Donaldson-Thomas Theory

Seminar, Universität Bonn, Wintersemester 2021

Organizer: Georg Oberdieck

Friday 12-2, Room N 0.003

Talk 1. (Oct 22) Introductory lecture (Georg Oberdieck)

Overview over the general constructions of Donaldson-Thomas invariants (moduli space, virtual class, descendents) [13]. We will also consider Pandharipande-Thomas theory concerning stable pairs theory [11]. A few example computations will be discussed.

Talk 2. (Oct 29) The perfect obstruction theory of the moduli space I (Maximilian Schimpf)

This talk will concern the construction of the perfect-obstruction theory on moduli spaces of simple complexes on projective threefolds. There are two main players involved, the Kodaira spencer class and the Atiyah class. Classically, these are quite easy to define: Given a first order deformation $\pi : \mathcal{X} \to B = \mathrm{Spec}(k[x]/x^2)$ of a complex manifold $X := \pi^{-1}(0)$, we have the sequence of tangent bundles

$$0 \to \mathrm{Tan}_X \to \mathrm{Tan}_{\mathcal{X}}|_X \to \pi^*(\mathrm{Tan}_{B,0}) = \mathcal{O} \to 0$$

whose extension class $\kappa \in \mathrm{Ext}^1(\mathcal{O}, T_X) = H^1(X, T_X)$ is the Kodaira-spencer class. The Atiyah class $\mathrm{At}(E) \in \mathrm{Ext}^1(E, E \otimes \Omega)$ of a vector bundle E is the class of its curvature, but is best defined in terms of its transition functions. For these classical notions a possible source is [7].

In this talk we follow Huybrechts-Thomas [8]. After introducing shortly the classical notion of Kodaira spencer and Atiyah class, introduce the generalizations in [8]. Then state the main theorem of [8] given in the introduction. If time permits, sketch a proof.

Talk 3. (Nov 5) The perfect obstruction theory of the moduli space II (Denis Nesterov)

This talk concerns the existence of perfect-obstruction theories on proper fine moduli space of simple complexes in the derived category of a projective threefold. Follow Section 4 of [8], in particular state Theorem 4.1. and discuss the proof. Discuss further when this obstruction theory is perfect, and hence gives a virtual cycle in the sense of Behrend-Fantechi (Section 4.3). Optional: Section 4.4.

Talk 4. (Nov 19) The Kawai-Yoshioka formula (Adam Dauser)

This is one of the few results in Pandharipande-Thomas theory which is both explicit, beautiful and easily accessible all at the same time. It concerns the Pandharipande-Thomas theory of $S \times \mathbb{P}^1$ in classes $(\beta, 0)$ where S is a K3 surface and $\beta \in H_2(S, \mathbb{Z})$ is an irreducible curve class.

Follow [10, Section 1.2] to relate the PT invariant in this case to the topological Euler number of the smooth moduli space $P_n(S, \beta)$. Then follow the argument of Kawai-Yoshioka, which can be written in a concise and modern form in Section 3.1 and Section 3.2 of [5].

Talk 5 (Nov 26) A short introduction to equivariant cohomology (Xianyu Hu)

This talk should be a quick introduction/reminder of equivariant cohomology and the Atiyah-Bott localization formula and its application. The main reference is [1], see also the reference there to extended notes by Fulton. The goal is to be able to state and work with the virtual localization formula in the next talk. In particular, discuss several examples such as how to compute the basic integral

$$\int_{\mathbb{P}^1} c_1(\mathcal{O}_{\mathbb{P}^1}(1))$$

using the localization formula. More generally, perform an integral over $\mathbb{P}^n$, e.g. find some Chern numbers on $\mathbb{P}^2$.

Talk 6. (Dec 3) Virtual localization formula (after Graber-Pandharipande) (Till Wehrhan)

Discuss the main result [6] and sketch the proof.

Talk 7. (Dec 10) Degeneracy and virtual cycles (Solomiya Mizyuk)

Follow the lines of [3, 4] and express the virtual class of the fixed locus (in the previous talk) in terms of tautological integrals of the Hilbert scheme.

This probably will require two talks, to be distributed to two people.

Talk 8. (Dec 17) Pandharipande-Thomas theory of local surfaces (Maximilian Schimpf)

Let S be a smooth projective surface with a line bundle $L \to S$. We consider the Pandharipande-Thomas theory of the threefold

$$X = \mathrm{Tot}(\mathcal{L}),$$

i.e. the total space of the line bundle L. For that we apply the virtual localization formula for the torus action which scales the fiber direction. Describe the fixed locus as a nested Hilbert schemes of the surface, and identify the virtual tangent bundle and the virtual normal bundle. This is parallel to the case discussed in [2].

!!! Winter break !!!

Talk 9. (Jan 2022)

Talk 10. (Jan 2022)

Talk 11. (Jan 2022)

References

[1] D. Anderson, *Introduction to Equivariant Cohomology in Algebraic Geometry*, `https://people.math.osu.edu/anderson.2804/papers/ecag_lectures.pdf`

[2] Gholampour, Amin; Sheshmani, Artan; Yau, Shing-Tung, Localized Donaldson-Thomas theory of surfaces. Amer. J. Math. 142 (2020), no. 2, 405442.

[3] Gholampour, Amin; Thomas, Richard P., Degeneracy loci, virtual cycles and nested Hilbert schemes, I. Tunis. J. Math. 2 (2020), no. 3, 633665.

[4] Gholampour, Amin; Thomas, Richard P., Degeneracy loci, virtual cycles and nested Hilbert schemes II. Compos. Math. 156 (2020), no. 8, 16231663.

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[7] Huybrechts, Daniel, Complex geometry. An introduction. Universitext. Springer-Verlag, Berlin, 2005. xii+309 pp. ISBN: 3-540-21290-6

[8] [1]

Huybrechts, Daniel; Thomas, Richard P., Deformation-obstruction theory for complexes via Atiyah and Kodaira-Spencer classes. Math. Ann. 346 (2010), no. 3, 545569.

[9] Kiem, Young-Hoon; Li, Jun, Localizing virtual cycles by cosections. J. Amer. Math. Soc. 26 (2013), no. 4, 10251050.

[10] Maulik, D.; Pandharipande, R.; Thomas, R. P., Curves on K3 surfaces and modular forms. With an appendix by A. Pixton. J. Topol. 3 (2010), no. 4, 937996.

[11] Pandharipande, R.; Thomas, R. P., Curve counting via stable pairs in the derived category. Invent. Math. 178 (2009), no. 2, 407447.

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[13] Thomas, R. P., A holomorphic Casson invariant for Calabi-Yau 3-folds, and bundles on K3 fibrations. J. Differential Geom. 54 (2000), no. 2, 367438.

[1] There is an error in the published version of [8]; its best to use the arXiv version.