S\times B}$ flat over B s.t. $\mathcal{I}_i|_{S\times\{b\}}$ has colength n_i's

Let $\mathcal{I}_1, \mathcal{I}_2$ be ideal sheaves of the univers. subsch Z_1, Z_2 of $S^{[n_1]}, S^{[n_2]}$, respect.

$$\begin{tikzcd} S \arrow[r, "v"] \arrow[d, "pr"] & S^{[n_1]}\times S^{[n_2]}\times S \arrow[d, "\pi"] \\ (I_1, I_2) \arrow[r, "i"] & S^{[n_1]}\times S^{[n_2]} \end{tikzcd}$$

Consider the cplx $R\pi_* R\mathcal{H}om(\mathcal{I}_1,\mathcal{I}_2) =: \mathbb{I}$

Next: apply Thm 1 to it and view $S^{[n_1,n_2]}$ as its deepest deg. locus.

Claim $R\pi_* R\mathcal{H}om(\mathcal{I}_1,\mathcal{I}_2)$ is q.iso to a 2-term cplx of v.b.

- It is perfect, as $R\mathcal{H}om(\mathcal{I}_1,\mathcal{I}_2)$ is perfect and π is flat
- Restricted to each fiber has cohom. in deg 0, 1

Indeed: $i^* R\pi_* R\mathcal{H}om(\mathcal{I}_1,\mathcal{I}_2) \overset{\text{flat b.c.}}{=} Rpr_* v^* R\mathcal{H}om(\mathcal{I}_1,\mathcal{I}_2) = Rpr_* R\mathcal{H}om(I_1,I_2) = R\mathrm{Hom}(I_1,I_2)$

Check it can have cohom. only in deg. 0 and 1, i.e. $\mathrm{Ext}^i(I_1,I_2)=0$ $\forall i \notin \{0,1\}$

It suff to check vanish. for $i=2$

$\mathrm{Ext}^2(I_1,I_2) \overset{SD}{=} \mathrm{Hom}(I_2, I_1\otimes\omega_S) \hookrightarrow \mathrm{Hom}(I_2, \mathcal{O}_S\otimes\omega_S)$ (from $0\to I_1\otimes\omega_S \to \mathcal{O}_S\otimes\omega_S \to \mathcal{O}_{Z_1}\otimes\omega_S$)

$0\to I_2 \to \mathcal{O}_S \to \mathcal{O}_{Z_2} \to 0$ Apply $\mathrm{Hom}(-,\mathcal{O}_S\otimes\omega_S)$

$$0\to \mathrm{Hom}(\mathcal{O}_{Z_2},\mathcal{O}_S\otimes\omega_S) \to \underset{\substack{\parallel \\ H^0(S,\omega_S)=2}}{\mathrm{Hom}(\mathcal{O}_S,\mathcal{O}_S\otimes\omega_S)} \to \mathrm{Hom}(I_2,\mathcal{O}_S\otimes\omega_S) \to \underset{\substack{\parallel \\ \mathrm{Ext}^1(\mathcal{O}_S,\mathcal{O}_{Z_2}) = H^1(\mathcal{O}_{Z_2})=0}}{\mathrm{Ext}^1(\mathcal{O}_{Z_2},\mathcal{O}_S\otimes\omega_S)}$$

$\Rightarrow \mathrm{Hom}(I_2,\mathcal{O}_S\otimes\omega_S) = 0 = \mathrm{Hom}(I_2, I_1\otimes\omega_S)$

Note that $\mathrm{Hom}(I_1, I_2) = \begin{cases} \mathbb{C} & \text{if } I_1 \subset I_2 \\ 0 & \text{otherwise} \end{cases}$

Indeed, 1) $\mathrm{Hom}(I_1,I_2) \hookrightarrow \mathrm{Hom}(I_1, \mathcal{O}_S)$ ← dim = 1

$$(I_1 \to I_2) \longmapsto \underbrace{(I_1 \to I_2 \xrightarrow{\text{inj}} \mathcal{O}_S)}_{\text{inj}}$$

$\Rightarrow$ • $\dim \mathrm{Hom}(I_1, I_2) \leq 1$

• If $\exists\ \varphi : I_1 \to I_2$ nonzero, then φ is injective

$\Rightarrow \mathrm{Hom}(I_1,I_2) = h^0(R\pi_* R\mathcal{H}om(I_1,I_2))$ "jumps" over $S^{[n_1,n_2]}$

Denote $p : S \times S^{[n_1,n_2]} \to S^{[n_1,n_2]}$

<u>Prop1</u> Let S be smooth proj cplex surf s.t. $h^{10}(S) = 0 = h^{20}(S)$. Then the nested Hilb. scheme $S^{[n_1,n_2]}$ carries a POT

$$\left(h^1(Rp_* R\mathcal{H}om(I_1,I_2))^* \to \Omega_{S^{[n_1]} \times S^{[n_2]}}|_{S^{[n_1,n_2]}}\right) \to \mathbb{L}_{S^{[n_1,n_2]}}$$

and virt cycle $[S^{[n_1,n_2]}]^{vir} \in A_{n_1+n_2}(S^{[n_1,n_2]})$ s.t. its pushf. to $S^{[n_1]} \times S^{[n_2]}$ is

$c_{n_1+n_2}(Rp_* R\mathcal{H}om(I_1,I_2)[1]) \in A_{n_1+n_2}(S^{[n_1]} \times S^{[n_2]})$.

<u>Pf</u>. Apply <u>Thm1</u> to the 2-term cplex of v.b. qiso to $R\pi_* R\mathcal{H}om(I_1,I_2)$.

Then its degeneracy locus is $Z = S^{[n_1,n_2]}$, ~~carrie~~

It carries a POT $\left(h^0 \otimes h^1 \to \Omega_{S^{[n_1]} \times S^{[n_2]}}|_{S^{[n_1,n_2]}}\right) \to \mathbb{L}_{S^{[n_1,n_2]}}$

$$\begin{array}{ccc} S & \longrightarrow & S^{[n_1,n_2]} \times S \\ \downarrow & & \downarrow p \\ (I_1 \subset I_2) & \longrightarrow & S^{[n_1,n_2]} \end{array}$$

Need to check: $h^0 = h^0(Rp_* R\mathcal{H}om(I_1,I_2)) = p_* \mathcal{H}om(I_1,I_2)$ is triv.

Have inj $\mathcal{O}_{S \times S^{[n_1]} \times S^{[n_2]}} \hookrightarrow \mathcal{H}om(I_1,I_2)$ $\rightsquigarrow$ $p_* \mathcal{O} \longrightarrow p_* \mathcal{H}om(I_1,I_2)$

$1 \longmapsto (I_1 \hookrightarrow I_2)$ $\qquad$ $p_*\mathcal{O} \simeq \mathcal{O}$

morph of v.b. of rk 1 $\Rightarrow$ they are iso

(on fibers: $H^0(\mathcal{O}_S) \hookrightarrow \mathrm{Hom}(I_1,I_2)$ so $\mathcal{O} \simeq p_* \mathcal{H}om(I_1,I_2)$)

By Thom Porteous formula, pushf. of $[S^{[n_1,n_2]}]^{vir}$ to $S^{[n_1]} \times S^{[n_2]}$ is $c_{\underbrace{\mathrm{rk}(Rp_* R\mathcal{H}om(I_1,I_2))}_{r_1 - r_0} \underbrace{+1}_{+1}}(Rp_* R\mathcal{H}om(I_1,I_2)[1])$

$\mathrm{rk}(Rp_* R\mathcal{H}om(I_1,I_2)) = -\mathrm{Ext}^1(I_1,I_2) + \mathrm{Hom}(I_1,I_2) = \chi(I_1,I_2) \overset{HRR}{=} \int_S \underbrace{ch(I_1)}_{(1,0,-n_1)} \underbrace{ch(I_2)}_{(1,0,-n_2)} \underbrace{td(T_S)}_{(1 + td_1 + \frac{1}{2}c_1 + \frac{1}{12}(c_1^2 - c_2))} =$

$0 \to I_1 \to \mathcal{O}_S \to \mathcal{O}_{Z_1} \to 0 \Rightarrow ch(I_i) = (1, 0, -n_i)$ $\qquad$ "$td_2(T_S)$"

$1 = \chi(\mathcal{O}_S) = \int_S td(T_S) = \deg td_2(T_S)$ $\qquad$ $= -n_1 - n_2 + 1$

$\Rightarrow r_1 - r_0 + 1 = n_1 + n_2 - 1 + 1 = n_1 + n_2$

$\Rightarrow$ pushf of $[S^{[n_1,n_2]}]^{vir}$ to $S^{[n_1]} \times S^{[n_2]}$ is $c_{n_1+n_2}(Rp_* R\mathcal{H}om(I_1,I_2)[1])$

§ Carlsson-Okounkov vanishing

Prop 2 S smooth projective surface, assume $h^{10}(S) = h^{20}(S) = 0$.

Then $c_{n_1+n_2+i}(R\pi_* R\mathcal{H}om(\mathcal{I}_1,\mathcal{I}_2)[1]) = 0$, $i>0$ on $S^{[n_1]} \times S^{[n_2]}$

$$S^{[n_1,n_2]} \xrightarrow{\iota} S^{[n_1]}\times S^{[n_2]}$$

Pf Follows from Higher Thom-Porteous formula & prev. discussion as:

$$\iota_* \big(c_1(h^{0*})^i \cap [S^{[n_1,n_2]}] \big) = c_{n_1+n_2+i}(R\pi_* R\mathcal{H}om(\mathcal{I}_1,\mathcal{I}_2)[1])$$

$\shortparallel$ for $i>0$

0 as $h^0 \cong \mathcal{O}$

$\boxtimes$

Rk Prop 2 holds also without assumptions $h^{10}(S) = h^{20}(S) = 0$

(Prop 1)

§ Without assumptions on $h^{10}(S)$, $h^{20}(S)$: idea

If $h^{10}(S) > 0 \rightsquigarrow [S^{[n_1,n_2]}]^{vir} = 0$ (Ob admits a cosection)

$h^{20}(S) > 0 \rightsquigarrow$ the perf complex $R\pi_* R\mathcal{H}om(\mathcal{I}_1,\mathcal{I}_2)$ becomes 3-term ($0 \neq h^2 = h^2(\cdots)$)

$\rightsquigarrow$ want to modify $R\pi_* R\mathcal{H}om(\mathcal{I}_1,\mathcal{I}_2)$ with $H^1(\mathcal{O}_S)$ and $H^2(\mathcal{O}_S)$ terms.

Consider the composition

$$H^2(\mathcal{O}_S) \otimes \mathcal{O}_{S^{[n_1]}\times S^{[n_2]}} \cong R^2\pi_*\mathcal{O} \cong R^2\pi_*\mathcal{I}_2 = h^2(R\pi_* R\mathcal{H}om(\mathcal{O},\mathcal{I}_2))$$

$$\downarrow$$

$$\longrightarrow h^2(R\pi_* R\mathcal{H}om(\mathcal{I}_1,\mathcal{I}_2))$$

$$\downarrow$$

$$h^3(R\pi_* R\mathcal{H}om(\mathcal{O}/\mathcal{I}_1,\mathcal{I}_2))$$

$\shortparallel$ 0 as π has relat dim 2

fibers $H^3(S, \mathcal{H}om(\mathcal{O}_{Z_1},\mathcal{I}_2))$ $\shortparallel$ 0 as $\dim S = 2$

$$\begin{tikzcd}
H^2(\mathcal{O}_S)\otimes\mathcal{O}[-2] \arrow[d, dashed, "\varphi"'] \arrow[dr] & \\
R\pi_* R\mathcal{H}om(\mathcal{I}_1,\mathcal{I}_2) \arrow[r] & h^2(R\pi_* R\mathcal{H}om(\mathcal{I}_1,\mathcal{I}_2))[-2]
\end{tikzcd}$$

If the lift (dotted arrow) φ exists, then consider $\mathrm{Cone}\,\varphi$. It doesn't have $h^2 \rightsquigarrow$ iso to 2-term cplx of v.b.

$\rightsquigarrow$ can replace $R\pi_* R\mathcal{H}om(\mathcal{I}_1,\mathcal{I}_2)$ by this cone.

One can prove: they have the same h^0 "jumping locus" $S^{[n_1,n_2]}$

have same Chern classes

Assuming same lift for $H^1(\mathcal{O}_S)\otimes\mathcal{O}[-1]$ can be found as well, applying Thm 1 to $\mathrm{Cone}\,\varphi$, would get

Thm S any proj surf. Then $S^{[n_1,n_2]} \hookrightarrow S^{[n_1]}\times S^{[n_2]}$ carries a POT and virt cycle $[S^{[n_1,n_2]}]^{vir} \in A_{n_1+n_2}(S^{[n_1,n_2]})$

s.t. $\iota_*[S^{[n_1,n_2]}]^{vir} = c_{n_1+n_2}(R\pi_* R\mathcal{H}om(\mathcal{I}_1,\mathcal{I}_2)[1])$

But the lifting φ doesn't exist in general one needs to pull back to a bigger space

$A \to S^{[n_1]}\times S^{[n_2]}$ where such a splitting exist