

# Virtual localization formula

Set up:  $T = \mathbb{C}^*$ ,  $X, Y$  varieties with torus action,  
 $Y$  nonsingular,  $X \rightarrow Y$   $T$ -eq. embedding.

$E^\bullet = [E^{-1} \rightarrow E^0]$  p.o.t. of  $X$ , so

(i)  $E^{-1}, E^0$  locally free sheaves

(ii)  $\exists \phi: E^\bullet \rightarrow L_X = [I_{X/Y} / I_{X/Y}^2 \rightarrow \mathcal{O}_Y]$  in  $D^{\text{vir}}(X)$

(a)  $\phi$  iso in coh degree 0

(b)  $\phi$  surjective in <sup>coh</sup> deg -1.

Notation:  $E^{\text{vir}} = [E_0 \rightarrow E_1]$ .

Assumption:  $\phi$  is a map of complexes

2.  $E^{-1}, E^0$   $T$ -eq. v.b. and  $\phi$  is also  $T$ -eq.

Facts (i)  $Y^T$  is nonsingular and  $X^T = X \cap Y^T$  (scheme theoretically).

(ii) Let  $Z$  be a variety with trivial  $T$ -action, then a  $T$ -eq. <sup>coh</sup> sheaf on  $Z$  is just a graded coh sheaf on  $Z$

$$S = \bigoplus_{k \in \mathbb{Z}} S^k$$

We denote  $S^0$  by  $S^t$  and  $\bigoplus_{k \geq 0} S^k$  by  $S^m$ .

(iv)  $Z$  as in (ii), then  $A_*^T(Z) = A_*(Z) \otimes_{\mathbb{Q}} \mathbb{Q}[t]$ .

Suppose  $Z$  smooth, then  $\alpha \in A_*^T(Z)$  is invertible <sup>in  $A_*(Z) \otimes \mathbb{Q}[t, t^{-1}]$</sup>  if its component  $\alpha$  in  $A^0(Z) \otimes_{\mathbb{Q}} \mathbb{Q}[t]$  is nonzero.

(iii)  $\mathcal{O}_{Y_i/Y_i}^f \cong \mathcal{O}_{Y_i}$  and  $\mathcal{O}_{X_i/X_i}^f = \mathcal{O}_{X_i}$

$\implies N_{X_i/Y}^f = T Y^m$ .  $Y_i^f$  components of  $Y^T$ ,  $X_i^f = X_i \cap X^T$ .

(v)  $Y_i^f$  components of  $Y^T$  localization for Chow:

$$[Y] = \sum_i z_i \frac{[Y_i]}{e(N_{Y_i/Y})}$$

$N_{Y_i/Y} = T Y^m$  implies that the characters in eigenbundle dec are all non-zero.

## Main Theorem:

$$[X]^{\text{vir}} = \sum_i z_i \left( \frac{[X_i]}{e(N_{X_i/X})} \wedge [X_i]^{\text{vir}} \right) \text{ Thus by (i) } e(N_{Y_i/Y}) \in A_*^T(Y_i) \text{ is invertible.}$$

$$e(N_i^{\text{vir}}) = e[E_0^m \rightarrow E_1^m]$$

$$= e(E_0^m) / e(E_1^m) \longleftarrow \text{invertible by the same reason.}$$



## P.o.t. for the fixed locus

$E$  p.o.t. of  $X$ ,  $E_i$  restriction to  $X_i$ ,  $E_i^{\circ, f}$  fixed part.

Construct  $\psi_i: E_i^{\circ, f} \rightarrow L_{X_i}$  as follows

$\phi_i^f: E_i^{\circ, f} \rightarrow L_{X_i|X_i}^f$  fixed part of restriction of  $\phi$ .

$\delta_i^f: L_{X_i|X_i} \rightarrow L_{X_i}$  can map,  $\delta_i^f$  its fixed part. Then  $\psi_i = \delta_i^f \circ \phi_i^f$

Prop:  $\psi_i$  is a p.o.t.

Proof: Show that  $\phi_i^f, \delta_i^f$  satisfy (a), (b).

1. A map  $\nu: A^{\circ} \rightarrow B^{\circ}$  satisfies (a), (b) if and only if

$$A^{-1} \oplus B^{-2} \rightarrow A^0 \oplus B^{-1} \rightarrow B^0 \rightarrow 0$$

is exact. Tensor product right exact  $\Rightarrow \phi_i$  satisfies (a), (b)

$\Rightarrow \phi_i^f$  satisfies (a), (b).

2. ~~(a)  $\mathcal{O}_{X_i} \otimes \mathcal{O}_{X_i} \rightarrow \mathcal{O}_{X_i}$~~

(a), (b)

$$I_{X \setminus Y} / I_{X \setminus Y}^2 \Big|_{X_i} \xrightarrow{f} \mathcal{O}_Y \Big|_{X_i}$$

$$\downarrow d^{-1}$$

$$I_{X \setminus Y_i} / I_{X_i \setminus Y_i}^2$$

$$\longrightarrow \mathcal{O}_{Y_i} \Big|_{X_i}$$

$$\cong \downarrow d^0$$

$$\mathcal{O}_{X_i|X_i}^f \cong \mathcal{O}_{X_i}$$

0th cohomology.

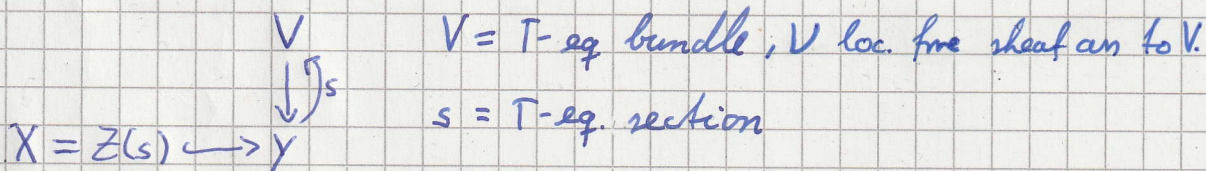
is iso.

$$X_i = X \cap Y_i \Rightarrow I_{X \setminus Y} / I_{X \setminus Y}^2 \Big|_{X_i} \twoheadrightarrow I_{X_i \setminus Y_i} / I_{X_i \setminus Y_i}^2$$

$\Rightarrow d^{-1}$  is surjective  $\xRightarrow{d^0}$   $d^{-1}$  reg in coh degree -1.



## Proof in the basic case



P.o.t.:  $V^v \xrightarrow{sv} \mathcal{O}_Y \rightarrow \mathcal{I}_X$

$$\begin{array}{ccc}
 E = [V^v \xrightarrow{sv} \mathcal{O}_Y] & & \\
 \downarrow & \Rightarrow & \downarrow \text{id} \\
 \mathcal{I}_X = [\mathcal{I}_X \xrightarrow{s} \mathcal{O}_Y] & & \mathcal{O}_Y
 \end{array}$$

Virtual fund. class:  $[X]^{\text{vir}} = \alpha'_V [\Gamma_s] = e_{\text{ref}}(V)$

$$\begin{array}{ccc}
 \Gamma_s & \rightarrow & Y \times V \\
 \uparrow & & \uparrow \alpha \\
 X & \rightarrow & Y
 \end{array}$$

Check: The induced p.o.t. for  $X$ :  $[(V_i^f)^v \rightarrow \mathcal{O}_Y] \rightarrow L_{X_i}$

coincides with the p.o.t. corresponding to  $X$  and  $s \in H^0(X, V_i^f)$ .

$\Rightarrow [X_i]^{\text{vir}} = e_{\text{ref}}(V_i^f)$

Virtual normal bundle:  $T_{Y|X}^m = N_{X|Y}$ , hence  $N_i^{\text{vir}} = [N_{X_i|Y} \rightarrow V_i^m]$

$\Rightarrow e(N_i^{\text{vir}}) = \frac{e(N_{X_i|Y})}{e(V_i^m)}$

Claim:  $e_{\text{ref}}(V) = \sum_i \frac{e(V_i^m) \cap e_{\text{ref}}(V_i^f)}{e(N_{X_i|Y})} = \sum_i t_{i*} \left( \frac{e(V_i^m)}{e(N_{X_i|Y})} \cap e_{\text{ref}}(V_i^f) \right)$

Proof: Localization formula for  $Y$ :

$[Y] = \sum_i t_{i*} \left( \frac{[X_i]}{e(N_{X_i|Y})} \right)$

Recall:  $A^*(Y) \otimes A^*(X) \xrightarrow{\alpha} A^*(X)$  gives  $A^*(X)$  an  $A^*(Y)$ -module structure.

We conclude

~~$$\begin{aligned}
 e_{\text{ref}}(V) = [Y] \cap e_{\text{ref}}(V) &= \sum_i \frac{1}{e(N_{X_i|Y})} (t_{i*} [X_i] \cap e_{\text{ref}}(V)) \\
 &= \sum_i \frac{1}{e(N_{X_i|Y})} (t_{i*} ([X_i] \cap e_{\text{ref}}(V))) \\
 &= \sum_i \frac{1}{e(N_{X_i|Y})} t_{i*} (e_{\text{ref}}(V_i)) \\
 &= \sum_i \frac{1}{e(N_{X_i|Y})} t_{i*} (e(V_i^m) \cap e_{\text{ref}}(V_i^f)).
 \end{aligned}$$~~

$$\begin{aligned}
 e_{\text{ref}}(V) = [Y] \cap e_{\text{ref}}(V) &= \sum_i t_{i*} \left( \frac{[X_i]}{e(N_{X_i|Y})} \right) \cap e_{\text{ref}}(V) \\
 &= \sum_i t_{i*} \left( \frac{[X_i]}{e(N_{X_i|Y})} \right) \cap e_{\text{ref}}(V) \\
 &= \sum_i t_{i*} \left( \frac{1}{e(N_{X_i|Y})} \cap [X_i] \cap e_{\text{ref}}(V) \right) \\
 &= \sum_i t_{i*} \left( \frac{1}{e(N_{X_i|Y})} \cap e_{\text{ref}}(V_i) \right) \\
 &= \sum_i t_{i*} \left( \frac{e(V_i^m)}{e(N_{X_i|Y})} \cap e_{\text{ref}}(V_i^f) \right).
 \end{aligned}$$



# Proof in the general case

$E^\bullet \rightarrow L_X$  p.o.t. for  $X$ ,  $E_i^\bullet \rightarrow L_{X_i}$  p.o.t. for  $X_i$ .

Virtual fund. class:

$$E^{-1} \rightarrow E^0 \oplus I/I^2 \xrightarrow{\gamma} \mathcal{P}_Y \rightarrow 0 \quad \text{exact seq. of coh. sheaves}$$

$$\Rightarrow 0 \rightarrow TY \rightarrow C(I/I^2) \times_X E_0 \rightarrow C(Q) \rightarrow 0 \quad \text{res of ab. cones.}$$

$Q = \ker \gamma$ ,  $Q$  quotient of  $E^{-1} \Rightarrow C(Q) \hookrightarrow E_1$  closed subcone.

$C_{X/Y} \hookrightarrow C(I/I^2)$  closed subcone.

Fact:  $D := C_{X/Y} \times_X E_0$  is a TY cone.

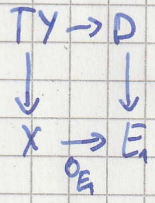
[BF]  $D^{\text{vir}} := C_{X/Y} \times_X E_0 / TY \hookrightarrow C(Q)$  closed subcone. Have a res

$$0 \rightarrow TY \rightarrow D \rightarrow D^{\text{vir}} \rightarrow 0$$

$[X]^{\text{vir}} := \sum_{E_1} \sigma_{E_1}^*(D^{\text{vir}})$

Same construction:  $[X_i]^{\text{vir}} := \sum_{E_1} \sigma_{E_1}^*(D_i^{\text{vir}})$   $D_i := C_{X_i/Y_i} \times_{X_i} E_0^m, D_i^{\text{vir}} = D_i / TY_i \hookrightarrow E_1^f$

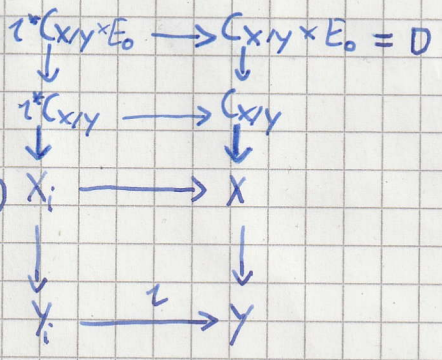
Alternatively: Observation:  $[X]^{\text{vir}} = \sum t_{i*} \left( \frac{[D_i]}{e(TY^m)} \cap [X_i]^{\text{vir}} \right)$  Follows from  $[Y] = \sum t_{i*} \frac{[D_i]}{e(TY)}$



Thus, to prove the main thm it suffices to show

$$\frac{[Y_i]}{e(TY^m)} \cap [X]^{\text{vir}} = \frac{e_1(E_1^m) \cap [X_i]^{\text{vir}}}{e(E_0^m)} \quad \frac{e_1(E_1^m)}{e_0(E_0^m)} \cap [X_i^{\text{vir}}] \quad (*)$$

$[X]^{\text{vir}} = \sum_{E_1} \sigma_{E_1}^*[D]$  Preparations: (1) cart. diagram:



(Vistoli)  $\Rightarrow i^*[C_{X/Y}] = [C_{X_i/Y_i}]$

$\Rightarrow i^*[D] = [C_{X_i/Y_i} \times E_0] = [D_i \times E_0^m]$  (\*\*)

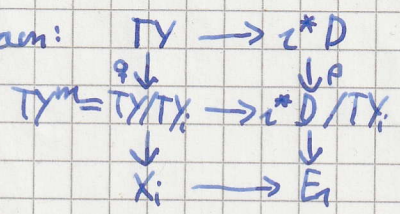
(2) Exact sequence of ab. cones

$$0 \rightarrow TY \rightarrow i^*(C(I/I^2) \times E_0) \rightarrow i^*(C(Q)) \rightarrow 0$$

and closed inclusion  $i^*(C(Q)) \hookrightarrow E_1$ .

Since  $i^*D \hookrightarrow i^*(C(I/I^2) \times E_0)$  is TY-invariant. Hence we have the quotient cones  $i^*D / TY_i \rightarrow i^*D / TY \hookrightarrow i^*(C(Q))$ .

We obtain a three level cart. diagram:





Lemma 1:  $[Y_i] \cap [X]^{vir} = s_{TY^m}^* o_{E_1}^! [D_i^{vir} \times E_0^m]$

Proof:  $[Y_i] \cap [X]^{vir} = i^! s_{TY}^* o_{E_1}^! [D]$   
 $= s_{TY}^* o_{E_1}^! i^! [D]$   
 $\stackrel{(**)}{=} s_{TY}^* o_{E_1}^! [D_i \times E_0^m]$   
 $= p^* [D_i^{vir} \times E_0^m]$   
 $= q^* s_{TY}^* o_{E_1}^! [D_i^{vir} \times E_0^m]$

Lemma 2: (i)  $[Y_i] \cap [X]^{vir} = \frac{e(TY^m)}{e(E_0^m)} \cap s_{E_0^m}^* (o_{E_1}^! [D_i^{vir} \times E_0^m])$

(ii)  $s_{E_0^m}^* (o_{E_1}^! [D_i^{vir} \times E_0^m]) = e(E_1^m) \cap [X_i]^{vir}$

$j: F \rightarrow E_0^m$

Note that Lemma 2 implies (\*) hence the main thm.

Proof: (i)  $F \rightarrow D_i^{vir} \times E_0^m$ . By three level cart diag,  $F \hookrightarrow TY^m$

$$\begin{array}{ccc} & & \\ & \downarrow & \downarrow \\ X_i & \longrightarrow & E_1 \end{array}$$

Moreover,  $D_i^{vir} \times E_0^m \rightarrow E_1$  is the product of the inclusion  $D_i^{vir} \subset E_1^m$  and the natural map from the obs. theory  $E_0^m \rightarrow E_1^m$ . Thus,

$F \hookrightarrow E_0^m$ . So we have a diagram:

$$\begin{array}{ccc} F & \xrightarrow{j} & E_0^m \\ \downarrow & & \downarrow \\ TY^m & \longrightarrow & X_i \end{array} \quad (***)$$

Prop:  $B_0, B_1$  T-eg. bundles over  $X_i$ ,  $Z$  scheme with two eq. inclusions  $j_0, j_1$  over  $X_i$ :  $\begin{array}{ccc} Z & \hookrightarrow & B_1 \\ \downarrow & & \downarrow \\ B_0 & \longrightarrow & X_i \end{array}$  Let  $\zeta \in A_*^1(Z)$ , then  $e(B_1) \cap s_{B_0}^* j_{0*}(\zeta) = e(B_0) \cap s_{B_1}^* j_{1*}(\zeta)$

Proof of Prop:

$$Z \times \mathbb{A}^1 \hookrightarrow B_0 \times B_1 \quad j_t = (1-t)j_0 + tj_1$$

Thus,  $s_{B_0 \times B_1}^* j_{0*}(\zeta) = s_{B_0 \times B_1}^* j_{1*}(\zeta)$ . Now the Prop. follows from the excess intersection formula. //

Applying Prop to (\*\*\*) with  $\zeta = o_{E_1}^! [D_i^{vir} \times E_0^m]$  gives (i).

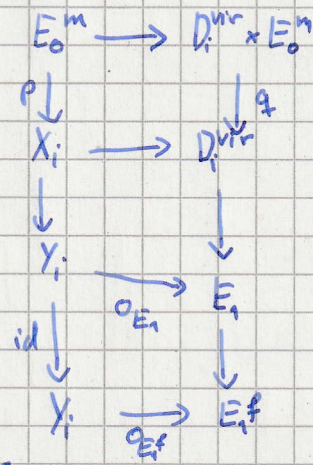
(ii) Consider the diagram  $E_0^m \xleftarrow{j_t} F_t \rightarrow D_i^{vir} \times E_0^m$   
 $\Rightarrow j_t \times o_{E_1}^! (can \times t\phi)$  does not depend on  $t$

$$\Rightarrow s_{E_0^m}^* j_t \times o_{E_1}^! (can \times t\phi) = s_{E_0^m}^* j_{0*} (can \times zero \text{ map})$$

So from now on, we assume  $\phi = zero \text{ map}$ .



We have a diagram:  
 Thus the result follows  
 from the exact intersection  
 formula.



$$\circ_{E_1}^! [D_i^{\text{vir}} \times E_0^m] = p^* \circ_{E_1}^! [D_i^{\text{vir}}].$$

$$\circ_{E_1}^! [D_i^{\text{vir}}] = e(E_1^m) \cap \circ_{E_f}^! (D_i^{\text{vir}}).$$