Let G be a complex linear algebraic group, and let X be a complex algebraic variety with a left G-action. Then we take a universal G-bundle $EG \to BG$ (i.e. for any principal G-bundle $E \to B$, there is a unique continuous map $f: B \to BT$, up to homotopy, such that $E \cong f^* B$.

$$\begin{array}{ccc} f^*E & \longrightarrow & \mathbb{E}T \\ \downarrow & & \downarrow \\ B & \xrightarrow{f} & \mathbb{B}T \end{array}$$

). $EG \times^G X / (e.g, x) \sim (e, g.x)$.

Definition 1. The equivariant cohomology of X (with respect to G) is the singular cohomology of $EG \times^G X$: $H^*_G X := H^*(\mathbb{E}G \times^G X)$. (For X is a point $H^*_G(pt) \cong H^*(\mathbb{B}G)$)

(The idea behind the definition is what when X is a free G-space. In this case $H^*_G X = H^*(G \backslash X)$. To get to this situation, we replace X with a free G-space of same homotopy type. (More modern point of view we should take it to be the quotient stack $[G \backslash X]$)

Example 2 Let $G = \mathbb{C}^*$, Then space $\mathbb{E}G = \mathbb{C}^\infty \setminus \{0\}$ is contractible and G acts freely, so $BG = \mathbb{E}G/G = \mathbb{P}^\infty$. So We can see $H^*_{\mathbb{C}^*}(pt) = H^* \mathbb{P}^\infty \cong \mathbb{Z}[t]$, where $t = c_1(\mathcal{O}_{\mathbb{P}^\infty}(-1))$ is the first chern class of the tautological bundle.

But however $\mathbb{E}G$ and $\mathbb{B}G$ space are typically infinite-dimensional. So they are not even algebraic varieties. However there are finite-dimensional, nonsingular algebraic varieties $\mathbb{E}_m \to \mathbb{B}_m = \mathbb{E}_m/G$ are "approximation" to $\mathbb{E}G \to \mathbb{B}G$.

Lemma 3. Suppose $\mathbb{E}_m$ is any (connected) space with a free right G-action and $H^i \mathbb{E}_m = 0$ for $0 < i < k(m)$ (for some integer $k(m)$). Then for any X, there are natural isomorphisms

$$H^i(\mathbb{E}_m \times^G X) \cong H^i(\mathbb{E}G \times^G X) =: H^i_G(X). \quad \text{for } i < k(m).$$

Example 4. For $G = \mathbb{C}^*$, take $\mathbb{E}_m = \mathbb{C}^m \setminus \{0\}$, so $\mathbb{B}_m = \mathbb{P}^{m-1}$. Since $\mathbb{E}_m$ is homotopy-equivalent to the sphere S^{2m-1}, it satisfies the conditions of the lemma, with $k(m) = 2m-1$ in above lemma. Note that $k(m) \to \infty$ as $m \to \infty$, so any given computation in $H^*_G X$ can be done in $H^*(\mathbb{E}_m \times^G X)$, for $m >> 0$.

So now we can forget the technical problem that $\mathbb{E}G$ or $\mathbb{B}G$ are not even algebraic varieties.

Example 5. Similiarly, for a torus $T \cong (\mathbb{C}^*)^n$, take $\mathbb{E}_m = (\mathbb{C}^m \setminus \{0\})^{\times n} \longrightarrow (\mathbb{P}^{m-1})^{\times n} =$

$\mathbb{B}_m$. We see that $H_T^*(pt) = H_T^*((\mathbb{P}^\infty)^n) = \mathbb{Z}[t_1,\cdots,t_n]$, with $t_i = c_1(\mathcal{O}_i(-1))$.

(Here $\mathcal{O}_i(-1)$ is the pullback of $\mathcal{O}(-1)$ by projection on the i-th factor).

Equivariant cohomology is functorial for equivariant maps: given a homo $G \xrightarrow{\varphi} G'$ and a map $X \xrightarrow{f} X'$ and $f(g.x) = \varphi(g).f(x)$. So we have a map $EG \times^G X \to EG' \times^{G'} X'$.

There are also equivariant chern classes and equivariant fundamental classes:

- If $\mathbb{E} \to X$ is an equivariant vector bundle, there are induced vector bundles $\mathbb{E}_m \times^G E \longrightarrow \mathbb{E}_m \times^G X$. Set $c_i^G(E) = c_i(\mathbb{E}_m \times^G X) \in H_G^{2i} X = H^{2i}(\mathbb{E}_m \times^G X)$, for $m >> 0$.

- When X is nonsingular variety, so is $\mathbb{E}_m \times^G X$. If $V \subseteq X$ is a G-invariant subvariety of codimension d, then $\mathbb{E}_m \times^G X \subseteq \mathbb{E}_m \times^G X$ has codimension d. We define $[V]^G = [\mathbb{E}_m \times^G V] \in H_G^{2d}(X) = H^{2d}(\mathbb{E}_m \times^G X)$, again for $m >> 0$.

Example 6. Let $\mathcal{L}_a = \mathbb{C}$ be the representation of $\mathbb{C}^*$ with the action $z \cdot v = z^a \cdot v$ where a is a fixed integer. Then $\mathbb{E}_m \times^{\mathbb{C}^*} \mathcal{L}_a \cong \mathcal{O}_{\mathbb{P}^{m-1}}(-a)$ as line bundles on $\mathbb{B}_m = \mathbb{P}^{m-1}$, so $c_1^{\mathbb{C}^*}(\mathcal{L}_a) = at \in \mathbb{Z}[t]$.

Example 7. Let $T = (\mathbb{C}^*)^n$ act on $E = \mathbb{C}^n$ by the standard action. Then $c_i^T(\mathbb{E}) = e_i(t_1,\cdots,t_n) \in H^*(pt) = \mathbb{Z}[t_1,\cdots,t_n]$, where e_i is the i-th elementary symmetric function. To see this, note that $\mathbb{E}_m \times^T \mathbb{E} \simeq \mathcal{O}_1(-1) \oplus \cdots \oplus \mathcal{O}_n(-1)$ as vector bundles on $\mathbb{B}_m = (\mathbb{P}^{m-1})^n$.

Example 8. (Equivariant cohomology of projective space) Let $T = (\mathbb{C}^*)^n$ acts on $\mathbb{C}^n$ in the usual way, defining an action on $\mathbb{P}^{n-1} = \mathbb{P}(\mathbb{C}^n)$. This makes $\mathcal{O}_{\mathbb{P}(\mathbb{C}^n)}(1)$ a T-equivariant line bundle.

By choosing a isomorphism.

Write $\zeta = c_1^T(\mathcal{O}_{\mathbb{P}^{n-1}}(1))$.

Claim 1: We have
$$H_T^* \mathbb{P}^{n-1} \simeq \mathbb{Z}[t_1,\cdots,t_n][\zeta]/(\zeta^n + e_1(t)\zeta^{n-1} + \cdots + e_n(t))$$
$$= \mathbb{Z}[t_1,\cdots,t_n][\zeta]/\left(\prod_{i=1}^n (\zeta + t_i)\right).$$

Proof: Pass from the vector space $\mathbb{C}^n$ to the vector bundle $E = \mathbb{E}_m \times^T \mathbb{C}^n$ on $\mathbb{B}_m$. We have $\mathbb{E}_m \times^T \mathbb{P}^{n-1} \cong \mathbb{P}(E)$ and $\mathbb{E}_m \times^T \mathcal{O}_{\mathbb{P}^{n-1}}(1) \simeq \mathcal{O}_{\mathbb{P}(E)}(1)$ all over $\mathbb{B}_m$. The claim follows from the well-known presentation of $H^*\mathbb{P}(E)$ over $H^*(\mathbb{B})$ since $e_i(t) = c_i^T(\mathbb{C}^n) = c_i(E)$ and $\zeta = c_1^T(\mathcal{O}_{\mathbb{P}^{n-1}}(1)) = c_1(\mathcal{O}_{\mathbb{P}(E)}(1))$, as in Example 7. $\square$

After this short introduction, we now move the key theorem: Atiyah-Bott-Berline-Vergne.

Now let $G = T \simeq (\mathbb{C}^*)^n$ and let $\Lambda = \Lambda_T = H_T^*(pt) \simeq \mathbb{Z}[t_1, \cdots, t_n]$.

To present the localization we need to introduce Gysin map in two cases.

Case 1: closed embeddings: If $\iota: Y \hookrightarrow X$ is a T-invariant closed embedding of codimension d, we have $\iota_*: H_T^* Y \to H_T^{*+2d} X$. This satisfies

(a) $\iota_*(1) = \iota_*[Y]^T = [Y]^T$ is the fundamental class of Y in $H_T^{2d} X$.

(b) (Self-intersection) $\iota^*(\iota_*(\alpha)) = C_d^T(N_{Y/X}) \cdot \alpha$, where $N_{Y/X}$ is the normal bundle.

Case 2. Integral. For a complete nonsingular variety X of dimension n, the map $\rho: X \to pt$ gives $\rho_*: H_T^* X \longrightarrow H_T^{*-2n}(pt)$

Example 9. If $p \in Y \subseteq X$, with X nonsingular, and p is a nonsingular point on the (possible singular) subvariety Y, then $\iota_p^*[Y]^T = C_d^T(N_p)$ in $\Lambda = H_T^*(p)$. And for a equivariant vector bundle E of rank r on Y, then $\iota_p^*(C_i^T(E)) = C_i^T(E_p)$. Because E_p is a representation of T, with weights $\chi_1, \cdots, \chi_r$. i.e. $t \cdot (v_1, \cdots, v_r) = (\chi_1(t) v_1, \cdots, \chi_r(t) v_r)$ for $\chi_i: T \to \mathbb{C}^*$. So $C_i^T(E_p) = e_i(\chi_1, \cdots, \chi_r)$ is the i-th elementary symmetric polynomial in the χ's. Especially $\iota_p^*(C_r(E)) = \chi_1 \cdots \chi_r$. So $\iota_p^*[Y]^T = C_d^T(N_p) = \chi_1 \cdots \chi_d$.

The localization theorems and the slogans

Assume that X is a nonsingular variety, with finitely many fixed points. We consider the sequences of maps $\bigoplus_{p \in X^T} \Lambda = H_T^* X^T \xrightarrow{\iota_*} H_T^* X \xrightarrow{\iota^*} H_T^* X^T = \bigoplus_{p \in X^T} \Lambda$. The composition map $\iota^* \iota_*: \oplus \Lambda \to \oplus \Lambda$ is diagonal, and is multiplication by $C_n^T(T_p X)$ on the summand corresponding to p.

Theorem 10. Let $S \subseteq \Lambda$ be a multiplicative set containing the element $c := \prod_{p \in X^T} C_n^T(T_p X)$.

(a) The map $S^{-1}\iota^*: \boxed{S^{-1} H_T^* X \longrightarrow S^{-1} H_T^* X^T}$ is surjective, and the cokernel of ι_* is annihilated by c; $(*)$

(b) Assume in addition that $H_T^* X$ is a free Λ-module of rank at most $\# X^T$. Then the rank is equal to $\# X^T$, and the above map $(*)$ is an isomorphism.

Proof: (a) It suffices to show that the composite map $S^{-1}(\iota^* \circ \iota_*) = S^{-1}\iota^* \circ S^{-1}\iota_*$ is surjective. This in turn follows from the fact that the determinant $\det(\iota^* \circ \iota_*) = \prod_p c_n^T(T_p X) = c$ become invertible after localization.

For (b), surjectivity of $S^{-1}\iota^*$ implies rank $H_T^* X \geq \# X^T$, and hence equality. Finally, since $S^{-1}\Lambda$ is noetherian, a surjective map of finite free modules of the same rank is an isomorphism. $\square$

showed by Nakayama.

The question arise of how to verify the hypothese of Theorem 10 (b). To the end, we consider the following condition on a T-variety X:

(EF) $H_T^* X$ is a free Λ-module, and has a Λ-basis that restricts to a $\mathbb{Z}$-basis for $H^*(X)$ (EF is often called equivariant formal)

One common situation in which EF-condition holds is when X is non-singular projective variety, with X^T finite. In this case, the Bialynicki-Birula decomposition yields a collection of T-invariant subvarieties, one for each fixed point, whose classes form a $\mathbb{Z}$-basis for $H^* X$. The corresponding equivariant classes form a Λ-basis for $H_T^* X$ restricting to the one for $H^* X$, so EF holds. So is (b) in Theorem 10.

Corollary 11. For a nonsingular projective T-variety with finitely many fixed points: $H_T^* X \twoheadrightarrow H^* X$ and $H_T^* X \hookrightarrow H_T^* X^T$.

Remark 12. In general, we can not expect either $H_T^* X \twoheadrightarrow H_T^* X^T$ nor $H_T^* X \twoheadrightarrow H^* X$. Let $T = X = \mathbb{C}^*$ acting on itself via left multiplication, then there are no fixed points, so the second statement obviously fails. On the other hand, $\mathbb{E}T \times_T T = \mathbb{E}T$ has vanishing cohomology, hence $H_T^*(X) = H^*(\mathbb{E}T) = 0$ whereas the singular cohomology $H^1(X) \cong \mathbb{Z}$. So the $H_T^* X \twoheadrightarrow H^* X$ fails.

Now we can finally present the Integration formula (Atiyah-Bott-Berline-Vergne).

Theorem 12. Let X be a complete singular variety of dimension n, with finitely T-fixed points. Then

$$\underset{\substack{\uparrow \\ \text{Gysin-map}}}{\rho_*}\alpha = \sum_{p \in X^T} \frac{\iota_p^* \alpha}{c_n^T(T_p X)} \quad \text{for all } \alpha \in H_T^* X.$$

Proof: Since $\iota_*: S^{-1}H_T^*\mathcal{X}^T \to H_T^*\mathcal{X}$ is surjective, it is enough to assume $\alpha = (\iota_p)_*\beta$, for some $\beta \in H_T^*(p) = \Lambda$. Then the LHS of the displayed equation is $\rho_*\alpha = \rho_*(\iota_p)_*\beta = \beta$. Because $H_T^*(p) \xrightarrow{(\iota_p)_*} H_T^*\mathcal{X} \xrightarrow{\rho_*} \Lambda$ is an iso. The RHS is $\displaystyle\sum_{q\in\mathcal{X}^T} \frac{\iota_q^*(\iota_p)_*\beta}{c_n^T(T_q\mathcal{X})} = \frac{\iota_p^*(\iota_p)_*\beta}{c_n^T(T_p\mathcal{X})} = \beta$, using the self-intersection formula for the last equality. □

Example 13. Consider the standard torus action on $\mathbb{P}^1$. Then $\rho^*([\infty]^T) = \int_{\mathbb{P}^1} [\infty]^T =$
$\dfrac{\iota^*[\infty]^T}{c_1^T(T_\infty\mathbb{P}^1)} = \dfrac{c_1^T(T_\infty\mathbb{P}^1)}{c_1^T(T_\infty\mathbb{P}^1)} = 1$ (by Example 9).

Example 14. Take $\mathcal{X} = \mathbb{P}^{n-1}$, with the standard action of T via character $t_1, \cdots, t_n$, and let $\zeta = c_1^T(\mathcal{O}(1))$. The one computes $\rho_*(\zeta^k) = \begin{cases} 0 & \text{if } k < n-1, \text{ by degree: } H_T^{2k-2(n-1)}(pt) = 0; \\ 1 & \text{if } k = n-1 \text{ by ordinary cohomology.} \end{cases}$

On the other hand, using the localization formula, we obtain

$$\rho_*(\zeta^k) = \sum_{i=1}^{n} \frac{(-t_i)^k}{\prod_{j\neq i}(t_j - t_i)}. \qquad \text{(It is a nontrivial algebraic identity!)}$$

The fact $c_n^T(T_{p_i}\mathcal{X}) = \prod_{j\neq i}(t_j - t_i)$ follows from $T_{p_i}\mathcal{X} = \mathrm{Hom}(\mathbb{C}e_i, \mathbb{C}^n/\mathbb{C}e_i) \cong \bigoplus_{i\neq j} L_i^\vee \otimes L_j$.

$p_i \downarrow [0 : \cdots \underset{\uparrow\ i\text{-th}}{1} : \cdots : 0]$

has weight $t_j - t_i$. So by Example 9, $c_{top}^T(T_{p_i}\mathcal{X}) = \prod_{j\neq i}(x_j - x_i)$..