Kawai-Yoshioka formula

Goal: We want to calculate the PT-invariants of CY-threefolds of the form $X = S\times\mathbb{P}^1$ for a K3 surface S and show their surprising (unless you are a physicist perhaps) modular behavior.

Upshot: We have a $\mathbb{C}^\times$-action on $S\times\mathbb{C}$, and if we work $\mathbb{C}^\times$-equivariantly, we can reduce to studying K3 surfaces.

Step 1: Moduli space of stable pairs in the case $S\times\mathbb{C}$

Fix the following for the rest of the presentation: Let S be a complex, projective K3 surface and $\beta\in H_2(S;\mathbb{Z})$ a primitive effective curve class. Then $X = S\times\mathbb{C}$ has a $\mathbb{C}^\times$-action by scaling the second factor. Denote by $\iota: S \xrightarrow{(id,\{0\})} S\times\mathbb{C}$ the map including $S\times\{0\}$. Denote by $P_n(X,\iota_*\beta)$ the moduli space of stable pairs (F,s) with $\chi(F)=n$, $c_1(F)=\beta$.

Just like with the GW-invariants of K3's, we have the problem, that there are deformations of S moving β, such that it isn't a curve class anymore. By deformation invariance then, the virtual fundamental class would be trivial.

Note: One can show that $P_n(X,\iota_*\beta)\cong P_n(S,\beta)\times\mathbb{C}$

as long as β is irreducible.

We want a reduced virtual class:

$$[P_n(X,\iota_*\beta)]^{red} \in A_1(P_n(X,\iota_*\beta),\mathbb{Q})$$

Problem: As X is not compact, $P_n(X,\iota_*\beta)$ will also not be compact. But $P_n(X,\iota_*\beta)^{\mathbb{C}^\times}$ is!

Solution: Use $\mathbb{C}^\times$-equivariant cohomology. The point has eq. cohomology generated by t, the first Chern class of the standard representation $id: \mathbb{C}^\times \to \mathbb{C}^\times$. Pushing forward to the point yields:

$$P_{n,\beta} := \int_{[P_n(X,\iota_*\beta)]^{vir}} 1 \qquad \in \mathbb{Q}(t)$$

(The details of equivariant cohomology will be explained in the next talk)

Sketch: Recall from the last presentations, that the obstruction bundle of the deformation theory on $P_n(X,\iota_*\beta)$ has fiber over $\{\mathcal{O}_X \xrightarrow{s} F\}$:

$$\mathrm{Ext}^2(I^\bullet,I^\bullet)_0 \underset{\text{Serre duality}}{\cong} \mathrm{Ext}^1(I^\bullet,I^\bullet\otimes K_X)_0^\vee \underset{\text{K3+?}}{\cong} \mathrm{Ext}^1(I^\bullet,I^\bullet)_0^\vee \otimes K_C$$

We have also seen, that $\mathrm{Ext}(I^\bullet,I^\bullet)_0$ is the tangent space of $P_n(S,\beta)\times\mathbb{C}$ at the point.

Hence the whole thing is isomorphic to $\Omega_{P_n(S,\beta)\times\mathbb{C}}$. Calling the standard representation of $\mathbb{C}^\times$, t, this yields the following:

The reduced obstruction bundle is $\Omega_{P_n(S,\beta)}\otimes t$, which is constructed from $(\Omega_{P_n(S,\beta)} \otimes t) \oplus \mathbb{C} \cong (\Omega_{P_n(S,\beta)} \oplus (-t)) \otimes t$ by removing the summand $\mathbb{C}$

dual of standard rep. from $\Omega_{\mathbb{C}}$

Lemma $P_{n,\beta} = \frac{1}{t}(-1)^{n+\langle\beta,\beta\rangle+1} e(P_n(S,\beta))$

topological Euler characteristics

proof We have that the dimension of $P_n(S,\beta)$ is $N := n+\langle\beta,\beta\rangle+1$.

$$P_{n,\beta} = \int_{P_n(X, \iota_*\beta)} e(\underbrace{\Omega_{P_n(S,\beta)} \otimes t}_{\text{reduced obstruction bundle}})$$

$$\underset{\substack{\text{Attiyah-Bott}\\\text{localization formula}}}{=} \int_{\substack{P_n(S,\beta)\\ \text{"}\\ \mathbb{C}^\times\text{-fixed points by the}\\ \text{discussion above}}} \frac{\overbrace{\sum_{i=0}^{N} c_i(\Omega_{P_n(S,\beta)})\, t^{N-i}}^{\text{expansion of } e(\Omega_P \otimes t)}}{t}$$

$$= \int_{P_n(S,\beta)} \frac{e(\Omega_{P_n(S,\beta)})}{t} = \frac{(-1)^N}{t} e(P_n(S,\beta)) \quad \boxtimes$$

because $T_{P_n(S,\beta)} = \Omega^\vee_{P_n(S,\beta)}$

Step 2 Calculation of the topological Euler characteristic of $P_n(S,\beta)$

Since we are calculating deformation invariant quantities, only $2g-2 = \langle\beta,\beta\rangle$ matters. So we can take β to be dC for an integer d and a minimizing curve class C for a polarization of S.

This means we have access to the following lemma by Lemma by Yoshioka:

<u>Lemma</u> (i) Assume F is μ-stable and $c_1(F) = C$, then every nontrivial extension $0 \to \mathcal{O}_S \to E \to F \to 0$ is μ-stable

(ii) Assume E is μ-stable, of positive rank, and $c_1(E) = C$. Then any non-zero section

$$0 \longrightarrow \mathcal{O}_S \xrightarrow{s} E \longrightarrow F \longrightarrow 0$$

yields that F is μ-stable.

$\Rightarrow$ we don't have to worry about stability too much

This motivates a stability condition, where a pair of a sheaf and a section are stable if the sheaf is stable.

<u>Definitions</u>: Let $\mathcal{M}(\underbrace{r, d, e}_{\text{this is the Chern Character of } F})$ be the moduli space of semistable sheaves of rank r, degree d and Euler number e

i.e. $c_1(F) = dC$

(We will assume this to consist of μ-stable sheaves only)

Let $\mathcal{P}^1(r,d,e)$ be the moduli space of coherent systems i.e. stable sheaves (rank r, degree d, Euler number e) together with a section.

There are two interesting maps between those:

1. $$\begin{array}{cc} \mathcal{P}^1(r,d,e) & [\mathcal{O}_S \xrightarrow{s} F] \\ \downarrow & \downarrow \\ \mathcal{M}(r,d,e) & [F] \end{array}$$ is a map with fiber $\mathbb{P}H^0(F)$ over $[F]$

② $$\begin{array}{ccc} \mathcal{P}^1(r+1, d, e+\underbrace{\chi(\mathcal{O}_S)}_{=2 \text{ in our case}}) & & [\mathcal{O}_S \xrightarrow{s} F] \\ \downarrow & & \downarrow \\ M(r,d,e) & & \operatorname{coker}(s) \end{array}$$

is a map with fibers $\mathbb{P}\operatorname{Ext}^1(\mathcal{O}_S, F)$ over $[F]$. Because these are exactly extensions $0 \to \mathcal{O}_S \to ? \to F \to 0$

By Serre duality, we have $\operatorname{Ext}^1(\mathcal{O}_S, F) = H^1(F)$

(Because of the lemma, all this is again stable)

<u>Example</u>. Similar to an example in the first talk, we get: $\mathcal{P}^1(0,1,e)$ parametizes

$$\left\{ \begin{array}{c} \mu\text{-stable sheaves } F \text{ of rank } 0, \text{ degree } 1 \\ \text{and Euler-number } e \text{ with a section} \end{array} \right\}$$

$$\updownarrow 1:1$$

$$\left\{ \begin{array}{c} \text{Torsion-free rank 1 sheaf } E \text{ supported} \\ \text{on a curve in class } C, \text{ with a section} \end{array} \right\}$$

We get a map:

$$\begin{array}{ccc} \mathcal{P}^1(0,1,e) & & [\mathcal{O}_S \xrightarrow{s} F] \\ \downarrow & & \downarrow \\ \operatorname{Hilb}(S, C) & & \operatorname{supp}(F) \end{array}$$

Hilbert scheme of subschemes in S in the class C

Again, just like in the example of an isolated curve in a CY 3-fold, this exhibits $\mathcal{P}^1(0,1,e)$ as relative Hilbertscheme of $e+g-1$ points i.e. the fibers over $[C' \hookrightarrow S]$ is isomorphic to $\mathrm{Hilb}^{e+g-1} C'$ where g is the genus of C'.

Now we want to use the maps ① and ② as relations for our calculation of the topological Euler characteristics. We first need some general theory

<u>Def</u>. The Grothendieck ring of varieties $K_0(Var/\mathbb{C})$ is generated by classes $[X]$ for varieties $X/\mathbb{C}$ subject to the relations $[X]=[Z]+[U]$ for closed $Z \subseteq X$, $U = X \setminus Z$ and $[X]\cdot[Y]=[X\times Y]$.

By excision + Künneth formula, there exists a unique ring homomorphism

$$e : K_0(Var/\mathbb{C}) \longrightarrow \mathbb{Z}$$
$$[X] \longmapsto e(X)$$ the topological Euler characteristics.

Denote the class of $[\mathbb{A}^n_{\mathbb{C}}] = [\mathbb{A}^1_{\mathbb{C}}]^n = \mathbb{L}^n$ and $[\mathbb{P}^{n-1}_{\mathbb{C}}] = [n]$.

<u>Def</u>. Let $\mathcal{M}(r,d,e)_s$ be the subscheme of $[F]$'s s.t. $h^0(F)=s$.

Then we have $[\mathcal{M}(r,d,e)] = \sum_s [\mathcal{M}(r,d,e)_s]$ and on these strata, ① is a projective bundle;

but so is (2) because $e = \chi(F) = h^0(F) - h^1(F) + h^2(F)$
$= h^0(F) - h^1(F)$ is also constant

$\Rightarrow$ From (1):

$$[P^1(r,d,e)] = \sum_i [e+i][\mathcal{M}(r,d,e)_{e+i}]$$

and from (2):

$$[P^1(r+1,d,e+\underbrace{\chi(\mathcal{O}_S)}_{=2 \text{ for K3}})] = \sum_i [i][\mathcal{M}(r,d,e)_{e+i}]$$

together $\Rightarrow$

$$[P^1(r,d,e)] = \sum_i [e+i][\mathcal{M}(r,d,e)_{e+i}]$$

$$= [e][\mathcal{M}(r,d,e)] + \mathbb{L}^e \sum [i][\mathcal{M}(r,d,e)_{e+i}]$$

$$= [e][\mathcal{M}(r,d,e)] + \mathbb{L}^e [P^1(r+1,d,e+2)]$$

As e is increased in every such step, one can see that this strictly decreases the dimension of $P^1(r+i,d,e+2i)$ and hence, at some step, produces the empty set

$\Rightarrow [P^1(r,d,e)] =$

$$\sum_{i=0}^{\infty} [e+2i] \mathbb{L}^{\sum_{j=0}^{i-1}(e+2j)} [\mathcal{M}(r+i,d,e+2i)]$$

<u>Step 3</u> <u>Conclusion</u>

Thm. [Yoshioka: Some examples of Mukai's reflections on K3 surfaces, Thm 0.2] $\mathcal{M}(r,d,e)$ is deformation equivalent to $\mathrm{Hilb}^{d-re}(S)$.

In particular:

Remember: $2g-2=\langle\beta,\beta\rangle$

$$e(\mathcal{M}(r,1,e)) = e(\mathrm{Hilb}^{g-re}(S))$$

Thm (Göttsche) Given a K3 surface S, we have the generating function generating function of the topological Euler characteristics is given by:

$$\sum_{n=0}^{\infty} \chi(\mathrm{Hilb}^n(S)q^n) = \frac{q}{\Delta(\tau)} \qquad q=e^{2\pi i\tau}$$

discriminant modular form

$$\sum_{n=0}^{\infty} (-1)^n e(\mathrm{Hilb}^n(S))\, q^n = q\,\frac{\eta(\tau)^{16}}{\eta(2\tau)^{20}}$$

(comes from a more general function for Hodge polynomials)

We want to calculate the case of $P^1(0,1,e)$ for a curve class β with $2g-2=(\beta,\beta)$:

$$\sum_{i=0}^{\infty} e(\underbrace{[e+2i]}_{=[\mathbb{P}^{e+2i-1}]})\, e(\amalg^{\sum_{j=0}^{i-1}(e+2j)})\, e([\mathcal{M}(i,1,e+2i)])$$

Thm (Kawai-Yoshioka formula)

$$\sum_{h=0}^{\infty} \sum_{n=1-h} e(P_n(S,h)) z^n q^{h-1} =$$

$$\left(\sqrt{z} - \frac{1}{\sqrt{z}}\right)^{-2} \frac{1}{\Delta(z,q)}$$

some explicit Jacobi modular form:

$$\Delta(z,q) = q \prod_{n=1}^{\infty} (1-q^n)^{20} (1-zq^n)^2 \cdot (1-z^{-1}q^n)^2$$

proof (sketch). We form the power series w.r.t. both the genus h of the curve class and the Euler number:

$$\sum_{h=0}^{\infty} \sum_{n=1-h} e(P_{\beta_h}(0,1,-n)) z^n q^{h-1} =$$

$$\sum_{h=0}^{\infty} \sum_{n=1-h} \sum_{i=0}^{\infty} e(\underbrace{[e+2i]}_{=[\mathbb{P}^{e+2i-1}]}) e(\amalg^{\sum_{j=0}^{i=1}(e+2j)}) e([M(i,1,e+2i)])$$

One can split this triple-power-series into two! Then, one can apply Göttsches Formula and various functional identities of modular forms $\square$