DT theory on orbifolds

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Preliminaries

$\mathcal{X}$ DM stack$/\mathbb{C}$, separated

$\mathcal{X}$ is an orbifold if

(i) smooth of finite type

(ii) generically trivial stabilizers.

$\mathcal{X}$ is ~~quasi~~projective if closed embeddable into a smooth proper DM stack

- $\pi: \mathcal{X} \to X$ coarse moduli space, X projective.

Ex:

- $[Y/G]$, $|G| < \infty$.
- $\sqrt[r]{(X,D)}$, $D \subset X$ smooth Cartier divisor.

From now on, we assume $\mathcal{X}$ is smooth & projective.

Classes:

$$N(\mathcal{X}) = K(\mathrm{Coh}(\mathcal{X}))/\ker \chi$$

$$\chi(E,F) = \sum_i (-1)^i \dim \mathrm{Ext}^i_{\mathcal{X}}(E,F)$$

$\rightsquigarrow N(\mathcal{X})$ free abelian group of finite rank.

$N_0(\mathcal{X})$, $N_{\leq 1}(\mathcal{X})$ are subgroups generated by sheaves supported in dim 0, ≤ 1

$N_1(\mathcal{X}) = N_{\leq 1}(\mathcal{X})/N_0(\mathcal{X})$, we pick a splitting $N_0 \subseteq N_{\leq 1}$, $\rightsquigarrow N_{\leq 1} \cong N_0 \oplus N_1 \ni (c, \beta)$

Stable pair

A stable pair is

$$\mathbb{E} = \big(\underset{-1}{\mathcal{O}_{\mathfrak{X}}} \longrightarrow \underset{0}{F} \big) \in D(\mathfrak{X})$$

- F pure of dim 1
- $\mathrm{coker}(s)$ is of dim at most 0.

We say $\mathbb{E}$ is of class (β, c) if $[\mathbb{E}] = (\beta, c)$.

Thm ~~let $\mathfrak{X}$ be a proj~~

$\exists$ fine proj. moduli ~~space~~ scheme $P_{\mathfrak{X}}(\beta, c) : Sch^{\circ} \longrightarrow Set$

parametrizing stable pairs of class (β, c).

$$\mathbb{E} = (\mathcal{O} \longrightarrow \mathbb{F}) \in D(\mathfrak{X} \times P_{\mathfrak{X}}(\beta, c))$$

univ. stable pair.

(If $\mathfrak{X}$ is only projective, for fixed Hilbert poly. P

$P_{\mathfrak{X}}(P)$ is a fine moduli scheme)

$$\begin{tikzcd} & \mathfrak{X} \times P_{\mathfrak{X}}(X, \beta) \arrow[dl] \arrow[d] \\ \mathfrak{X} & P_{\mathfrak{X}}(\beta, c) \end{tikzcd}$$

Thm For $\dim \mathfrak{X} = 3$,

$$\phi : R\pi_{*} \big(R\mathcal{H}om(\mathbb{E}, \mathbb{E})_0 \otimes \pi_{\mathfrak{X}}^{*}(\omega_{\mathfrak{X}}) \big)[2] \longrightarrow \mathbb{L}_{P_{\mathfrak{X}}(\beta, c)}$$

p.o.t.

$$P_{\mathfrak{X}}(\beta, c) \rightsquigarrow [P_{\mathfrak{X}}(\beta, c)]^{vir} \in A_{vd} (P_{\mathfrak{X}}(\beta, c))$$

<u>Defn</u> $\mathcal{X}$ is CY if $\omega_{\mathcal{X}} \cong \mathcal{O}_{\mathcal{X}}$.

(e.g. $[\mathbb{C}^n/G]$, $G \subset SL_n(\mathbb{Z})$, $|G| < \infty$)

Orbifold Chern character:

$$\begin{tikzcd} (x,g) \arrow[d, mapsto] & I\mathcal{X} \arrow[r] \arrow[d, "\pi_0"'] \arrow[dr, phantom, "\lrcorner"] & \mathcal{X} \arrow[d, "\Delta"] \\ x & \mathcal{X} \arrow[r, "\Delta"'] & \mathcal{X}\times\mathcal{X} \end{tikzcd}$$

$I\mathcal{X}$ <u>inertia stack</u>.

$$Ob(I\mathcal{X}) = \left\{ (x,g) \;\middle|\; \begin{array}{l} x \in Ob(\mathcal{X}) \\ g \in Aut(x) \end{array} \right\}$$

(a) $H^\bullet(\mathcal{X},\mathbb{Q}) = H^\bullet(X,\mathbb{Q})$

$\mathcal{X} = [Y/G]$, $H^\bullet(\mathcal{X},\mathbb{Q}) = H_G(Y,\mathbb{Q}) = H^\bullet(Y,\mathbb{Q})^G$.

in general defined by $[X_1 \underset{s}{\overset{t}{\rightrightarrows}} X_0]$ as $\mathrm{Ker}(H^\bullet(X_0) \xrightarrow{s^*-t^*} H^\bullet(X_1))$.

(b) $$H^\bullet_{orb}(\mathcal{X},\mathbb{Q}) := H^\bullet(I^\bullet\mathcal{X},\mathbb{Q})$$

If $\mathcal{X} = [Y/G]$, $$I\mathcal{X} = \coprod_{[g]} [Y^g/C(g)].$$

- Y^g is fixed locus of $g \in G$.
- $C(g) \subset G$, the centralizer subgroup of g, $C(g) = \{h \in G \mid hg = gh\}$.
- $[g]$ conjugacy class of g.

For S surface,

$$H^*(S^{[n]}) \cong H^*_{orb}([S^n/S_n], \mathbb{Q})$$

- $\exists$ a canonical involution

$$i : I\mathcal{X} \longrightarrow I\mathcal{X}$$

$$(x,g) \longmapsto (x,g^{-1})$$

- $I\mathcal{X} = \coprod_{j \in J} \mathcal{X}_j$, for $(x,g) \in \mathcal{X}_i$, one has a decomposition

of the tangent space $T_x\mathcal{X} = \bigoplus_{0 \le t \le r_i} U^{(t)}$

where $U^{(t)}$ is an eigenspace of $\langle g \rangle$ ↻ $T_x\mathcal{X}$ with

eigenvalue $\lambda_{r_i}^t$, $0 \le t < r_i$, where $\lambda_{r_i} = \exp\left(\frac{2\pi i}{r_i}\right)$.

$$age_i = \frac{1}{r_i} \sum_{0 \le t < r_i} t \cdot \dim(U^{(t)}) \in \mathbb{N}.$$

E.g. $I([S^n/S_n]) = \coprod_{\mu \vdash n} [S^{\mu}/C(\mu)]$ $\quad age_\mu = n - \ell(\mu)$.

For any W $(= \pi_0^*(V))$ on $I\mathcal{X}$, $\exists$ decomposition

$$W = \bigoplus_\lambda W^{(\lambda)}$$

$W^{(\lambda)}$ is eigenbundle to eigenvalue λ

$$\rho : K(I\mathcal{X}) \longrightarrow K(I\mathcal{X})_{\mathbb{C}}$$

$$\rho(W) = \sum_{\lambda} \lambda\, W^{(\lambda)}.$$

$$\widetilde{ch} : K(\mathcal{X}) \longrightarrow A^{\circ}(I\mathcal{X}).$$

$$\begin{array}{ccc} I\mathcal{X} & \longrightarrow & \mathcal{X} \\ \downarrow{\scriptstyle \pi_0} & & \\ \mathcal{X} & & \end{array}$$

$$\widetilde{ch}(V) = ch(\rho(\pi_0^*(V)))$$

$$= \sum_{\lambda} \lambda\, ch(\pi_0^*(V)^{(\lambda)})$$

$$A^*_{orb}(\mathcal{X}) = \bigoplus_i A^{*-age(i)}(\mathcal{X}_i)$$

Orbifold characters:

$$\left.ch_k^{orb}\right|_{\mathcal{X}_i} := \left.\widetilde{ch}_{k-age(i)}\right|_{\mathcal{X}_i}$$

this is the right defn because ch^{orb} ~~resp. multiplication~~. respects multiplication wrt orbifold cup product & tensor product.

$+ RR.(?)$

PT insertions

for any $\gamma_i \in A^{\ell}_{orb}(\mathcal{X})$ define operators:

$$\mathrm{ch}^{orb}_{k+2}(\gamma) : A_*(P_{\mathcal{X}}(\beta, c)) \longrightarrow A_{*-k+1-\ell}(P_{\mathcal{X}}(\beta, c))$$

to be

$$\mathrm{ch}^{orb}_{k+2}(\gamma)(\xi) = \pi_{P*}\left(\mathrm{ch}^{orb}_{k+2}(\mathbb{E}) \cdot \iota^* \pi^*_{I\mathcal{X}}(\gamma) \cdot \pi_P^*(\xi) \right)$$

$$\iota : I\mathcal{X} \longrightarrow I\mathcal{X}$$

$$\pi_P : I\mathcal{X} \times P_{\mathcal{X}}(\beta, c) \longrightarrow P_{\mathcal{X}}(\beta, c)$$

$$\pi_{I\mathcal{X}} : I\mathcal{X} \times P_{\mathcal{X}}(\beta, c) \longrightarrow I\mathcal{X}$$

$$\left\langle \prod_{i=1}^{r} \tau_{k_i}(\gamma_i) \right\rangle^{\mathcal{X}}_{c,\beta} := \int_{P_{\mathcal{X}}(c,\beta)} \prod_{i=1}^{n} \mathrm{ch}^{orb}_{k_i+2}(\gamma_i) \, [P_{\mathcal{X}}(c,\beta)]^{vir}$$

- ~~Relation~~ geo

- genus zero relative GW (X, D) = genus zero orbifold GW theory of $\sqrt[r]{(X, D)}$ for $r \gg 0$

- Crepant resolution conjectures:

$$\begin{tikzcd} \mathcal{X} \arrow[r] & X \\ & Y \arrow[u] \end{tikzcd} \qquad PT(\mathcal{X}) = \frac{PT(Y)}{PT_{exc}(Y)}$$