1. Setup: $/\mathbb{C}$

$X \to B$ connected smooth projective, of dim n

$\alpha \in \mathrm{Pic}(X)$

$\mathcal{M} \to B$ is a relative (algebraic space) fine moduli space of perfect simple complexes with fixed determinant α

- perfect & simple with ~~fixed determinant~~.

$\mathrm{Hom}(E, E) = \mathbb{C}$ $\quad E \cong 0 \to E^k \to \dots E^j \to 0 \dots$ E^i is locally free

- fine moduli space of complexes with fixed determinant, α

$$\exists\ \mathbb{E} \in D^b(X \times_B \mathcal{M})$$

s.t. $\det(\mathbb{E}) \cong \pi_{\mathcal{M}}^* M \otimes \pi_X^* \alpha$

s.t. $\mathcal{M}(\mathbb{C}) = \left\{ \begin{array}{l} E \text{ on } X_b \\ \det(E) \cong \alpha_b \end{array} \right\}$

$$\mathcal{M}(S) = \left\{ \begin{array}{l} \cancel{X \times_B} \ \mathbb{E} \in X \times_B S \\ \forall s \in S \ \cancel{E_s \cong E_m} \text{ for } \exists m \in \mathcal{M} \\ \text{s.t. } \mathbb{E}_s \cong \mathbb{E}_m \\ \det(\mathbb{E}) \cong \pi_S^* M_S \otimes \pi_X^* \alpha \quad M_S \in \mathrm{Pic}(S) \end{array} \right\} \Big/ \sim$$

$$\mathbb{E} \sim \mathbb{E}' \iff \mathbb{E} \cong \mathbb{E} \otimes \pi^* M'_S \quad M'_S \in \mathrm{Pic}(S) \}$$

$$f: S \to \mathcal{M} \iff (id \times f)^* \mathbb{E}$$

we also assume $ch(\mathbb{E}_m) = ch(\mathbb{E}_{m'})\ \forall m, m' \in \mathcal{M}$

2. Relative obstruction theory

$\mathrm{rk}(\mathbb{E}) \neq 0$

$\mathrm{id}: \mathcal{O}_{X\times_B \mathcal{M}} \to \mathcal{H}om(\mathbb{E},\mathbb{E})$

$\mathrm{tr}: \mathcal{H}om(\mathbb{E},\mathbb{E}) \to \mathcal{O}_{X\times_B \mathcal{M}}$

$\mathrm{tr}\circ \mathrm{id}$ ~~id∘tr~~ $= \mathrm{rk}(\mathbb{E})$.

$$\Downarrow$$

$$\mathcal{H}om(\mathbb{E},\mathbb{E}) \simeq \underbrace{\mathcal{H}om(\mathbb{E},\mathbb{E})_0}_{= \ker \mathrm{tr}} \oplus \mathcal{O}_{X\times_B \mathcal{M}}$$

$A(\mathbb{E}) \in \mathrm{Ext}^1_{X\times_B \mathcal{M}}(\mathbb{E}, \mathbb{E}\otimes \mathbb{L}_{X\times_B \mathcal{M}})$

- composing with $\mathbb{L}_{X\times_B \mathcal{M}} \to \mathbb{L}_{X\times_B \mathcal{M}/X} = \pi_{\mathcal{M}}^* \mathbb{L}_{\mathcal{M}/B}$
- and precomposing with $\mathcal{H}om(\mathbb{E},\mathbb{E})_0 \to \mathcal{H}om(\mathbb{E},\mathbb{E})$

$A(\mathbb{E}/X) \in \mathrm{Ext}^1_{X\times_B \mathcal{M}}(\mathcal{H}om(\mathbb{E},\mathbb{E})_0, \pi_{\mathcal{M}}^* \mathbb{L}_{\mathcal{M}/B})$

- applying Verdier duality we obtain

$$\mathrm{Ext}^{1-n}_{\mathcal{M}}(\pi_{\mathcal{M}*}(\mathcal{H}om(\mathbb{E},\mathbb{E})_0 \otimes \omega_{\pi_{\mathcal{M}}}), \mathbb{L}_{\mathcal{M}/B})$$

$$\rightsquigarrow \phi : \underbrace{\pi_{\mathcal{M}*}(\mathcal{H}om(\mathbb{E},\mathbb{E})_0 \otimes \pi_X^* \omega_{X/B})[n-1]}_{= F^\bullet} \to \mathbb{L}_{\mathcal{M}/B}$$

~~Set up~~ $/ \mathbb{k}$

~~X - smooth connected projective~~

~~$\alpha \in \mathrm{Pic}(X)$~~

$\mathcal{M}$ - ~~moduli~~ fine moduli

~~Theorem: ϕ is a perfect obstruction~~ theory

Theorem (Behrend-Fantechi)

$\phi : F^{\bullet} \to \mathbb{L}^{\bullet}_{X/B}$ is a perfect obstruction theory iff.

1. $\forall (S_0, S, f)$ s.t. $S_0 \hookrightarrow S$ is a square-zero extension defined by I

$$\begin{array}{ccc} S_0 & \hookrightarrow & S \\ f \downarrow & \swarrow f' & \\ X & & \end{array}$$

$$\exists f' : S \to X \iff \overset{o(f)}{\cancel{\phi^*(\omega(f))}} \in \mathrm{Ext}^1(f^* \mathbb{F}, I)$$

$$\Big(\underset{o(f)}{\cancel{\phi^*(\omega(f))}} = \Big\{ f^* \mathbb{F} \to \underbrace{f^* \mathbb{L}^{\bullet}_{X/B} \to \mathbb{L}^{\bullet}_{S_0/B} \to I[1]}_{\omega(f)\; K(S_0/S/B)}$$

2. if $o(f) = 0$ then extensions form a torser under $\mathrm{Hom}(g^* \mathbb{F}^{\bullet}, I)$

Theorem ϕ is a perfect obstruction theory.

Let $f : S_0 \to \mathcal{M}$, $S_0 \subset S$ be as above.

$$\overbrace{f^* \pi_{\mathcal{M}*}(\mathcal{H}om(\mathbb{E}, \mathbb{E})_0 \otimes \pi_X^* \omega_{X/B})[n-1] \to f^* \mathbb{L}_{\mathcal{M}/B}}^{*} \to \underbrace{\mathbb{L}_{S_0/B} \to I[1]}_{K(S_0/S/B)}$$

$**$

$$** = o(f) \in \mathrm{Ext}^{2-n}_{S_0}(f^* \pi_{\mathcal{M}*}(\mathcal{H}om(\mathbb{E}, \mathbb{E})_0 \otimes \pi_X^* \omega_{X/B}) I)$$

$= [\mathcal{M}_B]$

Let $\bar{f} = \mathrm{id} \times f : X \times_B S_0 \to X \times_B M$

~~and Let $\bar{\pi}$ be the projection~~

$\bar{\pi} : X \times_B S_0 \to S_0$

$$\begin{tikzcd} X \times_B S_0 \arrow[r] \arrow[d, "\bar{\pi}"'] & X \times_B M \arrow[d, "\bar{\pi}_M"] \\ S_0 \arrow[r] & M \end{tikzcd}$$

$\bar{\pi}$ is flat $\Rightarrow$

$$* = \bar{\pi}_{*}\left(\mathcal{H}om(\bar{f}^*\mathbb{E}, \bar{f}^*\mathbb{E})_0 \otimes \pi_X^* \omega_{X/B}\right)[n-1] \to \mathbb{L}_{S_0/B}$$

$= A(\bar{f}^*\mathbb{E}/X)$

by functoriality of Atiyah class

$$\Rightarrow ** = A(\bar{f}^*\mathbb{E}/X) \circ \bar{\pi}^* k\left(X \times_B S_0 / X \times_B S / X\right) = \bar{\pi}^* k(* S_0/S/B)$$

trace-free part of

$$= \varpi(X \times_B S_0 / X \times_B S)(\bar{f}^*\mathbb{E}) = \circledast$$

$$\hat{\mathcal{O}}(\cancel{f^*\mathbb{E}})$$

$$\left(\mathrm{Ext}^{2-n}_{S_0}\left(\bar{\pi}_*\left(\mathcal{H}om(\bar{f}\mathbb{E}, \bar{f}\mathbb{E})_0 \otimes \omega_{\bar{\pi}}\right), I\right)\right.$$

$$\left.\cong \mathrm{Ext}^2_{X \times_B S_0}\left(\bar{f}^*\mathbb{E}, \bar{f}^*\mathbb{E} \otimes \pi^* I\right)_0\right)$$

Verdier duality

~~trace of an obstruction~~ = obstruction to deform $\cancel{\det}(\bar{f}^*\mathbb{E})$

$\mathrm{tr}\, \bar{\varpi} = \varpi(\det(\bar{f}^*\mathbb{E}))$

but ~~by~~ we fixed the determinant

$\Rightarrow \circledast = 0$ iff $\bar{f}^*\mathbb{E}$ extends from $X \times_B S_0$ to $X \times_B S$

$\Leftrightarrow$ $f: S_0 \to \mathcal{M}$ extends to $f': S \to \mathcal{M}$
by the moduli problem.

deformations form a torsor under
$$\mathrm{Ext}^1_{X \times_B S_0}(\bar{f}^*\mathbb{E}, \bar{f}^*\mathbb{E} \otimes \pi^* \underline{I})_0$$

Virtual cycle.
We need a perfect obstruction theory

$\heartsuit$ $\mathrm{Ext}^i_{X_b}(\mathbb{E}_m, \mathbb{E}_m) = 0$ $i \neq 1,2$ for all $m \in \mathcal{M}$

($\mathbb{E}$ is perfect $\Rightarrow \pi_{\mathcal{M}*}(\mathcal{H}om(\mathbb{E}, \mathbb{E})_0)$ is perfect complex)
$\nRightarrow$

~~Corollary $\heartsuit$ If $\heartsuit$ is satisfied~~

$$\pi_{\mathcal{M}*}\left(\mathcal{H}om(\mathbb{E}, \mathbb{E})_0 \otimes \pi_X^* \omega_{X/B}\right)[n-1]$$
$$\cong \pi_{\mathcal{M}*} \mathcal{H}om(\mathbb{E}, \mathbb{E})_0^\vee [1]$$

$\nRightarrow$
Corollary if $\heartsuit$ is satisfied, then ϕ is a <u>perfect</u> obstruction theory.

$\forall b$ $\exists [\mathcal{M}_b]^{vir}$ $\mathrm{vdim} = \chi(\mathcal{H}om(\mathbb{E}_m, \mathbb{E}_m)_0 \; (m \in \mathcal{M})$ def. in b
i.e. $\exists [\mathcal{M}]^{vir}$ on $\mathcal{M}$ s.t. $i_b^! [\mathcal{M}]^{vir} = [\mathcal{M}_b]^{vir}$

(Locally $F^\cdot \cong F^{-1} \to F^0$
locally free

♡ cannot be reasonably satisfied in dimensions ≥ 4

automatically satisfied in dimensions ≤ 2 by simple complexes

$$♡ \iff (i)\ \mathrm{Ext}^i_{X_b}(E_m, E_n)_0 = 0 \quad i < 0$$

$$(ii)\ H^0(X_b, \omega_{X_b}) \to \mathrm{Hom}(E, E \otimes \omega_{X_b}) \text{ is an isomorphism}$$

For CY ($\omega_{X_b} \cong \mathcal{O}_{X_b}$) ~~ii) ⟹ E_n needs to be simple then i) and ii) are satisfie~~

only need i), because if E_n is simple then ii) is satisfied

(e.g. if E is a sheaf)

in this case $\chi(\mathcal{H}om(E_m, E_n)_0) = 0$

⟹ virtual count of points.

$t(2) \in \mathbb{H}^+$