Vortex pairs and dipoles on closed surfaces

Björn Gustafsson[1]

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Abstract

We set up general equations of motion for point vortex systems on closed Riemannian surfaces, allowing for the case that the sum of vorticities is not zero and there hence must be counter-vorticity present. The dynamics of global circulations which is coupled to the dynamics of the vortices is carefully taken into account.

Much emphasis is put to the study of vortex pairs, having the Kimura conjecture in focus. This says that vortex pairs move, in the dipole limit, along geodesic curves, and proofs for it have previously been given by S. Boatto and J. Koiller by using Gaussian geodesic coordinates. In the present paper we reach the same conclusion by following a slightly different route, leading directly to the geodesic equation with a reparametrized time variable. In a final section we explain how vortex motion in planar domains can be seen as a special case of vortex motion on closed surfaces.

Keywords: Point vortex, vortex pair, vortex dipole, geodesic curve, affine connection, projective connection, Robin function, Hamiltonian function, symplectic form, Green function.
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[1]Department of Mathematics, KTH, 100 44, Stockholm, Sweden.
Email: gbjorn@kth.se

1 Introduction

This paper extends previous results in [22] on multiple point vortex motion on closed Riemannian surfaces of arbitrary genus to cases in which the sum of the vorticities is not zero, and therefore a counter-vorticity must be present. This occurs for example when there is only one point vortex. For reasons dictated by a Hodge decomposition, the counter-vorticity is naturally chosen to be uniformly distributed on the surface.

Special for our investigations is that we, in the case of higher genus, carefully take the circulation around the holes into account and make sure that the Kelvin law of preservation of circulations is satisfied. This was done already in [22], but in the case of a counter-vorticity being present one has to take some extra steps. It is natural to choose fixed closed curves on the surface to make up a basis for the first homology group, but the circulation around such curves will in general not be conserved in time. We handle the so arising problem by considering these circulations as free variables in the phase space, in addition to the locations of the vortices. If there are n vortices and the genus of the surface is $\mathtt{g}$ the phase space will thus have real dimension $2n + 2\mathtt{g}$.

The Hamiltonian function will, as usual, be a renormalized kinetic energy for the flow, see (5.28), but in the presence of a counter-vorticity the symplectic form has to be accordingly adapted, see (6.1). With this done the dynamical equations come out to be the expected ones (Theorem 6.1). After having set up the vortex dynamics in general (Section 5 and 6) and after a short discussion of the single vortex case in genus zero (Section 7), we turn in Section 8 to the question of motion of vortex pairs, i.e. systems consisting of two point vortices of equal strength but rotating in opposite direction. In this case there is no counter-vorticity. The conjecture of Y. Kimura [26], saying that a vortex pair in the dipole limit moves along a geodesic curve, has been a major source of inspiration for the present paper. The same applies for papers [29, 2] by S. Boatto, J. Koiller. These latter papers actually contain a proof of Kimura's conjecture, and in the present paper we try to clarify the situation further by connecting the dipole trajectory directly to the geodesic equation associated to the metric.

Throughout this paper we think of a Riemannian surface (a Riemannian manifold of dimension two) as a Riemann surface provided with a metric which is compatible with the conformal structure, hence can be written on the form

$$ds = \lambda(z)|dz|. \tag{1.1}$$

We are then lead into more or less classical function theory on Riemann surfaces. Our treatment is based entirely on local holomorphic coordinates. For example, the logarithmic singularities in the stream function will look like, up to a constant factor,

$$\psi(z) = \log|z-w| - c_0(w) + \mathcal{O}(|z-w|)$$

near a point vortex located at w. Here the constant term $c_0(w)$ is a kind of "connection" (0-connection, up to a sign, in our terminology). In potential theory it is known as a capacity function, or "Robin function" if ψ is viewed as a Green function, and in fluid dynamics it goes under the name "Routh's stream function", a kind of stream function for the motion of the vortex. It has an inhomogeneous transformation law, which can be translated into saying that

$$ds = e^{-c_0(w)}|dw|$$

is a conformally invariant Riemannian metric (usually not identical with (1.1) though).

The corresponding flow field $\nu = (\partial\psi/\partial y)dx - (\partial\psi/\partial x)dy = -*d\psi$, considered here as a one-form or a covariant vector field, looks like

$$\nu(z) = \frac{dz}{z-w} - c_1(w)dz + \text{complex conjugates} + \mathcal{O}(|z-w|)$$

near the vortex point. The term named "complex conjugates" is inserted to make possible for ν to be real-valued. Also in this expansion the constant term is a "connection". More precisely,

$$r_{\rm robin}(w) = -2c_1(w)$$

is an affine connection (or 1-connection in our terminology) in the sense of differential geometry. It appears in expressions for covariant derivatives, and after subtraction of the affine connection

$$r_{\rm metric}(w) = 2\frac{\partial}{\partial w}\log\lambda(w)$$

derived from the metric (1.1) it gives the speed of the vortex. Indeed, citing from [22] (Lemma 7.1 there) this speed is given by

$$\lambda(w)^2 \frac{dw}{dt} = \frac{\Gamma}{4\pi \mathrm{i}} \big(\overline{r_{\mathrm{metric}}(w)} - \overline{r_{\mathrm{robin}}(w)}\big), \tag{1.2}$$

where Γ is the strength of the vortex. This formula is also implicitly contained in Theorem 6.1 in the present paper.

One step further, it is possible to proceed directly to dipole flow by differentiating with respect to w:

$$d_w \nu(z) = \frac{dz dw}{(z-w)^2} - 2c_2(w) dz dw + \text{complex conjugates} + \text{regular terms}.$$

Again the constant term defines a connection. To be precise, $q(w) = -6c_2(w)$ is a projective, or Schwarzian, connection (or 2-connection), see Section 4 below, as well as relevant parts of [22]. This term gives some information on the fine structure of the flow near the dipole, however it seems not to be directly related to the speed of the dipole. The differential dw is to contain information about the orientation of the dipole and can be used to specify in which direction it is going to move. Indeed, all evidence shows that the dipole moves with infinite speed along the geodesic directed perpendicular to its orientation. This is Kimura's conjecture, which was confirmed in [29, 2] and which we make attempts to further clarify in the present paper.

Specifically we show that the equations of motion for a vortex pair in a suitably scaled dipole limit reduce to the geodesic equation for the center w of the dipole. We write this geodesic equation on the form

$$\frac{d}{dt} \arg \frac{dw}{dt} + \mathrm{Im}\,\big(r_{\mathrm{metric}}(w) \frac{dw}{dt}\big) = 0, \tag{1.3}$$

with t being an arbitrary parameter, for example Euclidean arc length. See Theorem 8.1.

The Kimura conjecture and other matters related to dipole motion has been discussed also in [35, 36, 39, 28, 30], and from slightly different points of view in [6, 31, 25, 5]. It may be remarked that Kimura's conjecture is counter-intuitive. The reason to think so is that vortex motion in general is governed by global laws on the manifold, like the structure of Green functions and other harmonic functions, whereas the geometry of geodesics is an entirely local matter. If the geometry of the manifold is changed at one place then this will not affect what geodesics look like at another place. It will however change the structure of Green functions and the general vortex dynamics everywhere. The solution of this paradox is that vortex dipoles are highly singular. A vortex pair is partly governed by the global harmonic structure, but in the limit, when the vortex pair becomes a dipole, this harmonic structure is completely overruled by the differential geometric structure. In that limit, all terms in the dynamical equations containing harmonic functions become negligible compared to those terms depending on the metric only. This fact shows up in the two dynamical equations above: the motion (1.2) of a single vortex is governed by $r_{\rm metric}(w)$ and $r_{\rm robin}(w)$ to equal parts, whereas only $r_{\rm metric}(w)$ appears in the equation (1.3) for a dipole. See Section 8 for further details.

In the final section of the paper we discuss how point vortex motion in planar domains can be considered as a special case of vortex motion on surfaces. This is done by doubling the domain to a compact Riemann surface (the Schottky double), which is a standard tool as far as the conformal structure is concerned. What is special in our case is that we have to take the metric structure into account, and this becomes non-smooth in the doubling procedure. An interesting observation is that the boundary of the domain becomes a geodesic curve with respect to the natural planar metric (on each of the halves) of the Schottky double.

In general the present paper is, in addition to the papers by S. Boatto and J. Koiller already mentioned, much in spirit of papers [35, 36, 16, 17, 45] by S. G. Llewellyn Smith, R. J. Nagem, C. Grotta Ragazzo, H. Viglioni, and Q. Wang. The paper [3] by A. Bogatskiy contains ideas concerning the higher genus case which are somewhat similar to those in the present paper, see in particular Section A.3 in [3]. We wish to mention also the work [4] by A. V. Borisov and I. S. Mamaev, and the very early article [14] (cited in [4, 36]) by A. A. Fridman and P. Ya. Polubarinova. That work, from 1928, may be one of the first papers on motion of dipoles and other higher singularities.

2 Fluid dynamics on a Riemannian surface

We consider the dynamics of a non-viscous incompressible fluid on a compact Riemannian manifold of dimension two. We follow the treatments in [43, 1, 13] in the respects of treating the fluid velocity field as a one-form and in extensively using the Lie derivative. These sources also provide the standard notions and notations for differential geometry to be used. Other treatises in fluid dynamics, suitable for our purposes, are [37, 38].

Traditionally, non-viscous fluid dynamics is discussed in terms of the fluid velocity vector field $\mathbf{v}$, the density ρ, and the pressure p of the fluid. The basic equations are the equation of continuity (expressing conservation of mass) and Euler's equation (conservation of momentum):

$$\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \mathbf{v}) = 0, \tag{2.1}$$

$$\frac{\partial \mathbf{v}}{\partial t} + (\mathbf{v} \cdot \nabla)\mathbf{v} = -\frac{1}{\rho}\nabla p. \tag{2.2}$$

These are to be combined with a constitutive law giving a relationship between p and ρ. We shall take that to be the simplest possible: $\rho = 1$. Thus ρ disappears from discussion, and the equation of continuity becomes

$$\nabla \cdot \mathbf{v} = 0. \tag{2.3}$$

One can get rid also of p, because when ρ is constant then p appears only as a scalar potential in Euler's equation, this equation effectively saying that

$$\frac{\partial \mathbf{v}}{\partial t} + (\mathbf{v} \cdot \nabla)\mathbf{v} = \nabla(\text{something}), \tag{2.4}$$

where this $\text{“something”} = p$ afterwards can be recovered, up to a (time dependent) constant.

So everything is extremely simple (in theory), and even more so in two dimensions. As is well-known, every (oriented) Riemannian manifold of dimensions two can be made into a Riemann surface by choosing isothermal local coordinates $(x, y) = (x^1, x^2)$, by which $z = x + \mathrm{i}y$ becomes a holomorphic coordinate and the metric takes the form

$$ds^2 = \lambda(z)^2 |dz|^2 = \lambda(z)^2 (dx^2 + dy^2) \tag{2.5}$$

with $\lambda > 0$. Thus the metric tensor g_{ij}, as appearing in general in $ds^2 = g_{ij} dx^i dx^j$, becomes $g_{ij} = \lambda^2 \delta_{ij}$ in these coordinates. It turns out to be convenient to work with the fluid velocity 1-form (or covariant vector)

$$\nu = \nu_x \, dx + \nu_y \, dy$$

rather than with the corresponding vector field, which then becomes

$$\mathbf{v} = \frac{1}{\lambda^2}(\nu_x \frac{\partial}{\partial x} + \nu_y \frac{\partial}{\partial y}).$$

The Hodge star operator takes 0-forms into 2-forms and vice versa, and takes 1-forms to 1-forms. It acts on basic differential forms as

$$*1 = \lambda^2(dx \wedge dy) = \mathrm{vol} = \text{the volume (area) form},$$

$$*dx = dy, \quad *dy = -dx, \quad *(dx \wedge dy) = \lambda^{-2}.$$

Thus, for 1-forms,

$$*\nu = -\nu_y \, dx + \nu_x \, dy,$$

which can be interpreted as a rotation by ninety degrees of the corresponding vector.

In addition to the Hodge star it is useful to introduce the Lie derivative $\mathcal{L}_{\mathbf{v}}$ of a vector field $\mathbf{v}$. When acting on differential forms this is related to interior derivation (or "contraction") $i(\mathbf{v})$ and exterior derivation d by the homotopy formula

$$\mathcal{L}_{\mathbf{v}} = d \circ i(\mathbf{v}) + i(\mathbf{v}) \circ d. \tag{2.6}$$

The Hodge star and interior derivation are related by

$$i(\mathbf{v})\mathrm{vol} = *\nu, \tag{2.7}$$

where $\mathbf{v}$ and ν are linked via the metric tensor as above. Thus

$$d * \nu = d(i(\mathbf{v})\mathrm{vol}) = \mathcal{L}_{\mathbf{v}}(\mathrm{vol}) = (\nabla \cdot \mathbf{v})\mathrm{vol}.$$

See [13] for this identity, and for further details in general. We conclude that (2.3) is equivalent to the statement that $*\nu$ is a closed form:

$$d * \nu = 0. \tag{2.8}$$

Locally we can therefore write

$$*\nu = d\psi \tag{2.9}$$

for some (locally defined) *stream function* ψ. The *vorticity* 2*-form* for a fluid is, in terms of the flow 1-form ν,

$$\omega = d\nu = (\frac{\partial \nu_y}{\partial x} - \frac{\partial \nu_x}{\partial y}) \, dx \wedge dy. \tag{2.10}$$

The Euler equation (2.2) can be written

$$(\frac{\partial}{\partial t} + \mathcal{L}_{\mathbf{v}})(\nu) = d(\frac{1}{2}|\mathbf{v}|^2 - p). \tag{2.11}$$

Note that the left member involves the fluid velocity both as a vector field and as a form. The combination $\frac{\partial}{\partial t} + \mathcal{L}_{\mathbf{v}}$ can be viewed as a counterpart, for forms, to the more traditional "convective derivative" (or material derivative) $\frac{\partial}{\partial t} + \mathbf{v} \cdot \nabla$, implicitly used in (2.4), for vector fields. However, the two derivatives are not the same. The equivalence between (2.11) and (2.2) (when $\rho = 1$) is a consequence of the identity

$$\mathcal{L}_{\mathbf{v}}\nu = d(\frac{1}{2}|\mathbf{v}|^2) + (\mathbf{v} \cdot \nabla)\nu,$$

which can be directly verified by coordinate computations. See [43, 13] for details.

Since the pressure p does not appear in any other equation, the Euler equation on the form (2.11) only expresses, in analogy with (2.4), that the left member is an exact 1-form:

$$(\frac{\partial}{\partial t} + \mathcal{L}_{\mathbf{v}})(\nu) = \text{exact} = d\phi. \tag{2.12}$$

From the scalar ϕ the pressure p then can be recovered via

$$\phi = \frac{1}{2}|\mathbf{v}|^2 - p + \text{constant}. \tag{2.13}$$

On acting by d on (2.12), the local form of Helmholtz-Kirchhoff-Kelvin law of conservation of vorticity follows:

$$(\frac{\partial}{\partial t} + \mathcal{L}_{\mathbf{v}})(\omega) = 0. \tag{2.14}$$

Since the right member of (2.12) is not only a closed differential form, but even an exact form, a stronger, global, version of the Helmholtz-Kirchhoff-Kelvin law actually follows. This can be expressed by saying that

$$\frac{d}{dt}\oint_{\gamma(t)} \nu = 0$$

for any closed curve $\gamma(t)$ which moves with the fluid.

3 Green functions and harmonic forms

3.1 The one point Green function

In the sequel M will be a closed (compact) Riemann surface provided with a Riemannian metric on the form (2.5).

Given a 2-form ω on M one can immediately obtain a corresponding Green potential G^{ω} by Hodge decomposition, i.e. by orthogonal decomposition in the Hilbert space of square integrable 2-forms. The inner product for such forms is

$$(\omega_1, \omega_2)_2 = \int_M \omega_1 \wedge *\omega_2. \tag{3.1}$$

The given 2-form ω then decomposes as the orthogonal decomposition of an exact form and a harmonic form:

$$\omega = d(\text{something}) + \text{harmonic}.$$

This Hodge decomposition can more precisely be written as

$$\omega = -d * dG^{\omega} + \text{constant} \cdot \text{vol}, \tag{3.2}$$

where G^{ω} is normalized to be orthogonal to all harmonic 2-forms, namely satisfying

$$\int_M G^{\omega}\, \text{vol} = 0. \tag{3.3}$$

The constant in (3.2) necessarily equals the mean value of ω,

$$\text{constant} = \frac{1}{V} \int_M \omega, \tag{3.4}$$

where V denotes the total volume (=area) of M:

$$V = \int_M \text{vol}.$$

When ω is exact, as in (2.10), the second term in the right member of (3.2) disappears, hence

$$-d * dG^{\omega} = \omega \tag{3.5}$$

in this case. In the other extreme, when ω is harmonic, i.e. is a constant multiple of vol, the first term disappears. Indeed, the whole Green function disappears:

$$G^{\mathrm{vol}} = 0. \tag{3.6}$$

For 1-forms the inner product has the same expression as for 2-forms:

$$(\nu_1, \nu_2)_1 = \int_M \nu_1 \wedge *\nu_2. \tag{3.7}$$

If ν represents a fluid velocity, then $(\nu, \nu)_1$ is the kinetic energy of the flow (up to a factor). For a function (potential) u we consider the Dirichlet integral $(du, du)_1$ to be its energy. Thus constant functions have no energy.

The energy $\mathcal{E}(\omega, \omega)$ of any 2-form ω is defined to be the energy of its Green potential G^ω. Thus, for the mutual energy,

$$\mathcal{E}(\omega_1, \omega_2) = (dG^{\omega_1}, dG^{\omega_2})_1 = \int_M G^{\omega_1} \wedge \omega_2. \tag{3.8}$$

It follows, from (3.3) for example, that the volume form has no energy:

$$\mathcal{E}(\mathrm{vol}, \mathrm{vol}) = 0.$$

It was tacitly assumed above that the forms under discussion belong to the L^2-space defined by the inner product. However, the mutual energy sometimes extends to circumstances in which source distributions have infinite energy. This applies in particular to the Dirac current δ_a, which we consider as a 2-form with distributional coefficient, namely defined by the property that

$$\int_M \delta_a \wedge \varphi = \varphi(a)$$

for every smooth function φ. Certainly δ_a has infinite energy, but if $a \neq b$, then $\mathcal{E}(\delta_a, \delta_b)$ is still finite and has a natural interpretation: it is the *(one-point) Green function*, or "monopole" Green function:

$$G(a, b) = G^{\delta_a}(b) = \mathcal{E}(\delta_a, \delta_b). \tag{3.9}$$

Here the first equality can be taken as a definition, and then the second equality follows on using (3.2), (3.3):

$$\begin{aligned} \mathcal{E}(\delta_a, \delta_b) &= \int_M dG^{\delta_a} \wedge *dG^{\delta_b} = -\int_M G^{\delta_a} \wedge d * dG^{\delta_b} \\ &= \int_M G^{\delta_a} \wedge \left(\delta_b - \frac{1}{V}\,\mathrm{vol}\right) = G^{\delta_a}(b) = G(a, b). \end{aligned}$$

This shows in addition that $G(a,b)$ is symmetric.

Changing now notations from a, b to z, w, where later w will have the role of being the location of a point vortex, the Green function $G(z,w)$, as a function of z, has just one pole (at $z = w$). Its Laplacian, as a 2-form, is then

$$-d * dG(\cdot, w) = \delta_w - \frac{1}{V}\mathrm{vol}. \tag{3.10}$$

It is interesting to notice that among the two terms in the right member of (3.10), one has infinite energy and one has zero energy ($\mathcal{E}(\delta_w, \delta_w) = +\infty$, $\mathcal{E}(\mathrm{vol}, \mathrm{vol}) = 0$).

Remark 3.1. It is more common to treat the Dirac delta and the Laplacian (denoted Δ) as "densities" with respect to the volume form. However, we find our usage convenient. In any case, the relationships are

$$\delta_a = (\text{delta "function"})\,\mathrm{vol}, \quad d * d\varphi = (\Delta\varphi)\,\mathrm{vol}.$$

Remark 3.2 (Notational remark)**.** Letters like z, w will have the double roles of being complex-valued local coordinates on parts of the Riemann surface and of denoting the points on the surface for which the coordinates have the values in question. A more formal treatment could use, for example, $P \in M$ for a point and $z(P) \in \mathbb{C}$ for the corresponding coordinate value.

Remark 3.3 (Real versus complex notation)**.** For future needs we wish to clarify the relationship between real and complex coordinates in the context of tangent and cotangent vectors.

Let $z = x + \mathrm{i}y$ be a complex coordinate on M and consider a curve $t \mapsto z(t)$ in M (for example the trajectory of a vortex). Set $\dot{z} = dz/dt$, and similarly for $\dot{x}$ and $\dot{y}$, so that $\dot{z} = \dot{x} + \mathrm{i}\dot{y}$. The velocity of this moving point is primarily to be considered as a vector in the (real) tangent space of M (at the point under consideration). This gives, in the picture of viewing tangent vectors as derivations, the velocity vector

$$\mathbf{V} = \dot{x}\frac{\partial}{\partial x} + \dot{y}\frac{\partial}{\partial y} = \dot{z}\frac{\partial}{\partial z} + \dot{\bar{z}}\frac{\partial}{\partial \bar{z}}, \tag{3.11}$$

where it is understood that $\dot{\bar{z}} = \overline{\dot{z}}$.

The real tangent space used above can in a next step be complexified, which means that one breaks the connection between $\dot{z}$ and $\dot{\bar{z}}$ and consider them as independent complex variables. Equivalently, one allows $\dot{x}$

and $\dot{y}$ to be complex-valued. The so obtained complex tangent space can be decomposed as a direct sum of its holomorphic and anti-holomorphic subspaces, and then

$$\mathrm{proj}: \quad \mathbf{V} = \dot{z}\frac{\partial}{\partial z} + \dot{\bar{z}}\frac{\partial}{\partial \bar{z}} \mapsto \dot{z}\frac{\partial}{\partial z}$$

becomes a natural and useful identification of the real tangent space with the holomorphic part of the complex tangent space. See [15], in particular Section 2 of Chapter 0, for further discussions. With this identification $\dot{z}$ represents the velocity of $z(t)$. Still one need to keep in mind that $\dot{z}$ is just a complex number and that it is rather the preimage under proj above that is the true velocity, as a real tangent vector.

The vector $\mathbf{V}$ above corresponds, via the metric, to the covector

$$\lambda^2(\dot{x}dx + \dot{y}dy) = \frac{\lambda^2}{2}(\dot{\bar{z}}dz + \dot{z}d\bar{z}), \tag{3.12}$$

where $\lambda = \lambda(x, y) = \lambda(z)$ (depending on context). Note that

$$\frac{1}{2}\dot{\bar{z}} = \frac{1}{2}(\dot{x} - \mathrm{i}\dot{y}), \quad \frac{1}{2}\dot{z} = \frac{1}{2}(\dot{x} + \mathrm{i}\dot{y}),$$

as coefficients of dz and $d\bar{z}$ (respectively) have similar roles (and signs) as the Wirtinger derivatives $\partial/\partial z$, $\partial/\partial\bar{z}$. For a covector ν in general we therefore define

$$\nu_z = \frac{1}{2}(\nu_x - \mathrm{i}\nu_y), \quad \nu_{\bar{z}} = \frac{1}{2}(\nu_x + \mathrm{i}\nu_y),$$

so that

$$\nu = \nu_x dx + \nu_y dy = \nu_z dz + \nu_{\bar{z}} d\bar{z}.$$

The corresponding (contravariant) vector is then, as in our fluid dynamical contexts,

$$\mathbf{v} = \frac{1}{\lambda^2}\big(\nu_x\frac{\partial}{\partial x} + \nu_y\frac{\partial}{\partial y}\big) = \frac{1}{\lambda^2}\big(\nu_z\frac{\partial}{\partial z} + \nu_{\bar{z}}\frac{\partial}{\partial \bar{z}}\big).$$

3.2 Harmonic one-forms and period relations

For later use we record the formulas (differentiation with respect to z)

$$dG(z, w) = \frac{\partial G(z, w)}{\partial z}dz + \frac{\partial G(z, w)}{\partial \bar{z}}d\bar{z} = 2\,\mathrm{Re}\,\big(\frac{\partial G(z, w)}{\partial z}dz\big),$$

$$*dG(z,w) = -\mathrm{i}\frac{\partial G(z,w)}{\partial z}dz + \mathrm{i}\frac{\partial G(z,w)}{\partial \bar{z}}d\bar{z} = 2\,\mathrm{Im}\,\big(\frac{\partial G(z,w)}{\partial z}dz\big). \tag{3.13}$$

If γ is any closed oriented curve in M then $\oint_\gamma dG(\cdot,w) = 0$, while the conjugate period defines a function

$$U_\gamma(w) = \oint_\gamma *dG(\cdot,w) = \oint_\gamma \big(-\mathrm{i}\frac{\partial G(z,w)}{\partial z}dz + \mathrm{i}\frac{\partial G(z,w)}{\partial \bar{z}}d\bar{z}\big) \tag{3.14}$$

$$= -2\mathrm{i}\oint_\gamma \frac{\partial G(z,w)}{\partial z}dz = 2\mathrm{i}\oint_\gamma \frac{\partial G(z,w)}{\partial \bar{z}}d\bar{z}, \tag{3.15}$$

which, away from γ, is harmonic in w and makes a unit additive jump as w crosses γ from the left to the right. The harmonicity of $U_\gamma(w)$ is perhaps not obvious from outset since $G(\cdot,w)$ is not itself harmonic, but the deviation from harmonicity, namely the extra term in (3.10), is independent of w and therefore disappears under differentiation.

Differentiating (3.14) with respect to w gives

$$dU_\gamma(w) = -2\mathrm{i}\oint_\gamma \frac{\partial^2 G(z,w)}{\partial z\partial w}dzdw - 2\mathrm{i}\oint_\gamma \frac{\partial^2 G(z,w)}{\partial z\partial \bar{w}}dzd\bar{w},$$

$$*dU_\gamma(w) = -2\oint_\gamma \frac{\partial^2 G(z,w)}{\partial z\partial w}dzdw + 2\oint_\gamma \frac{\partial^2 G(z,w)}{\partial z\partial \bar{w}}dzd\bar{w}$$

(integration with respect ot z, Hodge star and d with respect to w). Adding and subtracting we obtain the analytic (respectively anti-analytic) differentials

$$dU_\gamma(w) + \mathrm{i} * dU_\gamma(w) = 2\frac{\partial U_\gamma(w)}{\partial w}dw = -4\mathrm{i}\oint_\gamma \frac{\partial^2 G(z,w)}{\partial z\partial w}dzdw, \tag{3.16}$$

$$dU_\gamma(w) - \mathrm{i} * dU_\gamma(w) = 2\frac{\partial U_\gamma(w)}{\partial \bar{w}}d\bar{w} = -4\mathrm{i}\oint_\gamma \frac{\partial^2 G(z,w)}{\partial z\partial \bar{w}}dzd\bar{w}. \tag{3.17}$$

Let now $\{\alpha_1,\dots,\alpha_{\mathrm{g}},\beta_1,\dots,\beta_{\mathrm{g}}\}$ be representing cycles for a canonical homology basis for M such that each β_j intersects α_j once from the right to the left and no other intersections occur (see [10] for details). Then there are corresponding harmonic differentials dU_{α_j}, dU_{β_j} obtained on choosing

$\gamma = \alpha_j, \beta_j$ in the above construction, and these constitute a basis of harmonic differentials associated with the chosen homology basis. Precisely we have, for $k, j = 1, \dots, \mathtt{g}$,

$$\oint_{\alpha_k} (-dU_{\beta_j}) = \delta_{kj}, \quad \oint_{\beta_k} (-dU_{\beta_j}) = 0, \tag{3.18}$$

$$\oint_{\alpha_k} dU_{\alpha_j} = 0, \qquad \oint_{\beta_k} dU_{\alpha_j} = \delta_{kj}. \tag{3.19}$$

The Kronecker deltas, δ_{kj}, come from the discontinuity (jump) properties of U_γ mentioned after (3.14).

The integrals of the basic harmonic differentials along non-closed curves become periods of two point Green functions:

$$\int_b^a dU_{\alpha_j} = \oint_{\alpha_j} *dG^{\delta_a - \delta_b}, \tag{3.20}$$

$$\int_b^a dU_{\beta_j} = \oint_{\beta_j} *dG^{\delta_a - \delta_b}. \tag{3.21}$$

Here the integration in the left member is to be performed along a path that does not intersect the curve in the right member. These formulas are immediate from the definition (3.14) of U_γ.

4 Affine connections and geodesic curves

Besides differential forms, and tensor fields in general, certain kinds of connections with inhomogeneous transformation laws are relevant for point vortex motion. These arise naturally under "renormalization" processes, more precisely as the first regular terms in expansions of a polar singularities. The affine connections have the same meanings as in ordinary differential geometry, used to define covariant derivatives for example, and they play an important role in many areas of mathematical physics.

Some general references for the kind of connections we are going to consider are [41, 19, 20, 21, 23]. We define them in the simplest possible manner, namely as quantities defined in terms of local holomorphic coordinates and transforming in specified ways when changing from one coordinate to another.

Let $\tilde{z} = \varphi(z)$ represent a holomorphic local change of complex coordinate on a Riemann surface M and define three nonlinear differential expressions $\{\cdot, \cdot\}_k$, $k = 0, 1, 2$, by

$$\{\tilde{z}, z\}_0 = \log \varphi'(z),$$
$$\{\tilde{z}, z\}_1 = (\log \varphi'(z))' = \frac{\varphi''}{\varphi'},$$
$$\{\tilde{z}, z\}_2 = (\log \varphi'(z))'' - \frac{1}{2}((\log \varphi'(z))')^2 = \frac{\varphi'''}{\varphi'} - \frac{3}{2}(\frac{\varphi''}{\varphi'})^2.$$

The last expression is the *Schwarzian derivative* of φ. For $\{\tilde{z}, z\}_0$ there is an additive indeterminacy of $2\pi\mathrm{i}$, so actually only its real part, or exponential, is completely well-defined. If z depends on w via an intermediate variable u then the following chain rule holds:

$$\{z, w\}_k (dw)^k = \{z, u\}_k (du)^k + \{u, w\}_k (dw)^k \quad (k = 0, 1, 2).$$

Definition 4.1. An *affine connection* (or 1-*connection*) on M is an object which is represented by local differentials $r(z)dz$, $\tilde{r}(\tilde{z})d\tilde{z}$,... (one in each coordinate variable, and not necessarily holomorphic) glued together according to the rule

$$\tilde{r}(\tilde{z})d\tilde{z} = r(z)dz - \{\tilde{z}, z\}_1 \, dz.$$

Definition 4.2. A *projective connection* (or *Schwarzian connection*, or 2-*connection*) on M consists of local quadratic differentials $q(z)(dz)^2$, $\tilde{q}(\tilde{z})(d\tilde{z})^2, \dots$, glued together according to

$$\tilde{q}(\tilde{z})(d\tilde{z})^2 = q(z)(dz)^2 - \{\tilde{z}, z\}_2 \, (dz)^2.$$

One may also consider 0-*connections*, assumed here to be real-valued. Such a connection $p(z)$ transforms according to

$$\tilde{p}(\tilde{z}) = p(z) - \mathrm{Re}\{\tilde{z}, z\}_0.$$

This means exactly that

$$ds = e^{p(z)}|dz|.$$

is a Riemannian metric.

For a metric in general, $ds = \lambda(z)|dz| = e^{p(z)}|dz|$, there is a natural affine connection $r(z) = r(z)_{\mathrm{metric}}$ associated to it by

$$r(z) = 2\frac{\partial}{\partial z} \log \lambda = 2\frac{\partial p}{\partial z} = \frac{\partial p}{\partial x} - \mathrm{i}\frac{\partial p}{\partial y}. \tag{4.1}$$

This can be identified with the Levi-Civita connection in general tensor analysis. The real and imaginary parts coincide (up to sign) with the components of the classical Christoffel symbols Γ^k_{ij}. The Gaussian curvature of the metric is

$$\kappa = -4\lambda^{-2}\frac{\partial^2}{\partial z \partial \bar{z}}\log\lambda = -2\lambda^{-2}\frac{\partial r}{\partial \bar{z}}.$$

The equation for geodesic curves $z = z(t)$ is

$$\frac{d^2 z}{dt^2} + r(z)(\frac{dz}{dt})^2 = 0,$$

where t is arclength with respect to the metric. Written in another way,

$$\frac{d}{dt}\log\frac{dz}{dt} + r(z)\frac{dz}{dt} = 0. \tag{4.2}$$

The first version is just a reformulation of the usual equation in terms of Christoffel functions in ordinary differential geometry (see [13]), and it is also discussed in a complex analytic context in [23].

The real part of (4.2) only contains information about the parametrization, while the imaginary part, namely

$$\frac{d}{dt}\arg\frac{dz}{dt} + \mathrm{Im}\,\big(r(z)\frac{dz}{dt}\big) = 0. \tag{4.3}$$

describes the geodesic curve in terms of an arbitrary real parameter t. The latter property is useful in the context of the motion of vortex pairs since the speed of such a pair tends to infinity as the distance between the two vortices goes zero, and time therefore has to be successively reparametrized.

5 Energy renormalization and the Hamiltonian

5.1 The renormalized kinetic energy

When ν is the flow 1-form of an incompressible fluid, the equation of continuity says that $d * \nu = 0$ (see (2.8)). The vorticity 2-form is $\omega = d\nu$, and (3.5) holds. Thus $d(\nu + *dG^\omega) = 0$, and setting

$$\eta = \nu + *dG^\omega \tag{5.1}$$

it follows that η is harmonic: $d\eta = 0 = d * \eta$. Locally we can write $*\eta = d\psi_0$ for some harmonic function ψ_0. Then

$$\psi = G^{\omega} + \psi_0 \tag{5.2}$$

becomes a locally defined stream function for the flow, so that

$$*\nu = d\psi = dG^{\omega} + d\psi_0.$$

Different local choices of ψ may differ by additive time dependent constants. The role of the additional term ψ_0 in the stream function will be clarified when discussing the hydrodynamic Green function for planar vortex motion in Section 9.

The relation (5.1), when written on the form $\nu = \eta - *dG^{\omega}$, is an orthogonal decomposition of the flow 1-form ν with respect to the inner product (3.7). Indeed,

$$(\eta, - * dG^{\omega})_1 = \int_M \eta \wedge dG^{\omega} = -\int_M d\eta \wedge G^{\omega} = 0,$$

since η is harmonic. It follows that (twice) the *total (kinetic) energy* $(\nu, \nu)_1$ of the flow is given by

$$\int_M \nu \wedge *\nu = \int_M dG^{\omega} \wedge *dG^{\omega} + \int_M \eta \wedge *\eta = \mathcal{E}(\omega, \omega) + \int_M \eta \wedge *\eta$$

$$= \int_M G^{\omega} \wedge \omega + \sum_{j=1}^{\mathtt{g}} (\oint_{\alpha_j} \eta \oint_{\beta_j} *\eta - \oint_{\alpha_j} *\eta \oint_{\beta_j} \eta).$$

We shall now specialize on the point vortex case, having point vortices of strengths Γ_k located at points $w_k \in M$ ($k = 1, \dots, n$). However, we shall not assume that these strengths add up to zero. Hence the sum

$$\Gamma = \sum_{k=1}^{n} \Gamma_k \tag{5.3}$$

may be non-zero and there will then be a compensating uniform counter vorticity. The total vorticity $\omega = d\nu$, which satisfies $\int_M \omega = 0$, appears as the right member in

$$-d * dG^{\omega} = \omega = \sum_{k=1}^{n} \Gamma_k \delta_{w_k} - \frac{\Gamma}{V} \text{vol}. \tag{5.4}$$

Explicitly we have (recall (3.6))

$$G^{\omega}(z) = \sum_{k=1}^{n} \Gamma_k G(z, w_k). \tag{5.5}$$

On using (3.13) the conjugate α-periods of G^{ω} can be expressed as

$$\oint_{\alpha_j} *dG^{\omega} = -\sum_{k=1}^{n} 2\mathrm{i}\Gamma_k \oint_{\alpha_j} \frac{\partial G(z, w_k)}{\partial z} dz. \tag{5.6}$$

By differentiation and using also (3.15), (3.16) (with $\gamma = \alpha_j$) we find that

$$\frac{\partial}{\partial w_k} \oint_{\alpha_j} *dG^{\omega} = \Gamma_k \frac{\partial U_{\alpha_j}(w_k)}{\partial w_k} = \mathrm{i}\Gamma_k \frac{\partial U^*_{\alpha_j}(w_k)}{\partial w_k}, \tag{5.7}$$

$$\frac{\partial}{\partial w_k} \big(\oint_{\alpha_j} *dG^{\omega} \big) dw_k = \frac{\Gamma_k}{2} (dU_{\alpha_j}(w_k) + \mathrm{i} * dU_{\alpha_j}(w_k)). \tag{5.8}$$

Here $U^*_{\alpha_j}$ denotes a harmonic conjugate of U_{α_j}, whereby $*dU_{\alpha_j} = d(U^*_{\alpha_j})$. Similar relations hold for periods around β_j, and for $\bar{w}_k$ derivatives.

The expression for the kinetic energy can in the point vortex case be written, at least formally,

$$(\nu, \nu)_1 = \sum_{k,j=1}^{n} \Gamma_k \Gamma_j G(w_k, w_j) + \sum_{j=1}^{\mathtt{g}} \big(\oint_{\alpha_j} \eta \oint_{\beta_j} *\eta - \oint_{\alpha_j} *\eta \oint_{\beta_j} \eta \big).$$

However, the presence of the terms $\Gamma_k^2 G(w_k, w_k)$ makes the first term become infinite. In order to renormalize this singular behavior we isolate the logarithmic pole in the Green function by writing

$$G(z, w) = \frac{1}{2\pi} (-\log |z - w| + H(z, w)), \tag{5.9}$$

and expand, for a fixed w, the regular part in a power series in z as

$$H(z, w) = h_0(w) + \frac{1}{2} \Big(h_1(w)(z - w) + \overline{h_1(w)}(\bar{z} - \bar{w}) \Big) + \tag{5.10}$$

$$+ \frac{1}{2} \Big(h_2(w)(z - w)^2 + \overline{h_2(w)}(\bar{z} - \bar{w})^2 \Big) + h_{11}(w)(z - w)(\bar{z} - \bar{w}) + \mathcal{O}(|z - w|^3).$$

We note from (5.10) that

$$H(w,w) = h_0(w), \quad \{\frac{\partial H(z,w)}{\partial z}\}_{z=w} = \frac{1}{2}h_1(w). \tag{5.11}$$

Thus the symmetry of $H(z,w)$ gives

$$h_1(w) = \frac{\partial h_0(w)}{\partial w}, \tag{5.12}$$

In [22] the behavior of the coefficients in the expansion (5.10) under conformal mapping was discussed. For example, the coefficient $h_{11}(w)$ transforms as the density of a metric, and it is indeed proportional to the given metric:

$$h_{11}(w) = \frac{\pi}{2V}\lambda(w)^2. \tag{5.13}$$

The coefficients $h_0(w)$, $h_1(w)$ and $h_2(w)$ are subject to inhomogeneous transform laws, expressing that they, in certain combinations, behave as connections under conformal mappings.

For $h_0(w)$ the law is

$$\tilde{h}_0(\tilde{w}) = h_0(w) + \mathrm{Re}\{\tilde{w}, w\}_0,$$

and this means that its exponential defines a metric, namely the *Robin metric* via

$$ds = e^{-h_0(w)}|dw|. \tag{5.14}$$

Metrics of this kind are implicit in the theory of capacity functions, as exposed in [40]. It should be pointed out that our version of the Robin metric depends on the given metric, since the Green function itself depends on it. The Robin metric can also be adapted to given global circulations, and then it becomes more intrinsically hydrodynamic in nature. The coefficient $h_0(w)$ is one example of a (coordinate) Robin function.

Now, letting w have the role of being a vortex point indicates that one could renormalize the kinetic energy by simply discarding the singular term $\log|z-w|$, as this seems at first sight to produce just a circular symmetric flow, not affecting the speed of the vortex. However, this is not fully correct in the case of curved surfaces. The term $\log|z-w|$ cannot be just removed, it need be replaced by a term which counteracts the above inhomogeneous transformation law of $h_0(w)$. Such a term comes naturally from the given metric $ds = \lambda(w)|dw| = e^{\log\lambda(w)}|dw|$. We see that minus $\log\lambda(w)$ has the right properties, and it combines with $h_0(w)$ into

$$R_{\rm robin}(w) = \frac{1}{2\pi}(h_0(w) + \log\lambda(w)). \tag{5.15}$$

This is indeed a function, the *Robin function*, and it appears naturally when writing the singularity of the Green function in terms of the distance with respect to the given Riemannian metric:

$$G(z,w) = -\frac{1}{2\pi}\log{\rm dist}(z,w) + R_{\rm robin}(w) + \mathcal{O}({\rm dist}(z,w)).$$

As for the infinite kinetic energy, we conclude that it should be renormalized by replacing the diagonal terms $G(w_k,w_k)$ by $R_{\rm robin}(w_k)$. This gives the same equations of motion as more direct approaches, or those available in the literature, like [24, 2, 9]. Thus the *renormalized energy* is

$$(\nu,\nu)_{1,{\rm renorm}} = \sum_{k=1}^{n}\Gamma_k^2 R_{\rm robin}(w_k) + \sum_{k\neq j}\Gamma_k\Gamma_j G(w_k,w_j) +$$

$$+\sum_{j=1}^{\mathsf{g}}\Big(\oint_{\alpha_j}\eta\oint_{\beta_j}*\eta - \oint_{\alpha_j}*\eta\oint_{\beta_j}\eta\Big). \tag{5.16}$$

This depends on the locations $w_1,\dots,w_n$ of the vortices (even the last term depends on these, although somewhat more indirectly).

Many authors start out directly with the Robin function (5.15), but there are some advantages with exposing the two terms in it as individual quantities. One is that it clarifies the structure of the vortex motion equations by separating harmonic contributions, such as $h_0(w_k)$ and $G(w_k,w_j)$, from differential geometric contributions, like $\log\lambda(w_k)$. The first category of terms can be classified as nonlocal, coming from solutions of elliptic partial differential equations on the entire surface, whereas the second category are purely local in nature. There is also the contribution from global circulations, represented by the final term in (5.16), and this may be considered to be truly global. So the vortex motion is in principle governed by a balance between different categories of contributions, local, nonlocal, and global in nature. As we shall see later this balance is drastically changed in the more singular case of dipole motion (Sections 8). Then only the local terms survive.

We need to elaborate further the expression (5.1) and relate it to the global circulations. The 1-form η is harmonic and hence can be expanded as

$$\eta = -\sum_{j=1}^{\mathrm{g}} A_j dU_{\beta_j} + \sum_{j=1}^{\mathrm{g}} B_j dU_{\alpha_j}. \tag{5.17}$$

The coefficients A_k, B_k are in view of (3.18), (3.19) given by

$$A_k = \oint_{\alpha_k} \eta, \quad B_k = \oint_{\beta_k} \eta.$$

The circulations of the total flow $\nu = \eta - *dG^{\omega}$ then become

$$a_k = \oint_{\alpha_k} \nu = A_k - \oint_{\alpha_k} *dG^{\omega},$$

$$b_k = \oint_{\beta_k} \nu = B_k - \oint_{\beta_k} *dG^{\omega}.$$

We shall treat the circulations $a_1, \dots, a_{\mathrm{g}}$, $b_1, \dots, b_{\mathrm{g}}$ as free (independent) variables, along with the locations $w_1, \dots, w_n$ of the vortices. These variables will be the coordinates of the phase space, and together they determine η and ν. The locations of the vortices are given by complex variables, in contrast to the circulations which are real.

If γ is a closed oriented curve in M, fixed in time and avoiding the point vortices, then according to the Euler equation (2.12), and in the notation used there, the circulation of ν around γ changes with speed

$$\frac{d}{dt}\oint_{\gamma} \nu = \oint_{\gamma} \frac{\partial \nu}{\partial t} = \oint_{\gamma} (d\phi - \mathcal{L}_{\mathbf{v}}\nu) = -\oint_{\gamma} i(\mathbf{v})d\nu =$$

$$= -\oint_{\gamma} i(\mathbf{v})\omega = \oint_{\gamma} i(\mathbf{v})\big(\frac{\Gamma}{V}\mathrm{vol}\big) = \frac{\Gamma}{V}\oint_{\gamma} *\nu.$$

Here we used also (2.6), (2.7), (5.4). It follows in particular, on choosing $\gamma = \alpha_k, \beta_k$, that

$$\frac{da_k}{dt} = \frac{\Gamma}{V}\oint_{\alpha_k} *\nu = \frac{\Gamma}{V}\oint_{\alpha_k} *\eta, \tag{5.18}$$

$$\frac{db_k}{dt} = \frac{\Gamma}{V}\oint_{\beta_k} *\nu = \frac{\Gamma}{V}\oint_{\beta_k} *\eta. \tag{5.19}$$

5.2 Matrix formalism and the Hamiltonian

The period matrix (written here in block form)

$$\begin{pmatrix} P & R \\ R^T & Q \end{pmatrix} = \begin{pmatrix} (-\oint_{\beta_k} *dU_{\beta_j}) & (\oint_{\beta_k} *dU_{\alpha_j}) \\ (\oint_{\alpha_k} *dU_{\beta_j}) & (-\oint_{\alpha_k} *dU_{\alpha_j}) \end{pmatrix} =$$

$$= \begin{pmatrix} (\int_M dU_{\beta_k} \wedge *dU_{\beta_j}) & (-\int_M dU_{\beta_k} \wedge *dU_{\alpha_j}) \\ (-\int_M dU_{\alpha_k} \wedge *dU_{\beta_j}) & (\int_M dU_{\alpha_k} \wedge *dU_{\alpha_j}) \end{pmatrix} \tag{5.20}$$

is symmetric and positive definite (see [10] in general). In particular, P and Q are themselves symmetric and positive definite. As for R and R^T we need to be explicit with what are row and column indices above: in all entries above, k is the row index, j the column index. Thus, for example, $R_{kj} = \oint_{\beta_k} *dU_{\alpha_j}$.

Next we write

$$dU_\alpha = \begin{pmatrix} dU_{\alpha_1} \\ \vdots \\ dU_{\alpha_g} \end{pmatrix}, \quad dU_\beta = \begin{pmatrix} dU_{\beta_1} \\ \vdots \\ dU_{\beta_g} \end{pmatrix}.$$

As mentioned, these two column matrices together define a basis of the harmonic forms. Another basis is provided by the corresponding Hodge starred column vectors $*dU_\alpha$, $*dU_\beta$, in similar matrix notation. The relation between the bases is

Lemma 5.1. *The two bases* $\{dU_\alpha, dU_\beta\}$ *and* $\{*dU_\alpha, *dU_\beta\}$ *are related by*

$$\begin{pmatrix} *dU_\alpha \\ *dU_\beta \end{pmatrix} = \begin{pmatrix} R^T & Q \\ -P & -R \end{pmatrix} \begin{pmatrix} dU_\alpha \\ dU_\beta \end{pmatrix}. \tag{5.21}$$

Proof. One simply checks that the two members in (5.21) have the same periods with respect to the homology basis $\{\alpha_j, \beta_j\}$. □

We arrange also the circulations of the flow ν into column vectors:

$$a = \begin{pmatrix} a_1 \\ \vdots \\ a_g \end{pmatrix} = \begin{pmatrix} \oint_{\alpha_1} \nu \\ \vdots \\ \oint_{\alpha_g} \nu \end{pmatrix}, \quad b = \begin{pmatrix} b_1 \\ \vdots \\ b_g \end{pmatrix} = \begin{pmatrix} \oint_{\beta_1} \nu \\ \vdots \\ \oint_{\beta_g} \nu \end{pmatrix},$$

briefly written as

$$a = \oint_{\alpha} \nu, \quad b = \oint_{\beta} \nu. \tag{5.22}$$

Similarly for other vectors of circulations, for example

$$A = \oint_{\alpha} \eta = \oint_{\alpha} \nu + \oint_{\alpha} *dG^{\omega} = a + \oint_{\alpha} *dG^{\omega}, \tag{5.23}$$

$$B = \oint_{\beta} \eta = \oint_{\beta} \nu + \oint_{\beta} *dG^{\omega} = b + \oint_{\beta} *dG^{\omega}. \tag{5.24}$$

The α-periods of the conjugated Green function were computed in (5.6), and from (5.7), (5.8) it follows that

$$\frac{\partial A}{\partial w_k} dw_k + \frac{\partial A}{\partial \bar{w}_k} d\bar{w}_k = \Gamma_k \, dU_{\alpha}(w_k).$$

Taking into account also the dependence of $a_1, \dots, a_{\mathrm{g}}$, and doing the same for B and the β-periods, gives the total differentials

$$dA = da + d \oint_{\alpha} *dG^{\omega} = da + \sum_{k=1}^{n} \Gamma_k dU_{\alpha}(w_k), \tag{5.25}$$

$$dB = db + d \oint_{\beta} *dG^{\omega} = db + \sum_{k=1}^{n} \Gamma_k dU_{\beta}(w_k). \tag{5.26}$$

In matrix notation (5.17) becomes

$$\eta = -A^T dU_{\beta} + B^T dU_{\alpha} = \begin{pmatrix} B^T, & -A^T \end{pmatrix} \begin{pmatrix} dU_{\alpha} \\ dU_{\beta} \end{pmatrix}, \tag{5.27}$$

and the contribution from η to the kinetic energy is the quadratic form

$$\int_M \eta \wedge *\eta = \sum_{k=1}^{\mathrm{g}} \Big(\oint_{\alpha_k} \eta \oint_{\beta_k} *\eta - \oint_{\beta_k} \eta \oint_{\alpha_k} *\eta \Big) =$$

$$= \begin{pmatrix} A^T, & B^T \end{pmatrix} \begin{pmatrix} P & R \\ R^T & Q \end{pmatrix} \begin{pmatrix} A \\ B \end{pmatrix}.$$

The last equality is based on straight-forward computations. Note that P, Q, R are fixed matrices, while A and B depend on $w_1, \dots, w_n$, $a_1, \dots, a_{\mathrm{g}}$, $b_1, \dots, b_{\mathrm{g}}$ via (5.23), (5.24).

We now define the *Hamiltonian function*, $\mathcal{H}$, as the renormalized kinetic energy of the flow considered as a function of the locations of the point vortices and of the circulations:

$$2\mathcal{H}(w_1,\dots,w_n;a_1,\dots,a_{\mathrm{g}},b_1,\dots,b_{\mathrm{g}})=(\nu,\nu)_{1,\mathrm{renorm}}= \tag{5.28}$$

$$\begin{pmatrix}\Gamma_1, & \Gamma_2 & \dots & \Gamma_n\end{pmatrix}\begin{pmatrix}R_{\mathrm{robin}}(w_1) & G(w_1,w_2) & \dots & G(w_1,w_n)\\ G(w_2,w_1) & R_{\mathrm{robin}}(w_2) & \dots & G(w_2,w_n)\\ \vdots & \vdots & \ddots & \vdots\\ G(w_n,w_1) & G(w_n,w_2) & \dots & R_{\mathrm{robin}}(w_n)\end{pmatrix}\begin{pmatrix}\Gamma_1\\ \Gamma_2\\ \vdots\\ \Gamma_n\end{pmatrix}+$$

$$+\begin{pmatrix}A^T, & B^T\end{pmatrix}\begin{pmatrix}P & R\\ R^T & Q\end{pmatrix}\begin{pmatrix}A\\ B\end{pmatrix}.$$

We have adjusted with a factor two in front of $\mathcal{H}$ in order to conform with standard formulas.

The left member of (5.28) exhibits the dependence of $\mathcal{H}$ on the points $w_1,\dots,w_n$, but considering the w_j as coordinates the dependence is not analytic. To exhibit analytic dependence one may therefore write

$$\mathcal{H}=\mathcal{H}(w_1,\dots,w_n,\bar{w}_1,\dots,\bar{w}_n;a_1,\dots,a_{\mathrm{g}},b_1,\dots,b_{\mathrm{g}}).$$

6 Hamilton's equations

6.1 Phase space and symplectic form

To formulate Hamilton's equation one needs, besides the Hamiltonian function itself, a phase space and a symplectic form on it. The phase space will in our case consist of all possible configurations of the vortices, collisions not allowed, together with all possible circulations of the flow around the curves in the homology basis. Thus we take it to be

$$\mathcal{F}=\{(w_1,\dots,w_n;a_1,\dots,a_{\mathrm{g}},b_1,\dots,b_{\mathrm{g}}):w_j\in M, w_k\neq w_j \text{ for } k\neq j\}.$$

Here the w_j shall be interpreted as points on M, but in most equations below they will refer to complex coordinates for such points. Compare Remark 3.2.

Assuming that $\Gamma \neq 0$ (recall (5.3)) the *symplectic form* on $\mathcal{F}$ is taken to be

$$\Omega = \sum_{k=1}^{n} \Gamma_k \mathrm{vol}(w_k) - \frac{V}{\Gamma} \sum_{j=1}^{\mathrm{g}} da_j \wedge db_j =$$

$$= -\frac{1}{2\mathrm{i}} \sum_{k=1}^{n} \Gamma_k \lambda(w_k)^2 dw_k \wedge d\bar{w}_k - \Gamma V \sum_{j=1}^{\mathrm{g}} \frac{da_j}{\Gamma} \wedge \frac{db_j}{\Gamma}. \tag{6.1}$$

The last expression makes sense also if $\Gamma = 0$, because by (5.18), (5.19) the factors da_j/Γ and db_j/Γ remain finite as $\Gamma \to 0$. Thus in both expressions above, the last term shall simply be removed if $\Gamma = 0$.

Let

$$\xi = \sum_{k=1}^{n} (\dot{w}_k \frac{\partial}{\partial w_k} + \dot{\bar{w}}_k \frac{\partial}{\partial \bar{w}_k}) + \sum_{j=1}^{\mathrm{g}} (\dot{a}_j \frac{\partial}{\partial a_j} + \dot{b}_j \frac{\partial}{\partial b_j})$$

denote a generic tangent vector of $\mathcal{F}$ viewed as a derivation. As for the first term, recall the conventions in Remark 3.3. *Hamilton's equations* in general say that

$$i(\xi)\Omega = d\mathcal{H}. \tag{6.2}$$

One main issue below (in Section 6.2) will be to verify that with our choices of phase space, symplectic form and Hamiltonian function, the equations (6.2) really produce the expected vortex dynamics.

To evaluate (6.2) we first compute the left member as

$$i(\xi)\Omega = -\frac{1}{2\mathrm{i}} \sum_{k=1}^{n} \Gamma_k \lambda(w_k)^2 \big(\dot{w}_k d\bar{w}_k - \dot{\bar{w}}_k dw_k\big) - \frac{V}{\Gamma} \sum_{j=1}^{\mathrm{g}} (\dot{a}_j db_j - \dot{b}_j da_j). \tag{6.3}$$

Explicitly (6.2) therefore says that

$$\Gamma_k \lambda(w_k)^2 \dot{w}_k = -2\mathrm{i} \frac{\partial \mathcal{H}}{\partial \bar{w}_k}, \tag{6.4}$$

$$\dot{a}_j = -\frac{\Gamma}{V} \frac{\partial \mathcal{H}}{\partial b_j}, \quad \dot{b}_j = +\frac{\Gamma}{V} \frac{\partial \mathcal{H}}{\partial a_j}. \tag{6.5}$$

Here the partial derivatives in the right members are implicit in

$$d\mathcal{H} = \sum_{k=1}^{n} \frac{\Gamma_k^2}{2} \big(\frac{\partial R_{\mathrm{robin}}(w_k)}{\partial w_k} dw_k + \frac{\partial R_{\mathrm{robin}}(w_k)}{\partial \bar{w}_k} d\bar{w}_k \big) + \tag{6.6}$$

$$+\sum_{k,j=1,k\neq j}^{n}\Gamma_k\Gamma_j\big(\frac{\partial G(w_k,w_j)}{\partial w_k}dw_k+\frac{\partial G(w_k,w_j)}{\partial \bar{w}_k}d\bar{w}_k\big)+$$

$$+\begin{pmatrix} A^T, & B^T \end{pmatrix}\begin{pmatrix} P & R \\ R^T & Q \end{pmatrix}\begin{pmatrix} \sum_{k=1}^n \Gamma_k dU_\alpha(w_k) \\ \\ \sum_{k=1}^n \Gamma_k dU_\beta(w_k) \end{pmatrix}+$$

$$+\begin{pmatrix} A^T, & B^T \end{pmatrix}\begin{pmatrix} P & R \\ R^T & Q \end{pmatrix}\begin{pmatrix} da \\ db \end{pmatrix}.$$

Recall the expressions (5.25), (5.26) for A, B in terms of the phase space variables a, b (as column matrices) and $w_1,\dots,w_n$.

In the partial derivatives of $R_{\rm robin}(w)$ (see (5.15)) we single out the two *affine connections*

$$r_{\rm metric}(w)=2\frac{\partial}{\partial w}\log\lambda(w), \tag{6.7}$$

$$r_{\rm robin}(w)\ =-2h_1(w)=-2\frac{\partial}{\partial w}h_0(w), \tag{6.8}$$

(see (5.12) for the last equality). We then have the following alternative expressions for (ingredients of) the first term in $d\mathcal{H}$:

$$\begin{aligned}\frac{\partial R_{\rm robin}(w_k)}{\partial w_k} &= \frac{1}{2\pi}\big(h_1(w_k)+\frac{\partial}{\partial w_k}\log\lambda(w_k)\big)\\ &= \frac{1}{4\pi}\big(r_{\rm metric}(w_k)-r_{\rm robin}(w_k)\big).\end{aligned} \tag{6.9}$$

6.2 Dynamical equations

The following theorem now makes the Hamilton equations (6.2) explicit.

Theorem 6.1. *The dynamical equations for a point vortex system as above are*

$$\lambda(w_k)^2\dot{w}_k = \frac{\Gamma_k}{2\pi \mathrm{i}}\big(\overline{h_1(w_k)} + \frac{\partial}{\partial \bar{w}_k}\log\lambda(w_k)\big) - 2\mathrm{i}\sum_{j=1, j\neq k}^{n}\Gamma_j\frac{\partial G(w_k, w_j)}{\partial \bar{w}_k} +$$

$$+ 2\begin{pmatrix} B^T, & -A^T \end{pmatrix}\begin{pmatrix} \partial U_\alpha(w_k)/\partial \bar{w}_k \\ \partial U_\beta(w_k)/\partial \bar{w}_k \end{pmatrix},$$

$$\begin{pmatrix} \dot{a} \\ \dot{b} \end{pmatrix} = \frac{\Gamma}{V}\begin{pmatrix} -R^T & -Q \\ P & R \end{pmatrix}\begin{pmatrix} A \\ B \end{pmatrix}.$$

Recall the column vectors (5.23), (5.24):

$$A = a + \oint_\alpha *dG^\omega, \quad B = b + \oint_\beta *dG^\omega.$$

Proof. The equations are obtained by identifying the expression (6.3) for $i(\xi)\Omega$ with the expression (6.6) for $d\mathcal{H}$. Thus, in (6.3), the k:th term in the first sum,

$$-\frac{1}{2\mathrm{i}}\Gamma_k\lambda(w_k)^2\big(\dot{w}_k d\bar{w}_k - \dot{\bar{w}}_k dw_k\big), \tag{6.10}$$

is to be identified with the corresponding parts in the right member of (6.6), namely the first two terms. Together with (6.9) this gives immediately the first two terms in the equation for $\dot{w}_k$.

The third term comes from the term

$$\begin{pmatrix} A^T, & B^T \end{pmatrix}\begin{pmatrix} P & R \\ R^T & Q \end{pmatrix}\begin{pmatrix} \sum_{k=1}^n \Gamma_k dU_\alpha(w_k) \\ \\ \sum_{k=1}^n \Gamma_k dU_\beta(w_k) \end{pmatrix}$$

in equation (6.6). On using (5.21) this can be rewritten as

$$\begin{pmatrix} B^T, & -A^T \end{pmatrix}\begin{pmatrix} R^T & Q \\ -P & -R \end{pmatrix}\begin{pmatrix} \sum_{k=1}^n \Gamma_k dU_\alpha(w_k) \\ \\ \sum_{k=1}^n \Gamma_k dU_\beta(w_k) \end{pmatrix} =$$

$$= \begin{pmatrix} B^T, & -A^T \end{pmatrix}\begin{pmatrix} \sum_{k=1}^n \Gamma_k * dU_\alpha(w_k) \\ \\ \sum_{k=1}^n \Gamma_k * dU_\beta(w_k) \end{pmatrix}.$$

Identifying here the coefficient for $d\bar{w}_k$ with the corresponding coefficient in (6.10) gives

$$\lambda(w_k)^2\dot{w}_k = 2\begin{pmatrix} B^T, & -A^T \end{pmatrix}\begin{pmatrix} \frac{\partial U_\alpha(w_k)}{\partial \bar{w}_k} \\ \frac{\partial U_\beta(w_k)}{\partial \bar{w}_k} \end{pmatrix},$$

as desired.

Finally, the term

$$\begin{pmatrix} A^T, & B^T \end{pmatrix}\begin{pmatrix} P & R \\ R^T & Q \end{pmatrix}\begin{pmatrix} da \\ db \end{pmatrix}$$

in (6.6) is to be identified with

$$-\frac{V}{\Gamma}\begin{pmatrix} -\dot{b}, & \dot{a} \end{pmatrix}\begin{pmatrix} da \\ db \end{pmatrix}$$

in (6.3). This gives

$$\begin{pmatrix} \dot{a} \\ \dot{b} \end{pmatrix} = \frac{\Gamma}{V}\begin{pmatrix} -R^T & -Q \\ P & R \end{pmatrix}\begin{pmatrix} A \\ B \end{pmatrix},$$

as desired. □

7 Motion of a single point vortex

In the case of a single vortex Theorem 6.1 simplifies a little. We may then denote the vortex point w_1 simply by w, and the strength Γ_1 agrees with the total vorticity Γ for the point vortices. If in addition $\mathtt{g} = 0$ then everything simplifies considerable. There is only one free variable, the location $w \in M$ for the vortex, and the dynamical equation for this is

$$\lambda(w)^2\dot{w} = \frac{\Gamma}{2\pi \mathrm{i}}\big(\overline{h_1(w)} + \frac{\partial}{\partial \bar{w}}\log\lambda(w)\big).$$

The Hamiltonian and the symplectic form are

$$\mathcal{H}(w) = \Gamma^2 R_{\mathrm{robin}}(w) = \frac{\Gamma^2}{2\pi}(h_0(w) + \log\lambda(w)),$$

$$\Omega = \Gamma \operatorname{vol}(w) = -\frac{1}{2\mathrm{i}}\Gamma\lambda(w)^2 dw \wedge d\bar{w}.$$

It follows that if (and only if) the two metrics

$$ds^2_{\text{metric}} = \lambda(w)^2|dw|^2 = \frac{2V}{\pi}h_{11}(w)|dw|^2, \tag{7.1}$$

$$ds^2_{\text{robin}} = e^{-2h_0(w)}|dw|^2 \tag{7.2}$$

are identical, up to a constant factor, then the vortex will never move, whatever its initial position is. In [2, 17] this is referred to as ds_{metric} being a "steady vortex metric", or being "hydrodynamically neutral".

Example 7.1. An obvious example of a hydrodynamically neutral metric is that of a homogenous sphere. Indeed, for a sphere of radius one we have, in coordinates obtained by stereographic projection into the complex plane, well-known formulas such as

$$G(z, w) = -\frac{1}{4\pi}\Big(\log\frac{|z-w|^2}{(1+|z|^2)(1+|w|^2)} + 1\Big),$$

$$h_0(w) = \log(1+|w|^2) - \frac{1}{2}, \quad h_1(w) = \frac{\bar{w}}{1+|w|^2}, \quad h_2(w) = \frac{\bar{w}^2}{2(1+|w|^2)^2},$$

$$e^{-2h_0(w)} = \frac{e}{(1+|w|^2)^2}, \quad \lambda(w)^2 = 8h_{11}(w) = \frac{4}{(1+|w|^2)^2}.$$

The Hamiltonian function is constant,

$$\mathcal{H}(w) = \frac{\Gamma^2}{2\pi}\big(\log(1+|w|^2) - \frac{1}{2} + \log\frac{2}{1+|w|^2}\big) = \frac{\Gamma^2}{4\pi}(2\log 2 - 1),$$

and there is no motion of the vortex.

8 Motion of a vortex pair in the dipole limit

For a vortex pair $\{w_1, w_2\}$ with $\Gamma_1 = -\Gamma_2$ we have $\Gamma = \Gamma_1 + \Gamma_2 = 0$, hence there is no compensating background vorticity. The circulations a and b will be time independent by the last equation in Theorem 6.1 and are not needed in phase space, which then simply becomes

$$\mathcal{F} = \{(w_1, w_2) : w_1, w_2 \in M, w_1 \neq w_2\},$$

with symplectic form

$$\Omega = -\frac{1}{2\mathrm{i}}\Gamma_1\big(\lambda(w_1)^2 dw_1 \wedge d\bar{w}_1 - \lambda(w_2)^2 dw_2 \wedge d\bar{w}_2\big). \tag{8.1}$$

The Hamiltonian is the same quantity as before, see (5.28), but it may now be considered as a function only of w_1 and w_2. The circulations a, b are fixed parameters, given in advance.

However the period vectors A and B still depend on time via w_1, w_2. Indeed, using (3.20), (3.21) we have

$$A = a + \Gamma_1 \oint_\alpha \big(* dG(\cdot, w_1) - *dG(\cdot, w_2)\big) = a + \Gamma_1 \int_{w_2}^{w_1} dU_\alpha,$$

$$B = b + \Gamma_1 \oint_\beta \big(* dG(\cdot, w_1) - *dG(\cdot, w_2)\big) = b + \Gamma_1 \int_{w_2}^{w_1} dU_{\beta_j}.$$

The Hamiltonian is given by

$$2\mathcal{H}(w_1, w_2) = \begin{pmatrix} \Gamma_1, & -\Gamma_1 \end{pmatrix} \begin{pmatrix} R_{\rm robin}(w_1) & G(w_1, w_2) \\ G(w_2, w_1) & R_{\rm robin}(w_2) \end{pmatrix} \begin{pmatrix} \Gamma_1 \\ -\Gamma_1 \end{pmatrix} +$$

$$+ \begin{pmatrix} A^T, & B^T \end{pmatrix} \begin{pmatrix} P & R \\ R^T & Q \end{pmatrix} \begin{pmatrix} A \\ B \end{pmatrix}, \tag{8.2}$$

and the dynamics of the vortex pair $\{w_1, w_2\}$ becomes, by (6.4),

$$\Gamma_1 \lambda(w_1)^2\, \dot{w}_1 = -2\mathrm{i}\, \frac{\partial \mathcal{H}(w_1, w_2)}{\partial \bar{w}_1},$$

$$\Gamma_1 \lambda(w_2)^2\, \dot{w}_2 = +2\mathrm{i}\, \frac{\partial \mathcal{H}(w_1, w_2)}{\partial \bar{w}_2}.$$

In place of w_1 and w_2 one may turn to $w = \frac{1}{2}(w_1 + w_2)$ and $u = \frac{1}{2}(w_1 - w_2)$ as coordinates. These are similar to the "center-arrow coordinates" used in [2, 36]. Then

$$\begin{cases} w_1 = w + u, \\ w_2 = w - u. \end{cases} \tag{8.3}$$

We are interested in the dipole limit $u \to 0$. More precisely, we shall let $|u| \to 0$ while requiring $\arg u$ to be stable and approach a definit limit. This limit then determines the orientation of the dipole.

In view of the symmetry $H(w_1, w_2) = H(w_2, w_1)$ we have expansions

$$H(w+u, w-u) = h_0(w) + \mathcal{O}(|u|^2),$$

$$h_0(w \pm u) = h_0(w) \pm \big(h_1(w)u + \overline{h_1(w)u}\big) + \mathcal{O}(|u|^2),$$

$$\log\lambda(w \pm u) = \log\lambda(w) \pm \frac{1}{2}\big(r(w)u + \overline{r(w)u}\big) + \mathcal{O}(|u|^2)$$

(coupled signs throughout). The latter equation uses the affine connection $r(w) = r_{\mathrm{metric}}(w)$, see (6.7) or (4.1). We record also

$$\lambda(w \pm u)^2 = \lambda(w)^2\Big(1 \pm \big(r(w)u + \overline{r(w)}\bar{u}\big) + \mathcal{O}(|u|^2)\Big). \tag{8.4}$$

Using these expansions we obtain, for the first matrix in the Hamiltonian (8.2),

$$2\pi \begin{pmatrix} R_{\mathrm{robin}}(w_1) & G(w_1, w_2) \\ G(w_2, w_1) & R_{\mathrm{robin}}(w_2) \end{pmatrix} = \tag{8.5}$$

$$= \begin{pmatrix} h_0(w+u) + \log\lambda(w+u) & -\log|2u| + H(w+u, w-u) \\ -\log|2u| + H(w+u, w-u) & h_0(w-u) + \log\lambda(w-u) \end{pmatrix} =$$

$$= \begin{pmatrix} \log\lambda(w) & -\log|2u| \\ -\log|2u| & \log\lambda(w) \end{pmatrix} + \begin{pmatrix} h_0(w) & h_0(w) \\ h_0(w) & h_0(w) \end{pmatrix} +$$

$$+ \begin{pmatrix} h_1(w) + \frac{1}{2}r(w) & 0 \\ 0 & -h_1(w) - \frac{1}{2}r(w) \end{pmatrix} u +$$

$$+ \begin{pmatrix} \overline{h_1(w)} + \frac{1}{2}\overline{r(w)} & 0 \\ 0 & -\overline{h_1(w)} - \frac{1}{2}\overline{r(w)} \end{pmatrix} \bar{u} + \mathcal{O}(|u|^2).$$

When acting with $(\Gamma_1, -\Gamma_1)$ on both sides of the matrix (8.5), the last three terms in the final expression disappear and the result becomes, up to $\mathcal{O}(|u|^2)$,

$$\begin{pmatrix} \Gamma_1 & -\Gamma_1 \end{pmatrix} \begin{pmatrix} \log\lambda(w) & -\log|2u| \\ -\log|2u| & \log\lambda(w) \end{pmatrix} \begin{pmatrix} \Gamma_1 \\ -\Gamma_1 \end{pmatrix} = \tag{8.6}$$

$$= 2\Gamma_1^2\big(\log\lambda(w) + \log|2u|\big).$$

For the full Hamiltonian (8.2) we therefore have, up to $\mathcal{O}(|u|^2)$,

$$2\mathcal{H}(w+u, w-u) = \frac{\Gamma_1^2}{\pi}\big(\log\lambda(w) + \log|2u|\big) +$$

$$+\left(\begin{array}{cc} (a+\Gamma_1\int_{w-u}^{w+u} dU_\alpha)^T, & (b+\Gamma_1\int_{w-u}^{w+u} dU_\beta)^T \end{array} \right) \left(\begin{array}{cc} P & R \\ R^T & Q \end{array} \right) \left(\begin{array}{c} a+\Gamma_1\int_{w-u}^{w+u} dU_\alpha \\ b+\Gamma_1\int_{w-u}^{w+u} dU_\beta \end{array} \right).$$

In the above expression the final term remains bounded as $u \to 0$, hence is negligible in this limit compared to the first term. Therefore the leading terms in this limit are given by

$$\mathcal{H}(w+u, w-u) = \frac{\Gamma_1^2}{2\pi}\big(\log\lambda(w) + \log|2u|\big) + \mathcal{O}(1) \quad (u \to 0). \tag{8.7}$$

This is essentially a constant factor times log of the distance (in the metric) between $w_1 = w+u$ and $w_2 = w-u$. Indeed, we recover the simple and beautiful formula

$$\mathcal{H}(w_1, w_2) = \frac{\Gamma_1^2}{2\pi}\log\operatorname{dist}(w_1, w_2) + \mathcal{O}(1)$$

of Boatto and Koiller. See equations (24), (25) in [2]. Compare also [17]. The distance is taken with respect to the Riemannian metric. The error term $\mathcal{O}(1)$ can be identified with a what is called the "Batman function" in [2].

To proceed towards the Hamilton equations (6.2) we differentiate the leading term in (8.7) and insert the metric affine connection $r(w) = r_{\mathrm{metric}}(w)$ as defined by (6.7),

$$\begin{aligned} d\mathcal{H} &= \frac{\partial\mathcal{H}}{\partial w}dw + \frac{\partial\mathcal{H}}{\partial\bar{w}}d\bar{w} + \frac{\partial\mathcal{H}}{\partial u}du + \frac{\partial\mathcal{H}}{\partial\bar{u}}d\bar{u} = \\ &= \frac{\Gamma_1^2}{4\pi}\Big(r(w)dw + \overline{r(w)}d\bar{w} + \frac{du}{u} + \frac{d\bar{u}}{\bar{u}}\Big). \end{aligned} \tag{8.8}$$

Using (8.4) we expand the symplectic 2-form given by (8.1) in terms of w and u as

$$\Omega = -\frac{\Gamma_1}{2\mathrm{i}}\big(\lambda(w+u)^2 d(w+u)\wedge d(\bar{w}+\bar{u}) - \lambda(w-u)^2 d(w-u)\wedge d(\bar{w}-\bar{u}\big) =$$

$$= \mathrm{i}\Gamma_1\lambda(w)^2\Big(dw\wedge d\bar{u} - d\bar{w}\wedge du + \big(r(w)u + \overline{r(w)u}\big)\big(dw\wedge d\bar{w} + du\wedge d\bar{u}\big)\Big) + \mathcal{O}(|u|^2).$$

Interior derivation by

$$\xi = \dot{w}\frac{\partial}{\partial w} + \dot{\bar{w}}\frac{\partial}{\partial\bar{w}} + \dot{u}\frac{\partial}{\partial u} + \dot{\bar{u}}\frac{\partial}{\partial\bar{u}}$$

then gives, up to terms of order $\mathcal{O}(|u|^2)$,

$$i(\xi)\Omega = \mathrm{i}\Gamma_1\lambda(w)^2\Big(-\big(\dot{\bar{u}}+(r(w)u+\overline{r(w)u})\dot{\bar{w}}\big)dw+\big(\dot{u}+(r(w)u+\overline{r(w)u})\dot{w}\big)d\bar{w}-$$

$$-\big(\dot{\bar{w}}+(r(w)u+\overline{r(w)u})\dot{\bar{u}}\big)du+\big(\dot{w}+(r(w)u+\overline{r(w)u})\dot{u}\big)d\bar{u}\Big).$$

Comparing with (8.8) we see that the dynamics of the vortex pair is described by the two equations

$$\frac{\Gamma_1}{4\pi\mathrm{i}}\,\overline{r(w)} = \lambda(w)^2\Big(\dot{u}+\big(r(w)u+\overline{r(w)u}\big)\dot{w}\Big), \tag{8.9}$$

$$\frac{\Gamma_1}{4\pi\mathrm{i}}\,\frac{1}{\bar{u}} = \lambda(w)^2\Big(\dot{w}+\big(r(w)u+\overline{r(w)u}\big)\dot{u}\Big), \tag{8.10}$$

valid up to order $\mathcal{O}(|u|^2)$ as $u\to 0$.

Equation (8.10) can used to eliminate $\dot{w}$ in (8.9), which then becomes

$$\frac{\Gamma_1}{4\pi\mathrm{i}}\overline{r(w)} = \lambda(w)^2\dot{u}+\big(r(w)u+\overline{r(w)u}\big)\Big(\frac{\Gamma_1}{4\pi\mathrm{i}\bar{u}}-\lambda(w)^2\big(r(w)u+\overline{r(w)u}\big)\dot{u}\Big),$$

again up to $\mathcal{O}(|u|^2$. Here the left member cancels with one of the terms in the right member, and the rest can be written, after division by u,

$$0=\lambda(w)^2\cdot\frac{\dot{u}}{u}+r(w)\frac{\Gamma_1}{4\pi\mathrm{i}\bar{u}}-\lambda(w)^2\big(r(w)u+\overline{r(w)u}\big)^2\cdot\frac{\dot{u}}{u},$$

now up to an error of order $\mathcal{O}(|u|)$. In this equation the final main term in the right member is of a smaller magnitude than the first two terms and is swallowed by the allowed general error $\mathcal{O}(|u|)$. Thus we arrive at

$$\lambda(w)^2\frac{d}{dt}\log u + r(w)\frac{\Gamma_1}{4\pi\mathrm{i}\bar{u}} = 0, \tag{8.11}$$

valid with an error of at most $\mathcal{O}(|u|)$ as $u\to 0$.

The above equation, (8.11), essentially comes from (8.9), and it is to be combined again with (8.10). For the latter it is enough to use the simplified form

$$\frac{\Gamma_1}{4\pi\mathrm{i}\bar{u}} = \lambda(w)^2\frac{dw}{dt}, \tag{8.12}$$

which only uses the leading terms, and for which the error is still at most $\mathcal{O}(|u|)$. Inserting (8.12) in (8.11) results in the master equation

$$\frac{d}{dt}\log u + r(w)\frac{dw}{dt} = 0. \tag{8.13}$$

One problem with (8.13) is that the speed dw/dt becomes infinite, along with the first term, in the dipole limit. However, this problem only affects the real part of the equation. For the imaginary part one can replace true time t by an arbitrary parameter, which may be scaled with u so that dw/dt remains finite as $|u| \to 0$. Alternatively one may scale Γ_1 with u so that the left member of (8.12) remains finite. Then one can still think of any new parameter t as a time variable. In any case, we take imaginary parts of (8.13) and obtain

$$\frac{d}{dt} \arg u + \operatorname{Im}(r(w)\frac{dw}{dt}) = 0. \tag{8.14}$$

Equation (8.12) shows that the directions of u and dw/dt are related as

$$\arg u = \arg \frac{dw}{dt} \pm \frac{\pi}{2}, \tag{8.15}$$

where the plus sign is to be chosen if $\Gamma_1 > 0$, the minus sign if $\Gamma_1 < 0$. Now (8.14) and (8.15) taken together give the final law for the motion of the center w of the vortex pair in the dipole limit:

$$\frac{d}{dt} \arg \frac{dw}{dt} + \operatorname{Im}(r(w)\frac{dw}{dt}) = 0. \tag{8.16}$$

As explained in Section 4 (see specifically equation (4.3)), (8.18) is exactly the equation for a geodesic curve when expressed in an arbitrary parameter t. One may notice that (8.14) can be written in the parameter-free form

$$d \arg u + \operatorname{Im}(r(w)dw) = 0$$

(similarly for (8.16)), confirming the fact that the geometry of the dipole trajectory has a meaning independent of any choice of parameter.

The real part of (8.13) says, in view of (6.7), that

$$\frac{d}{dt}\big(\log |u| + \log \lambda(w)\big) = 0,$$

hence that $|u|\lambda(w) = C$ (constant) along each trajectory. By (8.12) this also gives

$$\Big|\frac{dw}{dt}\Big| = \frac{C}{\lambda(w)}. \tag{8.17}$$

Thus along each trajectory $\lambda(w)$ is proportional to one over the velocity, and in this sense has the same role as the refraction index in optics.

We summarize the most essential parts of the above discussion as

Theorem 8.1. *The dynamical equations for a vortex pair in the dipole limit is identical with the geodesic equation for the metric $ds = \lambda(w)|dw|$, namely*

$$\frac{d}{dt}\arg\frac{dw}{dt} + \mathrm{Im}(r(w)\frac{dw}{dt}) = 0. \tag{8.18}$$

Here $w = w(t)$ is the location of the dipole, t is an arbitrarily scaled time parameter chosen such that dw/dt is finite.

Remark 8.1. It is possible to understand why dipole move along geodesics by thinking of vortex pair as a "wave front", in an optical analogy. Equation (8.15) says that the motion is perpendicular to the wave front (the line segment from $w - u$ to $w + u$). Equation (8.14) then expresses that if the front of a vortex pair is not aligned with the level line of $\lambda(w)$ then the direction of u (representing the wave front) changes in such a way that the curve $w(t)$ bends towards higher values of λ.

Being slightly more direct and exact, on taking t to be Euclidean arc length the first term in (8.18) is the ordinary curvature for the curve traced out by $w(t)$. The second term can be viewed as the inner product between the gradient of $\lambda(w)$ (this gradient can be identified with $\overline{r(w)}$) and dw/dt rotated 90 degrees to the right. Letting θ denote the angle between the gradient of $\lambda(w)$ and the velocity vector dw/dt we can write

$$\mathrm{Im}(r(w)\frac{dw}{dt}) = |\overline{r(w)}||\frac{dw}{dt}|\sin\theta.$$

This compatible with the laws of optics, for example Fermat's law, and also with "Snell's law" (see for example [18, 5]) in the somewhat singular case that $\lambda(w)$ jumps between two constant values.

9 Remarks on vortex motion in planar domains

Vortex motion in a planar domain can easily be treated as a special case of vortex motion on Riemann surfaces by turning to the *Schottky double* of the planar domain. For simplicity we shall only discuss the case of one single vortex in the planar domain. The case of several vortices will be similar in an obvious way. The ideas in this section extend to cases of vortex motion on general open Riemannian surfaces with analytic boundary.

Let $\Omega \subset \mathbb{C}$ be the planar domain, assumed to be bounded by finitely many real analytic curves. The Schottky double, first described in [42], is the compact Riemann surface $M = \hat{\Omega}$ obtained by completing Ω with a "backside" $\tilde{\Omega}$, having the opposite conformal structure, and glueing the two along the common boundary. Thus $\hat{\Omega} = \Omega \cup \partial\Omega \cup \tilde{\Omega}$ in a set theoretic sense, and the conformal structure becomes smooth over $\partial\Omega$, as can be seen from well-known reflection principles. If z is a point in $\Omega \subset M$, then $\tilde{z}$ will denote the corresponding (reflected) point on $\tilde{\Omega} \subset M$. Both z and $\tilde{z}$ can also be considered as points in $\mathbb{C}$, then serving as coordinates of the mentioned points in M (holomorphic respectively anti-holomorphic coordinates), and as such they are the same: $z = \tilde{z} \in \mathbb{C}$.

In our case we need also a Riemannian structure, with a metric. This is to be the Euclidean metric on each of Ω and $\tilde{\Omega}$. But these do not fit smoothly across curved parts of $\partial\Omega$, the match will only be Lipschitz continuous. However that is good enough for our purposes because the vortex will anyway never approach the boundary. (In the case of several vortices it is however possible to make up situations in which vortices do reach the boundary).

The metric on $M = \hat{\Omega}$ is thus to be

$$ds = \begin{cases} |dz|, & z \in \Omega, \\ |d\tilde{z}|, & \tilde{z} \in \tilde{\Omega}. \end{cases} \tag{9.1}$$

To see how this behaves across $\partial\Omega$ we need a holomorphic coordinate defined in a full neighborhood of this curve in M. A natural candidate can be defined in terms of the *Schwarz function* $S(z)$ for $\partial\Omega$, a function which is defined by its properties of being holomorphic in a neighborhood of $\partial\Omega$ in $\mathbb{C}$ and by satisfying

$$S(z) = \bar{z} \quad \text{on } \partial\Omega. \tag{9.2}$$

See [8, 44] for details about $S(z)$. We remark that $z \mapsto \overline{S(z)}$ is the (local) anti-conformal reflection map in $\partial\Omega$ and that $S'(z) = T(z)^{-2}$, where $T(z)$ is the positively oriented and holomorphically extended unit tangent vector on $\partial\Omega$.

The complex coordinate z in Ω extends, as a holomorphic function, to a full neighborhood of $\Omega \cup \partial\Omega$, both when this neighborhood is considered as a subset of $\mathbb{C}$ and when it is considered as a subset of M. The first case is trivial, and the second case depends on $\partial\Omega$ being analytic. In terms of the Schwarz function this second extension is given by

$$z = \begin{cases} z & \text{for } z \in \Omega \cup \partial\Omega, \\ \overline{S(\tilde{z})} & \text{for } \tilde{z} \in \tilde{\Omega}, \text{ close to } \partial\Omega. \end{cases} \tag{9.3}$$

In the latter expression, $\overline{S(\tilde{z})}$, $\tilde{z}$ is to be interpreted as a complex number. When the metric on M is expressed in the coordinate (9.3) it becomes

$$ds = \begin{cases} |dz| & \text{for } z \in \Omega \cup \partial\Omega, \\ |S'(z)||dz| & \text{for } z \in \mathbb{C} \setminus \overline{\Omega}, \text{ close to } \partial\Omega. \end{cases} \tag{9.4}$$

In the second case (with $z \in \mathbb{C} \setminus \overline{\Omega}$), $z = \overline{S(\tilde{z})}$, $\tilde{z} \in \tilde{\Omega}$, whereby $\tilde{z} = \overline{S(z)}$ and so $|d\tilde{z}| = |S'(z)||dz|$. Thus (9.4) is consistent with (9.1). We see from the coordinate representation (9.4) that the metric is only Lipschitz continuous across $\partial\Omega$. This is the best one can expect.

The associated affine connection (6.7) is in the coordinate (9.3) given by

$$r(z) = \begin{cases} 0 & \text{for } z \in \Omega \cup \partial\Omega, \\ \{S(z), z\}_1 & \text{for } z \in \mathbb{C} \setminus \overline{\Omega}, \text{ close to } \partial\Omega, \end{cases}$$

where

$$\{S(z), z\}_1 = \frac{S''(z)}{S'(z)} = -2\frac{T'(z)}{T(z)}. \tag{9.5}$$

See Section 4 for notations. Thus $r(z)$ is discontinuous across $\partial\Omega$ and it is natural to represent it on this curve be represented by its mean-value:

$$r_{\text{MV}}(z) = \frac{1}{2}\{S(z), z\}_1 = -\frac{T'(z)}{T(z)} \quad (z \in \partial\Omega). \tag{9.6}$$

Example 9.1. Let $\Omega = \mathbb{D}$. Then $S(z) = 1/z$ and the coordinate z in (9.3) extends to the entire complex plane, thus representing all of $M = \mathbb{D} \cup \partial\mathbb{D} \cup \tilde{\mathbb{D}}$ except for the point $\tilde{0} \in \tilde{\mathbb{D}}$. The metric expressed in this coordinate becomes

$$ds = \begin{cases} |dz|, & |z| \leq 1, \\ |z|^{-2}|dz|, & |z| > 1. \end{cases}$$

The affine connection similarly becomes

$$r(z) = \begin{cases} 0, & |z| < 1, \\ -z, & |z| = 1, \\ -2z, & |z| > 1. \end{cases}$$

The geodesic curves in Ω are of course the (Euclidean) straight lines (similarly in $\tilde{\Omega}$), geodesic curves crossing $\partial\Omega$ are straight lines reflecting into the other side under equal angles on $\partial\Omega$ (just as ordinary optical reflection), while $\partial\Omega$ is in itself a geodesic curve. The latter is intuitively obvious since at any point on $\partial\Omega$ there should be one geodesic in the tangential direction, and this has no other way to go than to follow the boundary.

To confirm the last statement analytically, let t be an arc length (with respect to the Euclidean metric) parameter along $\partial\Omega$, so that $T(z) = \frac{dz}{dt}$ on $\partial\Omega$. The curvature κ of $\partial\Omega$ is

$$\kappa = \frac{d}{dt}\arg\frac{dz}{dt} \quad (z \in \partial\Omega),$$

and using that $T(z)\overline{T(z)} = 1$ on $\partial\Omega$ one finds that

$$T'(z) = \mathrm{i}\kappa \quad (z \in \partial\Omega).$$

In particular $T'(z)$, and so $r_{\mathrm{MV}}(z)T(z)$ (by (9.6)), is purely imaginary on $\partial\Omega$. Combining with (9.6) it follows that

$$\frac{d}{dt}\arg\frac{dz}{dt} = \mathrm{i}r_{\mathrm{MV}}(z)T(z),$$

hence

$$\frac{d}{dt}\arg\frac{dz}{dt} + \mathrm{Im}\big(r_{\mathrm{MV}}(z)T(z)\big) = 0 \quad (z \in \partial\Omega).$$

Thus the geodesic equation (8.18) holds for the curve $\partial\Omega$, as claimed.

We remark that the curvature κ of the boundary curve $\partial\Omega$ appears also in the expression for the Gaussian curvature for the metric on M. That curvature vanishes on Ω and on $\tilde{\Omega}$, whereas on $\partial\Omega$ it has a singular contribution with density 2κ with respect arc-length measure on $\partial\Omega$.

A single vortex in a planar domain Ω moves along a level line of the appropriate Robin function, or Routh's stream function [32, 33, 34]). If the vortex is near the boundary then it follows the boundary closely, with high speed. From the perspective of the Schottky double the boundary conditions for planar fluid motion are such that there is automatically a mirror vortex on the other side in the double, thus we really have a vortex pair close to $\partial\Omega$ on the double. In the limit this becomes a vortex dipole, moving with infinite speed along the geodesic $\partial\Omega$.

Considering in some more detail such a symmetric vortex pair, with vortex locations $w \in \Omega$ and $\tilde{w} \in \tilde{\Omega}$, we first notice that the Green function $G^{\omega}(z)$ for $\omega = \delta_w - \delta_{\tilde{w}}$ simply is the anti-symmetric extension to the Schottky double of the ordinary Green function $G_\Omega(z, w)$ for Ω:

$$G^{\delta_w - \delta_{\tilde{w}}}(z) = G_\Omega(z, w) \quad (z \in \Omega).$$

Then the stream function ψ in (5.2) becomes what is sometimes called the *hydrodynamic Green function*, which depends on the prescribed periods. This function, which can be traced back (at least in special cases) to [27, 34], has more recently been discussed in for example [7, 12, 11, 23].

We wish to clarify how this hydrodynamic Green function is related to the modification, in the beginning of Section 5, of the general Green function flow $- * dG^{\omega}$ by an additional term η. To do so we fix, in the case of a Schottky double $M = \hat{\Omega}$, the homology basis in such a way that the curves β_j, $j = 1, \dots, \mathtt{g}$, closely follow the inner components of $\partial\Omega$, and each curve α_j goes from the j:th inner component of $\partial\Omega$ through Ω to the outer component, and then back again on the backside.

We also introduce the harmonic measures u_j, $j = 1, \dots, \mathtt{g}$, here defined to be those harmonic functions in Ω which takes the boundary value one on the designated (number j) inner component of $\partial\Omega$ and vanishes on the rest of $\partial\Omega$. Their differentials du_j extend harmonically to the Schottky double with $\oint_{\alpha_k} du_j = -2\delta_{kj}$, $\oint_{\alpha_k} du_j = 0$. Thus $du_j = -2dU_{\beta_j}$ in terms of our general notations (as in (3.18), (3.19)).

In the block matrix notation of (5.21) we have

$$- * dU_\beta = P\, dU_\alpha + R\, dU_\beta,$$

where $P = (P_{kj})$, $R = (R_{kj})$ and (see (3.18), (3.19))

$$P_{kj} = - \oint_{\beta_j} * dU_{\beta_k}, \quad R_{kj} = \oint_{\alpha_j} * dU_{\beta_k}.$$

The last integral can be written

$$R_{kj} = -\frac{1}{2} \oint_{\alpha_j} * du_k = -\frac{1}{2} \int_{\alpha_j \cap \Omega} \frac{\partial u_k}{\partial n} ds - \frac{1}{2} \int_{\alpha_j \cap \tilde{\Omega}} \frac{\partial u_k}{\partial n} ds,$$

and it is easy to see that it is zero because of cancelling contributions from Ω and $\tilde{\Omega}$ due to the symmetry of du_k and α_j going the opposite way on the backside.

As a general conclusion we therefore have that $R = 0$ in the matrix (5.21) when M is the Schottky double of a planar domain. As a consequence,

$$-*\, dU_\beta = P\, dU_\alpha.$$

Similarly, the other equation contained in (5.21) becomes

$$*dU_\alpha = Q\, dU_\beta. \tag{9.7}$$

Turning now to the flow η in (5.17), this is necessarily symmetric on $M = \hat{\Omega}$, hence

$$\oint_{\alpha_j} \eta = 0.$$

It follows that the coefficients A_j in (5.17) vanish, whereby that equation becomes

$$\eta = \sum_{j=1}^{\mathtt{g}} B_j dU_{\alpha_j}.$$

In terms of the stream function $\psi = G^\omega + \psi_0$ (recall (5.2)) this gives, on inserting also (9.7),

$$d\psi_0 = *\eta = \sum_{j=1}^{\mathtt{g}} B_j * dU_{\alpha_j} = \sum_{j=1}^{\mathtt{g}} C_j dU_{\beta_j} = d\big(-\frac{1}{2}\sum_{j=1}^{\mathtt{g}} C_j u_j \big),$$

with $C_j = \sum_{i=1}^{\mathtt{g}} B_i Q_{ij}$. It follows in particular that ψ_0, and hence all of ψ, is single-valued on Ω.

In summary, the total stream function is

$$\psi(z) = G_\Omega(z,w) + \sum_{j=1}^{\mathtt{g}} C_j U_{\beta_j}(z),$$

and it is single-valued when restricted to Ω. It is clear that the C_j will actually depend on w, and for symmetry reasons the above formula eventually takes the well-known form

$$\psi(z) = G_\Omega(z,w) + \sum_{i,j=1}^{\mathtt{g}} C_{ij} U_{\beta_i}(z) U_{\beta_j}(w)$$

for some constants C_{ij} subject to $C_{ij} = C_{ji}$.

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