Virtual localization formula

Set up: $T = \mathbb{C}^*$, X, Y varieties with torus action, Y nonsingular; $X \hookrightarrow Y$ T-eq. embedding.

$E^\bullet = [E^{-1} \to E^0]$ p.o.t. of X, so

(i) E^{-1}, E^0 locally free sheaves

(ii) $\exists \phi : E^\bullet \to L^\bullet_X = [I_{X/Y}/I^2_{X/Y} \to \Omega_Y]$ in $D^{-}(X)$

(a) ϕ iso in coh degree 0

(b) ϕ surjective in coh deg -1.

Notation: $E^{\vee\bullet} = [E_0 \to E_1]$.

Assumption: 1. ϕ is a map of complexes

2. E^{-1}, E^0 T-eq. v.b. and ϕ is also T-eq.

~~Main Thm:~~

Facts (i) Y^T is nonsingular and $X^T = X \cap Y^T$ (scheme theoretically).

(ii) Let Z be a variety with trivial T-action, then a T eq. coh. sheaf S on Z is just a graded coh ~~sheaf~~ on Z

$$S = \bigoplus_{k \in \mathbb{Z}} S^k.$$

We denote S^0 by S^f and $\bigoplus_{k \neq 0} S^k$ by S^m.

(iv) Z as in (ii), then $A^T_*(Z) = A_*(Z) \otimes_{\mathbb{Q}} \mathbb{Q}[t]$.

Suppose Z smooth, then $\alpha \in A^T_*(Z)$ is invertible in $A_*(Z) \otimes \mathbb{Q}[t, t^{-1}]$ if its component in $A^0(Z) \otimes_{\mathbb{Q}} \mathbb{Q}[t]$ is nonzero.

(iii) $\Omega_Y|^f_{Y_i} \cong \Omega_{Y_i}$ and $\Omega_X|^f_{X_i} = \Omega_{X_i}$

$\Rightarrow N_{Y_i/Y} = TY^m$. Y_i components of Y^T, $X_i = Y_i \cap X^T$.

(v) ~~Y_i components of Y^T~~ Classical localization for Chow:

$$[Y] = \sum_i \iota_{i*} \frac{[Y_i]}{e(N_{Y_i/Y})}$$

$N_{Y_i/Y} = TY^m$ implies that the characters in eigenbundle dec are all non-zero.

Thus by (i) $e(N_{Y_i/Y}) \in A^{*,T}(Y_i)$ is invertible.

Main Theorem:

$$[X]^{vir} = \sum_i \iota_{i*}\left(\frac{\cancel{[X_i]^{vir}}\ 1}{e(N_i^{vir})} \cap [X_i]^{vir}\right)$$

$$e(N_i^{vir}) = e[E_0^m \to E_1^m]$$
$$= e(E_0^m)/e(E_1^m)$$

invertible by the same reason.

P.o.t. for the fixed locus

$E^\bullet$ p.o.t. of X, $E_i^\bullet$ restriction to X_i, $E_i^{\bullet,f}$ fixed part.

Construct $\psi_i : E_i^{\bullet,f} \to L^\bullet_{X_i}$ as follows

$\phi_i^f : E^{\bullet,f} \to L^\bullet_{X|X_i}{}^f$ fixed part of restriction of ϕ.

$\delta_i^{\#} : L^\bullet_{X|X_i} \to L^\bullet_{X_i}$ can map, δ_i^f its fixed part. Then $\psi_i := \delta_i^f \circ \phi_i^f$

<u>Prop:</u> ψ_i is a p.o.t.

<u>Proof:</u> Show that ϕ_i^f, δ_i^f satisfy (a), (b).

1. A map $v : A^\bullet \to B^\bullet$ satisfies (a), (b) if and only if

$$A^{-1} \oplus B^{-2} \to A^0 \oplus B^{-1} \to B^0 \to 0$$

is exact. Tensor product right exact $\Rightarrow \phi_i$ satisfies (a), (b)

$\Rightarrow \phi_i^f$ satisfies (a), (b).

2. ~~(a) $H^0(L^\bullet_{X|X_i}{}^f) \cong \Omega^f_{X|X_i} \cong$~~

(a), (b)

$\Omega^f_{X|X_i} \cong \Omega_{X_i}$ zeroth cohomology is iso.

$$\begin{array}{ccc}
I_{X/Y}/I^2_{X/Y}\big|^f_{X_i} & \longrightarrow & \Omega_Y\big|^f_{X_i} \\
\downarrow d^{-1} & & \cong \downarrow d^0 \\
I_{X_i/Y_i}/I^2_{X_i/Y_i} & \longrightarrow & \Omega_{Y_i/X_i}
\end{array}$$

$X_i = X \cap Y_i \Rightarrow I_{X/Y}/I^2_{X/Y}\big|_{X_i} \twoheadrightarrow I_{X_i/Y_i}/I^2_{X_i/Y_i}$

$\Rightarrow d^{-1}$ surjective $\overset{d^0}{\underset{\text{coh iso}}{\Rightarrow}}$ d^{-1} surj in coh degree -1.

Proof in the basic case

$$\begin{array}{ccl} & & V \\ & & \downarrow \uparrow s \\ X = Z(s) & \hookrightarrow & Y \end{array}$$

$V = T$-eq bundle, $\mathcal{V}$ loc. free sheaf an to V.

$s = T$-eq. section

P.o.t.: $\begin{array}{ccc} \mathcal{V}^\vee & \xrightarrow{s^\vee} & \mathcal{O}_Y \\ & \searrow\!\!\!\!\searrow \; I \nearrow & \end{array}$ $\Rightarrow$ $\begin{array}{ccl} E_\bullet & = & [\mathcal{V}^\vee \xrightarrow{ds^\vee} \Omega_Y] \\ \downarrow & & \quad {\scriptstyle s^\vee}\downarrow \qquad\quad \downarrow {\scriptstyle id} \\ I^\bullet_X & = & [\, I/I^2 \xrightarrow{\delta} \Omega_Y] \end{array}$

Virtual fund. class: $[X]^{vir} = 0^!_V [\Gamma_s] =: e_{ref}(V)$ $\qquad \begin{array}{ccc} \Gamma_s & \longrightarrow & Y \times V \\ \uparrow & & \uparrow 0_V \\ X & \longrightarrow & Y \end{array}$

Check: The induced p.o.t. for X_i

$$[(V_i^f)^\vee \longrightarrow \Omega_{Y_i}] \longrightarrow L_{X_i}$$

coincides with the p.o.t. corresponding to Y_i and $s \in H^0(Y_i, V_i^f)$.

$\Rightarrow [X_i]^{vir} = e_{ref}(V_i^f)$.

Virtual normal bundle: $T^m_{Y|Y_i} = N_{Y_i/Y}$, hence $N_i^{vir} = [N_{Y_i/Y} \to V_i^m]$.

$\Rightarrow e(N_i^{vir}) = \dfrac{e(N_{Y_i/Y})}{e(V_i^m)}$.

Claim: $e_{ref}(V) = \cancel{\sum \iota'_{i*}\left(\frac{e(V_i^m) \cap e_{ref}(V_i^f)}{e(N_{Y_i/Y})}\right)} \; \sum \iota'_{i*}\left(\dfrac{e(V_i^m)}{e(N_{Y_i/Y})} \cap e_{ref}(V_i^f)\right)$

Proof: Localization formula for Y:

$$[Y] = \sum \iota_{i*}\left(\frac{[Y_i]}{e(N_{Y_i/Y})}\right)$$

Recall: $A^T(Y) \otimes A^T(X) \xrightarrow{\cap} A^T(X)$ gives $A^T(X)$ an $A^T(Y)$-module structure.

We conclude

$$e_{ref}(V) = [Y] \cap e_{ref}(V) = \sum \frac{1}{e(N_{Y_i/Y})} (\iota_* [Y_i]) \cap e_{ref}(V))$$

$$= \sum \frac{1}{e(N_{Y_i/Y})} (\iota'_{i*} ([Y_i] \cap e_{ref}(V))$$

$$= \sum \frac{1}{e(N_{Y_i/Y})} \iota'_{i*} (e_{ref}(V_i))$$

$$= \sum \frac{1}{e(N_{Y_i/N})} \iota'_{i*} (e(V_i^m) \cap e_{ref}(V_i^f)). \quad \blacksquare$$

$$e_{ref}(V) = [Y] \cap e_{ref}(V) = \sum \iota_{i*}\left(\frac{[Y_i]}{e(N_{Y_i/Y})}\right) \cap e_{ref}(V)$$

$$= \sum \iota'_{i*}\left(\left(\frac{[Y_i]}{e(N_{Y_i/Y})}\right) \cap e_{ref}(V)\right)$$

$$= \sum \iota'_{i*}\left(\frac{1}{e(N_{Y_i/Y})} \cap [Y_i] \cap e_{ref}(V)\right)$$

$$= \sum \iota'_{i*}\left(\frac{1}{e(N_{Y_i/Y})} \cap e_{ref}(V_i)\right)$$

$$= \sum \iota'_{i*}\left(\frac{e(V_i^m)}{e(N_{Y_i/Y})} \cap e_{ref}(V_i^f)\right). \quad \blacksquare$$

Proof in the general case

$E^\bullet \to L_X$ p.o.t. for X, $E^{f\bullet} \to L_{X_i}$ p.o.t. for X_i.

Virtual fund. class:

$$E^{-1} \to E^0 \oplus I/I^2 \xrightarrow{\gamma} \Omega_Y \to 0 \quad \text{exact seq. of coh. sheaves}$$

$\Rightarrow 0 \to TY \to C(I/I^2) \times_X E_0 \to C(Q) \to 0$ res of ab. cones.

$Q = \ker \gamma$; Q quotient of E^{-1} $\Rightarrow C(Q) \hookrightarrow E_1$ closed subcone.

$C_{X/Y} \hookrightarrow C(I/I^2)$ closed subcone

Fact: [BF] $D := C_{X/Y} \times_X E_0$ is a TY cone.

$D^{vir} := C_{X/Y} \times_X E_0 / TY \hookrightarrow C(Q)$ closed subcone.

Have a res $0 \to TY \to D \to D^{vir} \to 0$.

$[X]^{vir} := 0^!_{E_1} \cancel{s^*_{E_1}} (D^{vir})$.

Same construction: $[X_i]^{vir} := 0^!_{E_1^f} \cancel{s^*_{E_1}} (D_i^{vir})$

$D_i = C_{X_i/Y_i} \times_{X_i} E_0^m$, $D_i^{vir} = D_i / TY \hookrightarrow E_1^f$

Alternatively:

$$\begin{array}{ccc} TY & \to & D \\ \downarrow & & \downarrow \\ X & \xrightarrow[0_{E_1}]{} & E_1 \end{array}$$

$[X]^{vir} = s^*_{TY} 0^!_{E_1} [D]$

Observation: $[X]^{vir} = \sum \iota^!_{i*} \left(\frac{[Y_i]}{e(TY^m)} \cap [X]^{vir} \right)$

Follows from $[Y] = \sum \iota_{i*} \frac{[Y_i]}{e(TY^m)}$

Thus, to prove the main thm it suffices to show

$$\frac{[Y_i]}{e(TY^m)} \cap [X]^{vir} = \cancel{\frac{e_1(E_1^m) \cap [X_i]^{vir}}{e(E_0^m)}} \quad \frac{e_1(E_1^m)}{e_0(E_0^m)} \cap [X_i^{vir}] \qquad (*)$$

Preparations: (1) cart. diagram:

$$\begin{array}{ccc} \iota^* C_{X/Y} \times E_0 & \to & C_{X/Y} \times E_0 = D \\ \downarrow & & \downarrow \\ \iota^* C_{X/Y} & \to & C_{X/Y} \\ \downarrow & & \downarrow \\ X_i & \to & X \\ \downarrow & & \downarrow \\ Y_i & \xrightarrow{\iota} & Y \end{array}$$

Vistoli $\Rightarrow \iota^! [C_{X/Y}] = [C_{X_i/Y_i}]$

$\Rightarrow \iota^! [D] = [C_{X_i/Y_i} \times E_0] = [D_i \times E_0^m]$. $(**)$

(2) Exact sequence of ab. cones

$$0 \to TY \to \iota^* C(I/I^2) \times E_0 \to \iota^* C(Q) \to 0$$

and closed inclusion $\iota^* C(Q) \hookrightarrow E_1$.

Since $\iota^* D \hookrightarrow \iota^* C(I/I^2) \times E_0$ is TY-invariant. Hence we have the quotient cones $\iota^* D / TY_i \to \iota^* D / TY \hookrightarrow \iota^* C(Q)$.

We obtain a three level cart. diagram:

$$\begin{array}{ccc} TY & \to & \iota^* D \\ q \downarrow & & \downarrow p \\ TY^m = TY/TY_i & \to & \iota^* D / TY_i \\ \downarrow & & \downarrow \\ X_i & \to & E_1 \end{array}$$

Lemma 1: $[Y_i]\cap[X]^{vir} = s^*_{TY^m}\, 0^!_{E_1}[D_i^{vir}\times E_0^m]$.

Proof: $[Y_i]\cap[X]^{vir} = \imath^! s^*_{TY}\, 0^!_{E_1}[D]$

$$= s^*_{TY}\, 0^!_{E_1}\, \imath^![D]$$

$$\overset{(**)}{=} s^*_{TY}\, 0^!_{E_1}\underbrace{[D_i\times E_0^m]}_{=p^*[D_i^{vir}\times E_0^m]}$$

$$= q^* s^*_{TY}\, 0^!_{E_1}[D_i^{vir}\times E_0^m]$$ ∎

Lemma 2: (i) $[Y_i]\cap[X]^{vir} = \dfrac{e(TY^m)}{e(E_0^m)}\cap s^*_{E_0^m} j_*(0^!_{E_1}[D_i^{vir}\times E_0^m])$ $\quad j: F\to E_0^m$

(ii) $s^*_{E_0^m} j_*(0^!_{E_1}[D_i^{vir}\times E_0^m]) = e(E_1^m)\cap[X_i]^{vir}$.

Note that Lemma 2 implies $(*)$ hence the main thm.

Proof: (i)
$$\begin{tikzcd} F \arrow[r] \arrow[d] & D_i^{vir}\times E_0^m \arrow[d] \\ X_i \arrow[r] & E_1 \end{tikzcd}$$
. By three level cart diag, $F\hookrightarrow TY^m$. Moreover, $D_i^{vir}\times E_0^m\to E_1$ is the product of the inclusion $D_i^{vir}\subset E_1^f$ and the natural map from the ob. theory $E_0^m\to E_1^m$. Thus, $F\hookrightarrow E_0^m$. So we have a diagram:
$$\begin{tikzcd} F \arrow[r, hook, "j"] \arrow[d] & E_0^m \arrow[d] \\ TY^m \arrow[r] & X_i \end{tikzcd} \qquad (***)$$

Prop: B_0, B_1 T-eq. bundles over X_i, Z scheme with two eq. inclusions j_0, j_1 over X_i:
$$\begin{tikzcd} Z \arrow[r, hook] \arrow[d] & B_1 \arrow[d] \\ B_0 \arrow[r] & X_i \end{tikzcd}$$
. Let $\zeta\in A_*^T(Z)$, then $e(B_1)\cap s^*_{B_0} j_{0*}(\zeta) = e(B_0)\cap s^*_{B_1} j_{1*}(\zeta)$.

Proof of Prop:

$$Z\times A_1 \longrightarrow B_0\times B_1 \qquad j_t = (1-t)j_0 + tj_1$$

Thus, $s^*_{B_0\times B_1} j_{0*}(\zeta) = s^*_{B_0\times B_1} j_{1*}(\zeta)$. Now the Prop. follows from the excess intersection formula. //

Applying Prop to $(***)$ with $\zeta = 0^!_{E_1}[D_i^{vir}\times E_0^m]$ gives (i).

(ii) Consider the diagram
$$\begin{tikzcd} E_0^m \arrow[dr] & F_t \arrow[l, hook', "j_t"'] \arrow[r] \arrow[d] & D_i^{vir}\times E_0^m \arrow[d, "\ell \times t\phi"] \\ & X_i \arrow[r] & E_1 \end{tikzcd}$$

$\Rightarrow j_{t*}\, 0^!_{E_1}(\text{can}\times t\phi)$ does not depend on t

$\Rightarrow s^*_{E_0} j_*\, 0^!_{E_1}(\text{can}\times\phi) = s^*_{E_0} j_{0*}(\overset{\ell}{\cancel{\text{can}}}\times \text{zero map})$.

So from now on, we assume ϕ = zero map.

We have a diagram:

$$\begin{tikzcd}
E_0^m \arrow[r] \arrow[d, "p"'] & D_i^{vir} \times E_0^m \arrow[d, "q"] \\
X_i \arrow[r] \arrow[d] & D_i^{vir} \arrow[d] \\
Y_i \arrow[r, "0_{E_1}"'] \arrow[d, "id"'] & E_1 \arrow[d] \\
Y_i \arrow[r, "0_{E_1^f}"'] & E_1^f
\end{tikzcd}$$

Thus the result follows from the excess intersection formula.

$$0^!_{E_1}[D_1^{vir} \times E_0^m] = p^* 0^!_{E_1}[D_i^{vir}].$$

$$0^!_{E_1}[D_i^{vir}] = e(E_1^m) \cap 0^!_{E_1^f}(D_i^{vir}).$$