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On the Motion of a Vortex in Two-dimensional Flow of an Ideal Fluid in Simply and Multiply Connected Domains.

by

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I. INTRODUCTION

The present report is the result of some questions posed by Prof. Bengt Joel Andersson (BJA) (Dept. of Hydromechanics, KTH, Stockholm) and conveyed to me by Prof. Harold S. Shapiro (Mathematics, KTH) concerning the motion of vortices in two-dimensional incompressible potential flow. In a short note from 1958 ([BJA]*) BJA shows that a freely moving vortex in a simply connected region (in the complex plane) always moves along the level lines of a certain function. He has then posed questions about equilibrium points for a vortex, that is points where a free vortex is at rest, and it was from the attempts to answer these questions that this report came into being.

The first question was if there always is such an equilibrium point. It turns out that there always is, provided the domain is bounded (not necessarily otherwise), and this is in fact more or less a consequence of the fact that the motion of a vortex in that case always is along closed curves. BJA then asked if there were some conditions on the domain that would guarantee that there was not more one than one such equilibrium point. BJA proposed convexity, and he turned out to be right: in a bounded convex region there always is precisely one point of rest for a free vortex. The proof of this fact consists of some rather nice applications of complex function theory, along the lines of Schwarz's lemma.

This was originally thought to be the main result of this report. However, it later came to my knowledge, thanks to S. Richardson (Applied Mathematics, University of Edinbourgh), that, in its purely function theoretic form, this result was known, proven already 1950 by Hans H. Haegi (in [H]). Therefore the emphasis of this report has now shifted towards the general study of the motion of a vortex in a finitely connected region in the complex plane, with the "uniqueness theorem" for convex domains just as a nice application (with my proof included since it differs a lot from that of Haegi).

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* See References (p. 107)

Here is a summary of the contents of the report.

We are thus mainly concerned with a hydrodynamical system consisting of a two-dimensional incompressible fluid in a finitely connected, possibly unbounded, region in the complex plane. The flow there is to be locally irrotational except at a certain movable point z_0, where we have a vortex of constant strength. It is the force on and the motion of that vortex point that we are interested in.

In section IIa) we derive the fundamental equation which relates the force on the vortex with its velocity (2. 41). If, for example, the vortex is kept fixed in some way, then the surrounding fluid will exert a certain force on it. This force, denoted F_{rest} or F_β, depends on the position z_0 of the vortex and may be regarded as a vector field in the domain, Ω. It turns out that this vector field is a potential field, that is

$$F_\beta = \text{grad } u_\beta \tag{1}$$

for some (real) function u_β in Ω.

If on the other hand there are no outer constraints on the vortex so that it can move freely (this is perhaps the most natural situation), then its velocity will be

$$\frac{dz_0}{dt} = i \cdot (\text{real constant}) \cdot F_\beta(z_0) \tag{2}$$

($i = \sqrt{-1}$). Thus its velocity is always perpendicular to F_β , and it follows that it moves along a level line of u_β .

Thus we have two domain functions F_β and u_β to study. These are expressible in terms of the Green's function, or more correctly, in terms of a certain "modified" Green's function, $g_\beta(z,\zeta)$, for the domain. This differs from the ordinary Green's function, $g(z,\zeta)$, in that instead of being constantly equal to zero on the boundary it is free to take arbitrary constant values on the individual boundary components, and is determined by having its conjugate periods prescribed (together with a normalization condition). The subscript β in $g_\beta(z,\zeta)$ is just a short-hand notation for the list of prescribed periods ($\beta = \beta_1,\ldots,\beta_m$)). The presence of this variable parameter in the problem reflects the fact that for a flow in a multiply connected region one can prescribe the circulations around the "holes" of the domain When the domain is simply connected the two kinds of Green's functions coincide.

Expanding the analytic completion (with respect to z) $G_\beta(z,\zeta)$ of $g_\beta(z,\zeta)$ in a power series about $z = \zeta$,

$$G_\beta(z,\zeta) = -\log(z-\zeta) + c_{\beta 0}(\zeta) + c_{\beta 1}(\zeta)\cdot(z-\zeta) + c_{\beta 2}(\zeta)\cdot(z-\zeta)^2 + \dots, \tag{3}$$

it is shown that F_β and u_β can be expressed as

$$F_\beta(\zeta) = -\text{(positive constant)}\cdot \overline{c}_{\beta 1}(\zeta) \ , \tag{4}$$

$$u_\beta(\zeta) = -\text{(positive constant)}\cdot c_{\beta 0}(\zeta) \ . \tag{5}$$

($c_{\beta 0}(\zeta)$ is chosen to be real in ((3)).

Most of these things are done in IIa). In section IIb) we treat briefly a problem in two-dimensional electrodynamics in which a charged particle obeys the same (or a similar) law as the vortex in the hydrodynamical problem.

In chapter III the functions $c_{\beta 0}(\zeta)$, $c_{\beta 1}(\zeta)$,.. as well as the corresponding functions $c_0(\zeta)$, $c_1(\zeta)$,... defined in terms of the ordinary Green's function, are studied from a purely mathematical point of view (boundary behaviour, transformation properties under conformal mappings etc.) In chapter IV we specialize to simply connected regions and express $c_0(\zeta)$, $c_1(\zeta)$,.. ($= c_{\beta 0}(\zeta)$, $c_{\beta 1}(\zeta)$, .. in that case) in terms of Riemann mapping functions from the unit disc.

Among the results in chapter III is that $c_{\beta 0}(\zeta)$ (and $c_0(\zeta)$) is of the order of magnitude

$$c_{\beta 0}(\zeta) = \log d(\zeta) + O(1) \tag{6}$$

near the boundary, $d(\zeta)$ denoting the distance from ζ to the boundary (Proposition 3.3). This shows in particular that

$$u_\beta(\zeta) \to +\infty \quad \text{as} \quad \zeta \to \partial\Omega \tag{7}$$

(although not necessarily as $\zeta \to \infty$ if Ω is unbounded). It is also found that u_β is always subharmonic. More precisely

$$\Delta u_\beta(\zeta) = (\text{positive constant}) \cdot K_s(\zeta, \zeta), \tag{8}$$

where $K_s(z, \zeta)$ is a certain Bergman kernel (thus $K_s(\zeta, \zeta) > 0$). (Section IIIh).)

When the domain, Ω, is simply connected u_β satisfies a remarkable differential equation, namely

$$\Delta u_\beta = A e^{Bu_\beta}, \tag{9}$$

where A and B are positive constants. This equation has no obvious physical interpretation and is not valid (for any choice of A and B) when the domain is multiply connected (at least not in general). (9) comes from the fact that the Riemannian metric in Ω defined by

$$ds = e^{-c_{\beta 0}(z)} |dz| \tag{10}$$

has constant Gaussian curvature if Ω is simply connected (in which case it coincides with the Poincaré metric). (Sections IVc) and d).)

From (7) it is obvious that if Ω is bounded u_β must have at least one stationary point (a point of minimum). Such a point is a point where F_β vanishes, that is a point where a free vortex is at rest (equilibrium point). In general there is more than one such point, but in section Va) we show that if the domain is convex (but not an infinite strip) then the number of such points never exceeds one. A specific example in section IVe) shows that the condition of convexity for the above property to hold cannot be relaxed to starlikeness.

The treatment in Va) makes repeated use of Schwarz's lemma. It is well-known that Schwarz's lemma can be formulated in an invariant way, expressing then that, analytic mappings are distance decreasing with respect to the Poincare metric. In section Vb) we reformulate part of the treatment in Va) to make a more direct use of the Poincaré metric or, what is the same for simply connected regions, the metric

$$ds = e^{-c_0(z)} |dz| . \tag{11}$$

We also make a little digression and show that also for multiply connected regions analytic mappings are distance decreasing with respect to (11) and this even in a slightly stronger sense than for the Poincaré metric (for example, a universal covering map is not <u>strictly</u> distance decreasing for the Poincaré metric, but for the metric (11) it is). Part of these things are done in section IIIg), where we also show the relations

$$e^{-c_{\beta 0}(z)} \leq e^{-c_0(z)} \leq \rho(z) , \tag{12}$$

$\rho(z)$ referring to the Poincare metric ($ds = \rho(z)\ |dz|$).

Some historical remarks (postscript):

Physically: The literature on two dimensional vortex motion mostly deals with the motion of one or several vortices in the entire complex plane or in certain explicit simply connected subregions thereof. Whenever arbitrary regions are considered the methods are based on transferring the flow to, say, a half plane by means of a Riemann mapping function, and the motion of the vortices are then expressed in terms of this mapping function.

This applies for example to the textbooks [M-T] and [V] (and also to [BJA]). Here [V] seems to go a bit farther that [M-T] in that it establishes the existence of a "stream function", essentially our function u_β (or $c_{\beta 0}$), for the motion of a single vortex in an arbitrary simply connected region. (thus [V] contains, more or less, the result in [BJA].) The differential equation (9) for this stream function is however never mentioned (nor have I met it anywhere else in this hydrodynamical context).

As far as I know there is no literature on vortex motion in multiply connected regions.

Mathematically, the present study essentially comes down to investigating certain domain functions for simply and multiply connected regions in the complex plane, especially $g(z,\zeta)$, $c_0(\zeta)$, $c_1(\zeta)$, $g_\beta(z,\zeta)$, $c_{\beta 0}(\zeta)$, $c_{\beta 1}(\zeta)$. Of these the ordinary Green's function $g(z,\zeta)$ does not require any particular mentioning. The function $c_0(\zeta)$ (defined by $g(z,\zeta) = -\log|z-\zeta|+c_0(\zeta)+O(|z-\zeta|)$ as $z \to \zeta$) is also well-known. Thus, in simply connected regions $c_0(\zeta) = \log r(\zeta)$, where $r(\zeta)$ is the so called mapping radius with respect to the point ζ. (Among all holomorphic functions f defined on the domain in question and satisfying $f(\zeta) = 0$ and $f'(\zeta) = 1$, the one of minimal maximum modulus maps the domain univalently onto the disc with center 0 and radius $r(\zeta)$.)

This mapping radius is extensively studied in [H]. [H] investigates stationary points for $r(\zeta)$ and among other things proves a theorem (Satz 4) which is the same as my Theorem 5.1 (p.89). (This is what was indicated on p.3.) Also, my Proposition 3.3 (p.37) is inspired by, and in fact is just a slight extension of, inequalities for $r(\zeta)$ in [H]. Moreover, [H] obtains estimates for $\dot{r} = \sup_{\zeta} r(\zeta)$ in terms of other domain parameters (the Bloch-Landau constant e.g.).

For regions in general $c_0(\zeta)$ occurs in the context of potential theory, where it sometimes is called the "Robin constant". It is also related to the transfinite diameter of the complement of the region (after a variable transformation). See IIIi) for more details.

The function $e^{-c_0(\zeta)}$ is an example of a "capacity" (namely the capacity of the entire boundary), and the function $p(z,\zeta) = c_0(\zeta) - g(z,\zeta)$ is a "capacity function" in the terminology of [S-O]. In [S-O] functions such as $p(z,\zeta)$ and $c_0(\zeta)$ are defined and studied on arbitrary open Riemann surfaces, and one is for example interested in characterizing surfaces for which $c_0(\zeta) \equiv +\infty$. Compare also Appendix 1 (p.99).

The function $g_\beta(z,\zeta)$, in this text named a "modified Green's function", depends on a list of periods $\beta = (\beta_1,\dots,\beta_m)$ with $\sum_1^m \beta_j = -2\pi$, where m is the connectivity of the domain (the defining properties of $g_\beta(z,\zeta)$ are listed on p. 29). In mathematical literature this function mostly occurs with the choice $\beta = (-2\pi, 0,\dots,0)$ (modulo a permutation), and is then used for example in the construction of univalent mappings onto circular slit discs (the function $f(z) = e^{-G_\beta(z,\zeta)}$ performs such a mapping). See for example [SCH].

Also for the choice $\beta = (-2\pi, 0,\dots,0)$, the functions $p(z,\zeta) = c_{\beta 0}(\zeta) - g_\beta(z,\zeta)$ (capacity function) and $e^{-c_{\beta 0}(\zeta)}$ (capacity) are studied in [S-O] (on Riemann surfaces). See Appendix 1 (p.99).

As to regarding $ds = e^{-c_0(z)}|dz|$ as a conformally invariant metric I have recently found that the fact that this metric is distance decreasing for analytic mappings (my Lemma 3.7 p. 63) is stated and proved as a part of a "Lindelöf's theorem" in [J] (Ch IV, § 46). Also, the fact that this metric is smaller than the Poincaré metric may be put into a more general perspective. Namely, as is proved in [K], whenever a metric (given on some suitable class of domains to which the unit disc belongs) is distance decreasing for all analytic mappings and coincides with the Poincaré metric on the unit disc it falls between two extremal metrics, the largest being the Poincaré metric and the smallest a metric named the Caratheodory distance. This is not a deep theorem, but follows almost immediately from the definitions of the respective metrics. (My reasoning on p. 67-69 is essentially just a rather untransparent proof of the upper half of this theorem.)

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Some notations used

$\mathbb{D}(a;r) = \{ z \in \mathbb{C} : |z-a| < r \}$.

$\mathbb{D} = \mathbb{D}(0;1)$ = the open unit disc.

$\mathbb{P} = \mathbb{C} \cup \{ \infty \}$ = the Riemann sphere.

${}^{*}u$ = any harmonic conjugate of u, if u is a harmonic function.

${}^{*}du = d({}^{*}u)$ (u harmonic).

Thus, along a curve

${}^{*}du = \frac{\partial u}{\partial n} ds$, where $\frac{\partial}{\partial n}$ denotes derivation in the direction of the right-ward normal, and ds is the arc-length differential.

The numbering of formulae and equations starts from 1 at the beginning of each chapter (I - VI). When, say, formula (11) in Ch II is referred to we write just (11) if we are in Ch II, (2. 11) otherwise.

II Physical Background

a) The hydrodynamical problem

1) Notations *)

Let $\Omega \subset \mathbf{C}$ be a finitely connected, possibly unbounded, plane region with a sufficiently nice boundary $\partial\Omega$. We shall consider in Ω an incompressible time-dependent flow which shall be (locally) irrotational except for a vortex of constant strength at a moving point $z_0 = z_0(t)$ (t = time). We are primarily interested in two cases: firstly, that the vortex moves freely (is not influenced in any way by outer constraints) and, secondly, that the vortex is kept fixed in some way (i.e. $z_0(t) \equiv$ constant). It is however advantageous to at once deal with a more general situation. Namely, we assume (somewhat unphysically perhaps) that we are able to force the vortex to move in an arbitrary prescribed manner in Ω. For example, one could think of having a rod stuck down at the vortex point by which one "drags" the vortex. Thus $z_0(t)$ will be assumed to be an arbitrarily prescribed (smooth) function of t, and the major effort of this section (IIa)) will be to compute the drag force F_{ext} needed on the vortex to accomplish that motion.

Let (with $z = x + iy$)

$\mathbf{v} = \mathbf{v}(x,y,t)$ be the velocity field of the flow (vector notation),

$\overline{w} = \overline{w}(z,t)$ the same in complex notation (bar denotes complex conjugation)

$\varphi = \varphi(x,y,t)$ the velocity potential for the flow, that is

$\mathbf{v} = -\operatorname{grad}\varphi$ (vector notation), or

$\overline{w} = -2\dfrac{\partial\varphi}{\partial\overline{z}}$ (complex notation), where

$$\frac{\partial}{\partial\overline{z}} = \frac{1}{2}\left(\frac{\partial}{\partial x} + i\frac{\partial}{\partial y}\right), \quad \frac{\partial}{\partial z} = \frac{1}{2}\left(\frac{\partial}{\partial x} - i\frac{\partial}{\partial y}\right).$$

Let

$\psi = \varphi^*$ be the stream-function and at the same time the harmonic conjugate of φ, and finally

$\Phi = \varphi + i\psi$ the complex potential of the flow.

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*) As a general reference for the kind of hydrodynamics we are dealing with we can recommend [M-T] , which (among other things) treats two-dimensional hydrodynamics with complex variable methods very thoroughly. [L-L 6] is also useful.

The existence (locally) of a potential φ for the flow is a consequence of the assumption that **v** shall be irrotational. The incompressibility assumption then gives that φ is harmonic, although additively multiple-valued. The harmonic conjugate ψ of φ is found to also be the stream function for the flow, and since there are no sources or sinks in the flow it is single-valued harmonic. $\Phi(z)$ and $w(z)$ are analytic functions*) (Φ multiple-valued) and

$$w(z) = -2\frac{\partial\varphi}{\partial z} = -\Phi'(z) . \tag{1}$$

φ, ψ and Φ are determined only up to additive constants.

2. The potential and the velocity field

The flow shall have a vortex of constant strength α (say) at the point $z_0 = z_0(t)$, where $z_0(t)$ is a prescribed (smooth) function of t. This means that Φ for each fixed t shall have the singularity

$$\Phi(z) = i\,\alpha\cdot\log(z-z_0) + \text{regular terms at } z = z_0. \tag{2}$$

For then $w(z) = -\frac{i\alpha}{z-z_0}$ + regular terms, and using polar coordinates $z-z_0 = re^{i\varphi}$ the velocity vector becomes

$$\overline{w}(z) = \frac{i\alpha}{\overline{z}-\overline{z}_0} + \text{regular terms} = \frac{\alpha}{r}\, e^{i(\varphi+\frac{\pi}{2})} + \text{regular terms}$$

which is seen to describe the velocity field of a vortex at z_0 .

The boundary condition for the flow is that $\overline{w}$ shall be tangent to $\partial\Omega$, i.e. that $\partial\Omega$ shall consist of streamlines:

$$\psi = \text{constant} \quad \text{along each component of } \partial\Omega \tag{3}$$

(the constants depend on time in general).

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*) All functions actually depend on t, but this dependence will often be suppressed in the notation. Thus $\Phi = \Phi(z) = \Phi(z,t)$ for example.

If Ω is simply connected (2) and (3) determine φ up to an additive constant and therefore determine the flow (for all t). If however Ω is multiply connected φ is determined only up to a harmonic measure (i.e. a harmonic function which is constant on each component of $\partial\Omega$) which means that the flow is determined only up to a (singularity-free) circulating flow in Ω. In order to specify the flow completely one therefore has to give some additional parameters, for example all but one of the "circulations" about the boundary components.
If $\Gamma_1,\dots,\Gamma_m$ denote the components of $\partial\Omega$ (positively oriented with respect to Ω) these are the numbers

$$C_j = \int_{\Gamma_j} \mathbf{v} \cdot dr = -\int_{\Gamma_j} d\varphi = \int_{\Gamma_j} {}^*d\psi = \int_{\Gamma_j} \frac{\partial\psi}{\partial n}\, ds \qquad (j = 1,\dots,m). \tag{4}$$

The law of "conservation of circulation" (or "Kelvins theorem" ; [M-T], § 3.51 or [L-L6], § 8) asserts that the C_j are constants (in time). Moreover, they must satisfy the consistency relation

$$C_1 + \dots + C_m = 2\pi\alpha \;, \quad {}^{*)} \tag{5}$$

but otherwise they may be arbitrarily prescribed. Thus, specifying $m - 1$ of the constants C_j, the flow will be completely determined for all time.

Let $B_j = B_j(t)$ denote the constant in (3) so that

$$\psi(z,t) = B_j(t) \quad \text{on } \Gamma_j \qquad (j = 1,\dots,m) \;. \tag{6}$$

The significance of these constants is that $B_k - B_j$ is proportional to the amount of fluid which per unit time passes through any curve in Ω connecting Γ_k and Γ_j. Since the circulations C_j determine the flow they also determine the $B_j(t)$ up to a common additive constant. Conversely specifying the $B_j(t)$ at some fixed $t = t_0$ clearly determine $\psi(z,t_0)$ and thereby the C_j. Thus the flow is determined for all time by giving the B_j at any fixed instance.

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*) The necessity of this relation follows by $0 = \int_D \Delta\psi = \int_{\partial D} {}^*d\psi = C_1+\dots+C_m - 2\pi\alpha$, where $D = \Omega \setminus$ (a small disc about z_0) (so that ψ is regular harmonic in D).

From (2), (6) and (4) we have that

$$\psi(z,t) = \operatorname{Im} \Phi(z,t) = -\alpha\, g_\beta(z,z_0(t)), \tag{7}$$

where $g_\beta(z,z_0)$ is a function which for each fixed $z_0 \in \Omega$ satisfies

$$g_\beta(z,z_0) = -\log|z-z_0| + \text{regular harmonic function} \quad \text{in } \Omega \tag{8}$$

$$g_\beta(z,z_0) = \text{constant} = b_j(z_0) \quad (\text{say}) \quad \text{on } \Gamma_j \quad (j = 1,\dots,m) \tag{9}$$

$(b_j(z_0(t)) = -\frac{1}{\alpha} B_j(t))$, and

$$\int_{\Gamma_j}^{*} dg_\beta(z,z_0) = \beta_j \qquad j = 1,\dots,m\,, \tag{10}$$

where the constants $\beta_j = -\frac{1}{\alpha} \cdot C_j$ are independent of z_0 and satisfy

$$\beta_1 + \dots + \beta_m = -2\pi\,. \tag{11}$$

The conditions (8) - (10) determine $g_\beta(z,z_0)$ up to an additive function of z_0. Because of (11) this function can be chosen so that

$$\sum_{j=1}^{m} b_j(z_0) \cdot \beta_j = 0 \qquad \text{for all } z_0 \in \Omega\,. \tag{12}$$

Thus for each $\beta = (\beta_1,\dots,\beta_m)$ satisfying (11) we have a unique function $g_\beta(z,z_0)$ satisfying (8), (9), (10) and (12). $g_\beta(z,z_0)$ is moreover found to have the symmetry property

$$g_\beta(z,z_0) = g_\beta(z_0,z)\,. \tag{13}$$

$g_\beta(z,z_0)$ is closely related to the Green's function $g(z,z_0)$ (which satisfies (8) and (9) with all constants = 0) for Ω and we shall later (p 29 ff) construct $g_\beta(z,z_0)$ from $g(z,z_0)$.

Let us write

$$g_\beta(z,z_0) = -\log|z-z_0| + h_\beta(z,z_0)\,, \tag{14}$$

where thus the function $h_\beta(\cdot,z_0)$ is harmonic in Ω. By (13)

$$h_\beta(z,z_0) = h_\beta(z_0,z)\,. \tag{15}$$

For each z_0 we can form the analytic completion $H_\beta(z,z_0)$ of $h_\beta(z,z_0)$ with respect to z. Although $H_\beta(z,z_0)$ will be multiple-valued (in general), its power series expansion about z_0,

$$H_\beta(z,z_0) = c_{\beta 0}(z_0) + c_{\beta 1}(z_0)\cdot(z-z_0) + c_{\beta 2}(z_0)(z-z_0)^2 + \dots, \tag{16}$$

makes perfectly good sense and is uniquely determined by the normalization

$$\operatorname{Im} c_{\beta 0}(z_0) = 0 \ . \tag{17}$$

Thus

$$c_{\beta 0}(z_0) = \operatorname{Re} H_\beta(z_0,z_0) = h_\beta(z_0,z_0) \ . \tag{18}$$

The analytic completion of $g_\beta(z,z_0)$ (with respect to z) is

$$G_\beta(z,z_0) = g_\beta(z,z_0) + i\overset{*}{g}_\beta(z,z_0) = -\log(z-z_0) + H_\beta(z,z_0) \ , \tag{19}$$

and is always multiple-valued.

By (7), (1), (19) and (16) we get the following expressions for the complex potential and the conjugate of the velocity vector of the flow:

$$\Phi(z,t) = -\, i\,\alpha\, G_\beta(z,z_0(t)\,) = -\, i\,\alpha\,[-\log(z-z_0) + c_{\beta 0}(z_0) + c_{\beta 1}(z_0)(z-z_0) + \dots\,], \tag{20}$$

$$w(z,t) = i\,\alpha\, G'_\beta(z,z_0(t)\,) = i\,\alpha\,[-\frac{1}{z-z_0} + c_{\beta 1}(z_0) + 2c_{\beta 2}(z_0)(z-z_0) + \dots\,] \ . \tag{21}$$

3. The force on the vortex

As we have said earlier the vortex point $z_0(t)$ shall be allowed to move in an arbitrary prescribed manner through the fluid. Such a motion cannot exist by itself, but there must be a certain (variable) force F_{ext} acting on the vortex to keep it moving (or even to keep it still). We are going to compute this force (or rather the negative of it) as a function of the motion of the vortex.

We start from Bernoulli's equation in the form ([M-T], § 3.60 or [L-L6], § 2)

$$-\frac{1}{\rho}\operatorname{grad} p = \frac{\partial \mathbf{v}}{\partial t} + \operatorname{grad}\frac{|\mathbf{v}|^2}{2} \quad , \tag{22}$$

valid for an irrotational flow in the absence of outer forces. Here p denotes the hydrostatic pressure and ρ the density of the fluid. (22) is nothing else than Newton's law of motion in infinitesimal form, the factor $-\operatorname{grad} p$ at the left-hand-side being the force on, and the right-hand-side the acceleration of an infinitesimal element of the fluid.

In our case (22) applies outside the vortex point. Since

$$\mathbf{v} = -\operatorname{grad}\varphi \quad \text{and} \tag{23}$$

$$\rho = \text{constant} \tag{24}$$

(22) can be written

$$\operatorname{grad}\left(\frac{p}{\rho} - \frac{\partial\varphi}{\partial t} + \frac{1}{2}|\mathbf{v}|^2\right) = 0 \qquad \text{in } \Omega \setminus \{z_0\} . \tag{25}$$

Here we must carefully notice that $\varphi = \varphi(x,y,t)$ is only locally well-defined (being additively multiple-valued). It however follows from (25) that for each fixed t

$$\frac{p}{\rho} - \frac{\partial\varphi}{\partial t} + \frac{1}{2}|\mathbf{v}|^2 = \text{globally constant in } \Omega = A(t) \text{ , say .} \tag{26}$$

Therefore, since p, ρ and $\mathbf{v}$ are all well-defined (single-valued) functions in $\Omega\setminus\{z_0\}$, it is a consequence of (25) that

$$\frac{\partial\varphi}{\partial t} \text{ is well-defined (single-valued) in } \Omega\setminus\{z_0\} . \tag{27}$$

This also implies that

$$\frac{\partial\Phi}{\partial t} \text{ is single-valued in } \Omega\setminus\{z_0\} \tag{28}$$

since $\frac{\partial\Phi}{\partial t} = \frac{\partial\varphi}{\partial t} + i\frac{\partial\psi}{\partial t}$, and ψ , hence $\frac{\partial\psi}{\partial t}$, is single-valued in $\Omega\setminus\{z_0\}$ due to the absence of sources and sinks in the flow.

Let us remark in passing that (27) actually proves that form of the law of "conservation of circulation" we used earlier (p.11), namely that the circulations $C_j = \int_{\Gamma_j} \mathbf{v} \cdot dr$ are constant in time. For since Γ_j are closed time-independent curves and $\frac{\partial \varphi}{\partial t}$ is a single-valued function we have

$$\frac{d}{dt} \int_{\Gamma_j} \mathbf{v} \cdot dr = - \frac{d}{dt} \int_{\Gamma_j} \operatorname{grad} \varphi \cdot dr = - \int_{\Gamma_j} \operatorname{grad} \frac{\partial \varphi}{\partial t} \cdot dr = 0 \,. \tag{29}$$

By (26) we get the following expression for the pressure:

$$p = \rho\left(\frac{\partial \varphi}{\partial t} - \frac{1}{2} |\mathbf{v}|^2 + A(t) \right) . \tag{30}$$

Now, to compute the force F exerted on the vortex (by the surrounding fluid) it is natural to consider small regions $D = D_t$ about $z_0(t)$ which "move with the fluid", i.e. consist of the same fluid elements all the time. The total force exerted on such a D by the surrounding fluid is easily seen to be (cf. fig. 2.1)

$$F_D = \int_{\partial D} p \cdot i \, dz \,, \tag{31}$$

and the force in question is obtained by letting D shrink to zero:

$$F = \lim_{\substack{|D| \to 0 \\ (D \ni z_0)}} F_D \,. \tag{32}$$

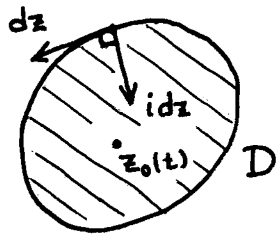

Fig. 2.1

Now there turns out to arise a slight complication in the computation of the integral in (31), caused by the fact that the flow is time-dependent. To avoid this complication we shall deviate a little from the above indicated way, and by a little trick convert the integral in (31) to an integral along the time-independent boundary $\partial\Omega$.

Namely, since the region Ω is fixed in time and the fluid has constant density the center of mass of the fluid must be at rest during the motion of the fluid. Therefore, by Newton's law of motion, the sum of all external forces acting on the fluid must be zero. This means that the force F that the fluid exerts on the vortex point must be equal, but opposite in direction, to the force the fluid exerts on the "walls" $(\partial\Omega)$. This gives that

$$F = -\int_{\partial\Omega} p\cdot(-\ i\ dz) = \int_{\partial\Omega} p\cdot i\ dz\ , \tag{33}$$

where now the integral is fairly easy to compute.

By (30), (33) becomes

$$F = i\ \rho \int_{\partial\Omega} \frac{\partial\varphi}{\partial t}\ dz - \frac{i\rho}{2} \int_{\partial\Omega} \overline{w}\cdot w\ dz + i\ \rho \int_{\partial\Omega} A\ dz\ . \tag{34}$$

To compute the first term we first note that since the stream-function ψ is constant on each component of $\partial\Omega$ ((3)) we have

$$\int_{\partial\Omega} \psi\ dz = 0\ , \tag{35}$$

and therefore also

$$\int_{\partial\Omega} \frac{\partial\psi}{\partial t}\ dz = 0\ . \tag{36}$$

Hence

$$\int_{\partial\Omega} \frac{\partial\varphi}{\partial t}\ dz = \int_{\partial\Omega} \left(\frac{\partial\varphi}{\partial t} + i\ \frac{\partial\psi}{\partial t} \right) dz = \int_{\partial\Omega} \frac{\partial\Phi}{\partial t}\ dz\ . \tag{37}$$

By (19), (20) we get

$$\frac{\partial\Phi}{\partial t} = -\ i\ \alpha\ \frac{d}{dt}\ G_\beta\ (z,z_0(t)\ ) = -\ \frac{i\ \alpha}{z-z_0(t)} \cdot \frac{dz_0(t)}{dt} - i\ \alpha\ \frac{d}{dt}\ H_\beta(z,z_0(t)\ )\ . \tag{38}$$

Here the last term $i\ \alpha\ \frac{d}{dt}\ H_\beta(z,z_0(t)\ )$ is a single-valued analytic function of z in Ω ; it is analytic because $H_\beta(z,z_0(t)\ )$ is, and it is single-valued (despite that $H_\beta(z,z_0(t)\ )$ is not, in general) because the remaining two terms in (38) are ($\frac{\partial\Phi}{\partial t}$ is single-valued by (28)). Therefore

$$\int_{\partial\Omega} \frac{\partial\Phi}{\partial t}\ dz = \int_{\partial\Omega} \left(-\ \frac{i\alpha}{z-z_0} \cdot \frac{dz_0}{dt} \right) dz = 2\ \pi\ \alpha \cdot \frac{dz_0}{dt}\ . \tag{39}$$

As to the second term in (34) we observe that along $\partial\Omega$ the velocity vector $\overline{w}$ is parallel to dz, hence $w\ dz$ is real and therefore equal to $\overline{w}\ d\overline{z}$. Thus

$$\int_{\partial\Omega} \overline{w} \cdot w \, dz = \int_{\partial\Omega} \overline{w} \cdot \overline{w} \, d\overline{z} = \overline{2 \pi i \operatorname{Res}_{z=z_0} w(z)^2} = (\text{by } (21)) =$$

$$= \overline{2 \pi i \cdot (-\alpha^2) \cdot [-2c_{\beta 1}(z_0)]} = -4 \pi i \alpha^2 \cdot \overline{c}_{\beta 1}(z_0) . \tag{40}$$

Since the last term in (34) clearly is zero (37), (39) and (40) give the final result

$$F = 2 \pi \rho \cdot [i \alpha \cdot \frac{dz_0}{dt} - \alpha^2 \cdot \overline{c}_{\beta 1}(z_0)] . \tag{41}$$

This is thus the force exerted by the fluid on the vortex point when this moves with the velocity $\frac{dz_0}{dt}$. We see that it is composed of two terms,

$$F_{vel} = 2 \pi \rho \cdot i \alpha \frac{dz_0}{dt} \quad \text{and} \tag{42}$$

$$F_{rest} = F_\beta = -2 \pi \rho \cdot \alpha^2 \cdot \overline{c}_{\beta 1}(z_0) . \tag{43}$$

The first of these, F_{vel}, is just a linear function of the velocity of the vortex and is quite uncomplicated. It is however interesting to notice that its direction is not opposite to that of the motion, but perpendicular to it.

The other term, $F_{rest} = F_\beta$, is the total force on the vortex when it is at rest. It involves the non-trivial domain function $c_{\beta 1}(z_0)$. Since it is a function of position only it can be regarded as a vector field in Ω. The study of this vector field is the main object of the present report. It turns out that it is a potential field, that is, there is a real function u_β in Ω such that

$$F_\beta = \operatorname{grad} u_\beta . \tag{44}$$

In fact, we will find that (Lemma 3.1)

$$u_\beta(z_0) = -\pi \rho \, \alpha^2 \cdot c_{\beta 0}(z_0) + \text{constant} . \tag{45}$$

We shall also see that (Proposition 3.3)

$$u_\beta(z) \to +\infty \quad \text{as} \quad z \to \partial\Omega \tag{46}$$

(although not necessarily as $z \to \infty$ if Ω is unbounded).

4. Discussion

Let us discuss and explain the results (41), (44), (45) and (46) a little already at this point.

We shall mostly be concerned with the case that the vortex is free to move. This means that there are no external forces on it, that is that

$$F = - F_{ext} = 0 \tag{47}$$

in (41). Thus

$$\frac{dz_0}{dt} = - i\, \alpha \cdot \overline{c}_{\beta 1}(z_0) \quad . \tag{48}$$

Comparing with (43) we see that $\frac{dz_0}{dt}$ is proportional to, but directed 90° to the left of, F_β. On the other hand by (44) F_β is also perpendicular the level lines

$$u_\beta(z) = \text{constant.} \tag{49}$$

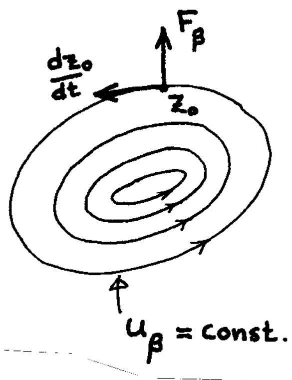

Fig. 2.

Thus it follows that $\frac{dz_0}{dt}$ is always tangent to the level lines of u_β, that is that <u>these level lines actually are the orbits along which the free vortex moves</u>. This can be rephrased by saying that $u_\beta(z)$ is a complete first integral for the system of a freely moving vortex, the motion of it being given by

$$u_\beta(z_0(t)\,) = \text{constant} \; . \tag{50}$$

Thus, in particular, a free vortex only moves along fixed paths (the level lines of u_β) in Ω, a fact which is not at all obvious from the beginning. Moreover, since $u_\beta \to + \infty$ as $z \to \partial\, \Omega$ ((46)) these paths never lead to the boundary of Ω and therefore, if Ω is bounded, in general consist of closed loops (exceptional level lines of u_β

may contain selfintersections or other types of "singular points"). This makes one expect that the motion of a free vortex in a bounded region in general is periodic (the vortex returns to each point on its orbit regularly with a certain time interval). These matters will be discussed more fully in section IIIe).

The above indicated periodicity of motion (under suitable circumstances) is a rather remarkable property for the system of a freely moving vortex, not shared by most other similar physical systems. Consider for example the mechanical system consisting of a single mass particle moving in a central field *) about a fixed point. In that case the particle in general moves along a non-closed curve which is dense in an open subset of the plane and which moreover contains infinitely many selfintersections. In particular the motion is non-periodic (fig. 2.3)

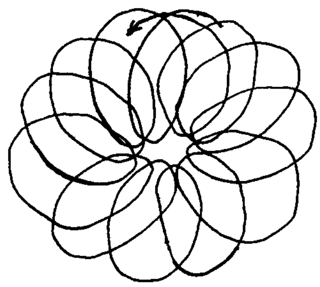

Fig. 2.3

There is another important difference between the vortex motion and, for example, the above described mechanical system. Namely, the velocity of a free vortex is determined by its position, whereas for a mass particle at a given point the velocity can be prescribed arbitrarily. This difference is in principle quite independent of the difference "closed vs. non-closed orbits" and the mathematical aspect of it is of course just that the equations of motion for the vortex are first order differential equations ((48)) while those for the mass particle are second order ones (here are meant the equations for the position variables as functions of time).

In order to see what lies behind these special properties of the vortex motion we consider the energy E associated with it. From a general mechanical standpoint we are dealing with a two-dimensional system with ,

*) By this it is meant that there is a potential V which is a function of the distance r between the particle and the fixed point only, so that the force acting on the particle is $F = -\operatorname{grad} V$.

for example the position z_0 of the vortex representing its coordinates in "configuration space". In general the energy of such a system is a function of both the position coordinates and some momentum coordinates, so that in our case E would be a function of z_0 and (say) $\frac{dz_0}{dt}$.

Now the only kind of energy which we need to take account of is the kinetic energy. The total kinetic energy for the flow is however found to be infinite, for it would be obtained by integrating $\rho|\mathbf{v}|^2$ over Ω, and $|\mathbf{v}|^2$ is not integrable over the vortex point ($|\mathbf{v}|^2 \sim \frac{1}{r^2}$ there)*) Therefore the "infinite part" of the kinetic energy has first to be subtracted off in a correct way to get a finite energy E to play with. We do not go into the details of this subtraction but just write the result symbolically as

$$E = \iint_{\Omega} \rho \; |\mathbf{v}|^2 dxdy - (\text{its "infinite part"}) \; . \tag{51}$$

The important thing for us is just that E only depends on the instantaneous velocity field $\mathbf{v}$.

Now this velocity field is the same whether or not the vortex moves, that is the energy E is independent of the velocity $\frac{dz_0}{dt}$ of the vortex and depends only on its position z_0. This is the critical fact. For since the system is closed the energy is conserved (is an integral of motion), giving the equation

$$E = E(z_0(t)) = \text{constant} \; , \tag{52}$$

and E now being a function only of the two position coordinates this equation alone determines the motion of the system (i.e. it determines its path in configuration space).

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*) This situation is completely analogous to the one in electricity of having infinite internal energy of point charges.

Thus the circumstance that E depends only on the position variables is responsible for the fact that one single integral, E or u_β, determines the motion of a free vortex. As indicated earlier, this in turn explains why the motion in certain cases (namely more or less when each orbit is part of a compact subset of Ω) is periodic. More generally, whenever there exists a complete set (i.e. sufficiently many to determine the motion) of globally well defined integrals of motion for a physical system (whose evolution in time is governed by differential equations) one expects that the motion in such orbits that are compactly contained in the space in question is periodic in general.

The fact that one single integral determines the motion of a vortex also explains, to some extent, why the differential equation for a free vortex is of the first order in the position coordinates. For if it were of higher order one would be able to prescribe the velocity of a vortex at any given point, that is through each point in Ω there would pass orbits in all directions, in obvious conflict with (52) (unless E is identically constant, which it is not).

The fact that no momentum variables enter E should perhaps also be compared with the fact, already employed in the derivation of the force F (p. 15), that the total momentum for the flow is identically zero, due to the circumstance that the center of gravity of the fluid necessarily is at rest.

As to the relation between the two integrals of motion considered, u_β and E, it is obvious that they must be "functionally related" (that is, say, $E = f(u_\beta)$) since they have the same level lines. In fact it is a consequence of (44) that

$$E = - u_\beta + \text{constant} . \tag{53}$$

Thus $- u_\beta$ represents the energy of the system.

In contrast to the above discussions, suppose we attach a mass m to the vortex point (the core of the vortex consisting of a rod or something with mass $m > 0$) . Then the motion of the vortex is associated with an extra kinetic energy $\frac{1}{2} m \left| \frac{dz_0}{dt} \right|^2$, so that E depends effectively on both z_0 and $\frac{dz_0}{dt}$, that is on four real variables. Therefore three integrals of motion are needed to guarantee a periodic motion, which one in general does not have.

As another example consider a system of two (or more) freely moving vortices. Just as for one vortex the energy E is a function only of the positions $z_1, z_2, \ldots$ of the vortices, but in this case the single equation

$$E(z_1, z_2, \ldots) = \text{constant} \tag{54}$$

is not enough to ensure periodicity of the motion, and there is no obvious reason to expect the existence of any integrals of motion besides E . *)
(As to the example on p.19 with a particle in a central field there are in general two integrals of motion (besides the energy, also the angular momentum with respect to the fixed point), whereas three are needed to have a periodic motion, just as in the case of a vortex with mass. For the special case of Newtonian gravitational forces, that is with the central field $V = -\frac{\alpha}{r}$ $(\alpha > 0)$, there actually is a third integral of motion, accounting for example for the (approximately) periodic motion of the planets in ellipses around the sun. See [L-L1], § 14-15.)

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*) This is meant "in general" for a system confined to a certain region $\Omega \subset \mathbf{C}$. If however Ω has special symmetry properties (for example is a disc), there are more integrals, corresponding to conservation laws related to the symmetry (conservation of angular momentum about the center, for a disc, for example). In particular if Ω is the entire plane there are <u>three</u> more integrals of motion, namely those given by the laws of conservation of (linear) momentum (two components) and of angular momentum. Together with energy conservation this gives <u>four</u> integrals of motion which is enough to determine the motion of a system of two vortices but not of three or more. Thus a two-vortex system in the entire plane should exhibit a periodic kind of motion which is also actually the case (except if the strength of the vortices are equal but of opposite signs, in which case they move to infinity. See [M-T], 13-23).

As a final point of this section we wish to mention a much shorter way of proving the differential equation (48) for a freely moving vortex, a way which however gives less information (e.g. it gives no information about the force on a non-free vortex, i.e. equation (41)).
Consider the expression

$$\overline{w}(z) = \frac{i\alpha}{\overline{z}-\overline{z}_0} - i\,\alpha\,\overline{c}_{\beta 1}(z_0) - 2\,i\,\alpha\cdot\overline{c}_{\beta 2}(z_0)\cdot(\overline{z}-\overline{z}_0) - \ldots \tag{55}$$

for the velocity field of the flow near the vortex point z_0. We see that the first term describes a completely rotationally symmetric flow around z_0, while the terms after the second one describe a flow which vanishes at z_0. Thus all these terms (i.e. all but the second one) contribute nothing to the flow at the point z_0, and so the flow there is given by the second term, $-\,i\,\alpha\,\overline{c}_{\beta 1}(z_0)$. If the vortex is free this should therefore be the velocity by which it moves, in other words

$$\frac{dz_0}{dt} = -\,i\,\alpha\,\overline{c}_{\beta 1}(z_0) \quad , \tag{56}$$

which is (48). (With a little effort this "proof" can be made more convincing.) This is the standard way of arguing when deriving the equations of motion for free vortices (used for example in [M-T]).

b) The electrodynamical problem

We shall consider a situation in two-dimensional electrodynamics which is governed by equations similar to those in the hydrodynamical problem. [L-L2] and [L-L8] may serve as general references here.

Let $\Omega \subset \mathbf{C}$ be a finitely connected region as before. The complement $K = \mathbf{C} \setminus \Omega$ of Ω is to be a perfect conductor and Ω itself shall just be empty space. At a point $z_0 \in \Omega$ we place a particle with electrical charge α.

This charge gives rise to a certain electrical field E in Ω, the potential φ of which is a harmonic function with the singularity

$$\varphi(z) = -\frac{\alpha}{2\pi}\log|z - z_0| + \text{regular harmonic} \qquad \text{at } z_0, \tag{57}$$

and which is constant

on each component of $\partial\Omega$. At this point one has to decide between two cases. Either the components $K_1,\ldots,K_m$ of K shall be electrically isolated from each other or they shall be put to a common ground.

In the first case each of the K_j has a certain total charge α_j which is not changed if the point z_0 is moved. Since the total charge of K must be $-\alpha$ the α_j must satisfy

$$\alpha_1 + \cdots + \alpha_m = -\alpha \ . \tag{58}$$

α_j is obtained from φ by

$$\alpha_j = -\int_{\Gamma_j} \frac{\partial\varphi}{\partial n}\, ds = -\int_{\Gamma_j} {}^*d\varphi \ , \tag{59}$$

where Γ_j is the component of $\partial\Omega$ belonging to K_j. It follows from (57) and (59) that φ must be (up to an additive constant)

$$\varphi(z) = \frac{\alpha}{2\pi}\, g_\beta(z,z_0) \qquad \text{with} \tag{60}$$

$$\beta_j = -\frac{2\pi}{\alpha}\cdot \alpha_j \ , \qquad j = 1,\ldots,m \ , \tag{61}$$

where $g_\beta(z,z_0)$ is the "modified Green's function", described on p. 12.

Similarly, the second case, with all K_j grounded, gives

$$\varphi(z) = \frac{\alpha}{2\pi}\, g(z,z_0) \ . \tag{62}$$

We shall limit ourselves to this case in the following, the first case being obtained from it by obvious modifications.

In writing

$$g(z,z_0) = -\log|z - z_0| + h(z,z_0) \ , \tag{63}$$

$h(z,z_0)$ being regular harmonic, we get $\varphi(z)$ decomposed into the self-potential of the charge at z_0, $-\frac{\alpha}{2\pi}\log|z - z_0|$, and the part of the potential coming from the induced charge distribution on K (actually on $\partial\Omega$), $\frac{\alpha}{2\pi}\, h(z,z_0)$. The electrical field of this latter part at z_0

is $\{-\operatorname{grad}_z \frac{\alpha}{2\pi} h(z,z_0)\}_{z=z_0}$ and it therefore gives rise to the Coulomb force

$$F = \alpha \cdot \{-\operatorname{grad}_z \frac{\alpha}{2\pi} h(z,z_0)\}_{z=z_0} \tag{64}$$

acting on the charge at z_0. Since the rotationally symmetric self-field $\operatorname{grad}_z \frac{\alpha}{2\pi} \log|z - z_0| = \frac{\alpha}{2\pi} \cdot \frac{1}{\overline{z-z_0}}$ of the charge obviously does not produce any net force on the charge itself, (64) gives the total electrical force exerted on the charge.

In analogy with page 13 we introduce the (multiple-valued) analytic completions (with respect to z) $G(z,z_0)$, $H(z,z_0)$ of $g(z,z_0)$ and $h(z,z_0)$, and develop $H(z,z_0)$ in its power series about $z = z_0$,

$$H(z,z_0) = c_0(z_0) + c_1(z_0)(z-z_0) + c_2(z_0)(z-z_0)^2 + \dots \quad . \tag{65}$$

With the normalization

$$\operatorname{Im} c_0(z_0) = 0 \tag{66}$$

we have

$$c_0(z_0) = \operatorname{Re} H(z_0,z_0) = h(z_0,z_0) \quad . \tag{67}$$

Since
$$\operatorname{grad}_z h(z,z_0) = 2\frac{\partial}{\partial \bar z} h(z,z_0) = \frac{\partial}{\partial \bar z}[H(z,z_0) + \overline{H(z,z_0)}] = \overline{H'(z,z_0)} \quad ,$$

(64) becomes

$$F = -\frac{\alpha^2}{2\pi} \cdot \bar c_1(z_0) \quad . \tag{68}$$

Thus we see that the charge at z_0 is subject to a force of a similar kind as was a vortex at rest at z_0 in the hydrodynamical problem (F_{rest}, (43)). If we had worked with the case of $K_1,\dots,K_m$ being isolated from each other instead, we would have obtained (68) with $c_1(z_0)$ replaced by $c_{\beta 1}(z_0)$, that is exactly as in the hydrodynamical case.

One can carry the analogy with the hydrodynamical problem one step further by trying to imitate also the force F_{vel}, (42). Since that force is proportional to α and to $\left|\frac{dz_0}{dt}\right|$, and perpendicular to $\frac{dz_0}{dt}$, this is achieved just by introducing a constant magnetic field perpendicular to the z-plane. If the strength of this field is B (taken positive when the field is directed downwards) the expression

$$F = i\,\alpha\,B \cdot \frac{dz_0}{dt} - \frac{\alpha^2}{2\pi} \cdot \overline{c}_1(z_0) , \tag{69}$$

analogous to (41), for the total electromagnetic force on the moving charge results.

It should however be remarked that formula (69) is only approximate the approximation being good only if the velocity $\frac{dz_0}{dt}$ is small. This is because we have not taken into account second order effects, such as the fact that when the charge at z_0 moves also the induced charges on $\partial\Omega$ move, thereby producing an extra magnetic field which has to be added to B in (69). This additional field must however be proportional to $\frac{dz_0}{dt}$, so assuming that $\frac{dz_0}{dt}$ is small (or B large) it can be neglected. There are other higher order effects as well, due to the interaction between time-dependent electrical and magnetic fields, which we have disregarded, but assuming $\frac{dz_0}{dt}$ is small enough they can be neglected.

Supposing that we are within the ranges of the above approximations and moreover supposing that the charged particle has zero (or very small) mass, the charge will move according to the same rules as a free vortex (but leaving out the β:s in the formulas), that is in closed orbits (if the region is bounded)

$$u(z) = \text{constant}, \tag{70}$$

where

$$u(z) = -\frac{\alpha^2}{4\pi} \cdot c_0(z) + \text{constant} \tag{71}$$

is a potential function for the force F in (68),

$$F = \text{grad}\ u . \tag{72}$$

III. The Green's Function

a) Preliminaries

Having expressed the physical quantities we are interested in (viz. the vector fields F_β (2.43) and F (2.68) in terms of certain mathematical domain functions in section II, we shall now begin a more detailed study of these latter from a purely mathematical point of view. Let us therefore begin by gathering some definitions from section II, making more precise assumptions about the domains involved.

Although we are primarily interested in finite domains Ω (i.e. $\Omega \subset \mathbb{C}$) it will sometimes be convenient to allow Ω to contain ∞ as an interior point. Thus letting

$$\mathbb{P} = \mathbb{C} \cup \{\infty\}$$

denote the Riemann sphere we shall consider domains $\Omega \subset \mathbb{P}$. *) As to boundary regularity we shall always assume that Ω is bounded by finitely (≥ 1) many continua, that is that

$$\mathbb{P} \setminus \Omega = K_1 \cup .. \cup K_m , \quad m \geq 1$$

where $K_1, \ldots, K_m$, the components of $\mathbb{P} \setminus \Omega$, are closed connected sets, each consisting of more than one point. These assumptions will be in force throughout this report (except when otherwise explicitly stated), even when words such as "arbitrary domain" or "any domain" are used. The components of the boundary $\partial\Omega$ of Ω will usually be denoted $\Gamma_1, \ldots, \Gamma_m$, so that $\Gamma_j \subset K_j$. It is well known that a region of the above kind is always conformally equivalent to a region bounded by analytic curves.

Now, for such regions Ω the Green's function $g(z,\zeta)$ exists, and it is characterized by the properties

$$\text{i)} \quad g(z,\zeta) = -\log|z-\zeta| + h(z,\zeta) , \quad z, \zeta \in \Omega \setminus \{\infty\} , \tag{1}$$

---

*) On a few occasions (IIIg)) and Appendix 2) we shall also allow Ω to be non-schlicht, that is to be a Riemann surface lying over $\mathbb{P}$.

where $h(z,\zeta)$ is a harmonic function of z (and of ζ),

$$\text{ii)}\ g(z,\zeta) \to 0 \quad \text{as} \quad z \to z_0 \quad \text{for each} \quad z_0 \in \partial\Omega \ . \tag{2}$$

When z or ζ equals ∞ i) has to be modified but we do not bother about this.

The function $h(z,\zeta)$ is defined for $z, \zeta \in \Omega \setminus \{\infty\}$ by i). Since $g(z,\zeta)$ is symmetric,

$$\text{iii)}\ g(z,\zeta) = g(\zeta,z) \quad , \tag{3}$$

so is $h(z,\zeta)$,

$$h(z,\zeta) = h(\zeta,z) \ . \tag{4}$$

Besides the Green's function we will need the m harmonic measures $\omega_1(z),\dots,\omega_m(z)$ of Ω, characterized by being harmonic in Ω and taking the boundary values

$$\omega_k(z) = \begin{cases} 1 & \text{on} \quad \Gamma_k \\ 0 & \text{on} \quad \Gamma_j \ , \quad j \neq k \ . \end{cases} \tag{5}$$

The $\omega_k(z)$ obviously satisfy

$$\omega_1(z) + \dots + \omega_m(z) \equiv 1 \tag{6}$$

and this is the only linear relation between them (that is, if $\mu_1\omega_1(z) + \dots + \mu_m\omega_m(z) \equiv \mu$ for constants $\mu_1,\dots,\mu_m,\mu$ then $\mu_1=\dots=\mu_m=\mu$).

$\omega_k(z)$ can be obtained from the Green's function by means of the following well-known formula ([NEH], Ch I, Sec 10 e.g.)

$$\omega_k(z) = -\frac{1}{2\pi}\int_{\Gamma_k}^{*} dg(\cdot,z) = -\frac{1}{2\pi}\int_{\Gamma_k} \frac{\partial g(\zeta,z)}{\partial n_\zeta}\, ds_\zeta \quad . \tag{7}$$

(If the boundary curves Γ_k are not smooth enough the paths of integration in (7) have to be moved a little into Ω.)

b) Construction of the "modified Green's functions"

In the hydrodynamical problem we encountered (p. 12) a family of functions $g_\beta(z,\zeta)$, closely related to the Green's function. These "modified Green's functions" will now be constructed from the ordinary Green's function and the harmonic measures.*

Thus, let a list

$$\beta = (\beta_1,\dots,\beta_m) \qquad (\beta_j \in \mathbb{R}) \tag{8}$$

of conjugate periods for $g_\beta(z,\zeta)$ be given, satisfying the consistency requirement

$$\beta_1 + \dots + \beta_m = -2\pi . \tag{9}$$

Then $g_\beta(z,\zeta)$ shall have the properties

i) $$g_\beta(z,\zeta) = -\log|z-\zeta| + h_\beta(z,\zeta) \tag{10}$$

where $h_\beta(\cdot,\zeta)$ is harmonic in $\Omega \setminus \{\infty\}$

ii) $$g_\beta(\cdot,\zeta) = \text{constant} = b_j(\zeta) \quad \text{on } \Gamma_j , \tag{11}$$

iii) $$\int_{\Gamma_j} {}^*dg_\beta(\cdot,\zeta) = \beta_j , \tag{12}$$

iv) $$\sum_{j=1}^{m} b_j(\zeta)\cdot\beta_j = 0 , \tag{13}$$

v) $$g_\beta(z,\zeta) = g_\beta(\zeta,z) . \tag{14}$$

Here (i) - (iii) are the essential properties, determining $g_\beta(z,\zeta)$ up to an arbitrary additive function of ζ. That function is fixed by condition (iv), which is a kind of normalization and (v) is a consequence of (i) - (iv).

Of course, $g_\beta(z,\zeta)$ will differ from $g(z,\zeta)$ only if Ω is multiply connected ($m = 1$, $\beta_1 = -2\pi$, $b_1(\zeta) \equiv 0$ if Ω is simply connected.).

---

* The functions $g_\beta(z,\zeta)$, with the particular choice $\beta = (-2\pi,0,\dots,0)$, are often considered and constructed in the context of finding univalent mapping functions onto canonical domains such as circular slit discs. See for example [SCH]. Compare also Appendix 1 (p.99) in the present report. The construction here is given mostly as a matter of convenience. Actually, the existence and uniqueness of a function $g_\beta(z,\zeta)$ having the properties i) - iv) below follow from general existence and uniqueness principles for harmonic functions.

One easily finds that $g_\beta(z,\zeta)$ must be of the form

$$g_\beta(z,\zeta) = g(z,\zeta) + \sum_{k,j=1}^{m} a_{kj}\omega_k(z)\omega_j(\zeta) \tag{15}$$

for some matrix $(a_{kj})_{k,j=1}^{m}$. Having set this up, one need only check that (a_{kj}) can be chosen so that (i) - (v) are satisfied.

Now, (i) and (ii) are automatically satisfied, with

$$h_\beta(z,\zeta) = h(z,\zeta) + \Sigma\, a_{kj}\, \omega_k(z)\, \omega_j(\zeta) \qquad \text{and} \tag{16}$$

$$b_k(\zeta) = \sum_{j=1}^{m} a_{kj}\, \omega_j(\zeta) \tag{17}$$

respectively.
Putting

$$p_{kj} = \int_{\Gamma_k} {}^{*}d\, \omega_j = \int_{\Gamma_k} \frac{\partial \omega_j}{\partial n} ds \tag{18}$$

and using (7), the left-hand-side of (iii) becomes

$$\int_{\Gamma_j} {}^{*}dg_\beta(\cdot,\zeta) = -\, 2\pi\, \omega_j(\zeta) + \sum_{k,i=1}^{m} a_{ki}p_{jk}\omega_i(\zeta) =$$

$$= \sum_{i=1}^{m} \Big(\sum_{k=1}^{m} p_{jk}a_{ki} - 2\pi\, \delta_{ji}\Big)\omega_i(\zeta) \quad . \tag{19}$$

In view of (6) (and the statement following it) (iii) therefore reduces to the linear system of equations

$$\sum_{k=1}^{m} p_{jk}a_{ki} - 2\pi\, \delta_{ji} = \beta_j \quad , \qquad i,j = 1,\dots,m \quad . \tag{20}$$

Similarly, using (17) and (7), (iv) becomes

$$\sum_{k=1}^{m} \beta_k a_{kj} = 0 \quad , \qquad j = 1,\dots,m. \tag{21}$$

Now I <u>claim</u>:

The system of equations (20) - (21) has a unique solution (a_{kj}). This solution moreover has the properties

1 $\quad a_{kj} = a_{jk} \qquad$ and $\qquad$ (22)

2 (a_{kj}) is positive semi-definite; more precisely, for any $\lambda_j \in \mathbb{R}$, $(j = 1,\dots,m,)$, $\sum_{k,j=1}^{m} a_{kj}\lambda_k \lambda_j \geq 0$, with equality only if

$$\lambda_j = (\text{constant})\cdot\beta_j \quad , \quad j = 1,\dots,m \quad . \tag{23}$$

It is clear that the required properties (i) - (v) of $g_\beta(z,\zeta)$ follows from this claim, since (v) follows from 1.

To prove the claim we first note that p_{kj} equals the Dirichlet inner produkt between ω_k and ω_j :

$$p_{kj} = \int_{\Gamma_k} {}^*d\,\omega_j = \int_{\partial\Omega} \omega_k {}^*d\,\omega_j = \int_{\Omega} d\,\omega_k {}^*d\omega_j \tag{24}$$

From this it follows that

$$p_{kj} = p_{jk} \tag{25}$$

and, since any $m-1$ of $d\,\omega_1,\dots,d\,\omega_m$ are linearly independent, that the submatrix obtained from (p_{kj}) by, for any r, deleting the r:th row and the r:th column is positive definite *), in particular nonsingular. Thus the relation

$$\sum_{k=1}^{m} p_{kj} = 0 \quad , \quad j = 1,\dots,m \quad , \tag{26}$$

obtained immediately from the definition of (p_{kj}), is the only linear relation between the rows of (p_{kj}). Since the right member in

$$\sum_{k=1}^{m} p_{jk}a_{ki} = \beta_j + 2\pi\,\delta_{ji} \quad , \quad i,j = 1,\dots,m \, , \tag{27}$$

by (9) satisfies the same linear relation, it follows that (27), i.e. (20), for each $i = 1,\dots,m$ has a one-dimensional solution space, and in that solution space there is exactly one solution $(a_{ki})_{k=1}^{m}$ which also satisfies (21). This proves the unique solvability of (20) - (21) .

---

*) a slightly stronger result follows, namely, using (6) and the statement following it, for any $\mu_j \in \mathbb{R}$ $(j = 1,\dots,m)$

$$\sum_{k,j=1}^{m} p_{kj}\mu_k\mu_j = \int_{\Omega} \sum_{1}^{m} \mu_k \, d\,\omega_k {}^*\Big(\sum_{1}^{m} \mu_j d\,\omega_j\Big) \geq 0,$$

with strict inequality unless all μ_j are equal.

The symmetry of (a_{kj}) follows for example by multiplying (27) by a_{jr} and summing over j:

$$\sum_{j,k=1}^{m} a_{jr} p_{jk} a_{ki} = \sum_{j=1}^{m} \beta_j\, a_{jr} + 2\pi\, a_{ir} = 2\pi\, a_{ir} \,. \tag{28}$$

Since the left-hand-side of (28) is symmetric in i and r (using (25)), so is the right-hand-side, that is (a_{ir}) is symmetric.

If we multiply equation (28) by $\lambda_i \lambda_r$ $(\lambda_j \in \mathbb{R}\,,\ j = 1,\dots,m)$ and sum we get

$$2\pi \sum_{i,r=1}^{m} a_{ir}\lambda_i\,\lambda_r = \sum_{j,k} p_{jk} \cdot \sum_r a_{jr}\,\lambda_r \cdot \sum_i a_{ki}\,\lambda_i =$$

$$= \sum_{j,k} p_{jk}\,\mu_j\,\mu_k \geq 0\,, \tag{29}$$

where $\mu_k = \sum_{i=1}^{m} a_{ki}\,\lambda_i$. Here, as we know, we have equality only if all μ_k are equal. But in that case (26) and (27) give

$$0 = \sum_{k=1}^{m} p_{jk}\,\mu_k = \sum_{k,i} p_{jk}\, a_{ki}\,\lambda_i = \beta_j \cdot \sum_{i=1}^{m} \lambda_i + 2\pi \cdot \lambda_j \quad, \tag{30}$$

from which statement 2 in the claim follows.

This finishes the construction of $g_\beta(z,\zeta)$.

There are two further observations concerning the function

$$\omega_\beta(z,\zeta) = \sum_{k,j=1}^{m} a_{kj}\, \omega_k(z)\, \omega_j(\zeta) = g_\beta(z,\zeta) - g(z,\zeta) \quad . \tag{31}$$

which we will need later on.

The first is that if the period list $\beta = (\beta_1,\dots,\beta_m)$ is changed, then the matrix (a_{kj}) changes by a matrix of the form $(a_k + a_j)$. This is rather easily seen from equations (20) - (21) . It follows that $\omega_\beta(z,\zeta)$ changes by a function of the form

$$\sum_{k,j} (a_k + a_j)\, \omega_k(z)\, \omega_j(\zeta) = \sum_{k=1}^{m} a_k \omega_k(z) + \sum_{j=1}^{m} a_j \omega_j(\zeta) = \omega(z) + \omega(\zeta)\,, \tag{32}$$

where $\omega(z) = \Sigma a_k \omega_k(z)$ is a harmonic function which is constant on each boundary component.

The other is the inequality

$$\omega_\beta(\zeta,\zeta) \geq 0 , \tag{33}$$

which is an immediate consequence of the positive semi-definiteness of the matrix (a_{kj}). In fact, (33) is just (23) with $\lambda_k = \omega_k(\zeta)$. (23) also shows that equality holds in (33) if and only if $\omega_k(\zeta) = (\text{constant})\cdot\beta_k$, that is if and only if

$$\beta = -2\pi(\omega_1(\zeta),\dots,\omega_m(\zeta)). \tag{34}$$

Thus equality in (33) can occur for some $\zeta \in \Omega$ only if $-2\pi < \beta_j < 0$ for all $j = 1,\dots,m$. If $\beta_j = -2\pi\delta_{kj}$ $(j = 1,\dots,m)$ for some k, we have equality on Γ_k and nowhere else. In general, for $m \geq 3$, the right hand side of (34) generates some 2-dimensional variety in the cube $-2\pi \leq x_j \leq 0$, $j = 1,\dots,m$, of $\mathbb{R}^m$ as ζ varies over Ω, and we have equality in (33) at some point only if β happens to lie on that variety. (34) combined with (21) also shows that $\omega_\beta(\zeta,\zeta) = 0$ is equivalent to the vanishing of $\omega_\beta(z,\zeta)$ identically in z.

Having now the functions $g(z,\zeta)$, $h(z,\zeta)$, $g_\beta(z,\zeta)$ and $h_\beta(z,\zeta)$ in our hands we form their analytic completions, $G(z,\zeta)$, $H(z,\zeta)$, $G_\beta(z,\zeta)$ and $H_\beta(z,\zeta)$ respectively, with respect to the z variable (they are multiple-valued in general). Thus

$$G(z,\zeta) = -\log(z-\zeta) + H(z,\zeta) \tag{35}$$

$$G_\beta(z,\zeta) = -\log(z-\zeta) + H_\beta(z,\zeta) . \tag{36}$$

In the power series expansions

$$H(z,\zeta) = c_0(\zeta) + c_1(\zeta)(z-\zeta) + c_2(\zeta)(z-\zeta)^2 + \dots \tag{37}$$

$$H_\beta(z,\zeta) = c_{\beta 0}(\zeta) + c_{\beta 1}(\zeta)(z-\zeta) + c_{\beta 2}(\zeta)(z-\zeta)^2 + \dots \tag{38}$$

everything except $\mathit{Im}\, c_0(\zeta)$ and $\mathit{Im}\, c_{\beta 0}(\zeta)$ is well determined. We shall always choose

$$\operatorname{Im} c_0(\zeta) = 0 \tag{39}$$

$$\operatorname{Im} c_{\beta 0}(\zeta) = 0. \tag{40}$$

Thus $c_0(\zeta)$, $c_{\beta 0}(\zeta)$ are real, and

$$c_0(\zeta) = H(\zeta,\zeta) = h(\zeta,\zeta) \tag{41}$$

$$c_{\beta 0}(\zeta) = H_\beta(\zeta,\zeta) = h_\beta(\zeta,\zeta) \quad . \tag{42}$$

Also, by (16)

$$c_{\beta 0}(\zeta) = c_0(\zeta) + \omega_\beta(\zeta,\zeta) = c_0(\zeta) + \sum_{k,j=1}^{m} a_{kj}\, \omega_k(\zeta)\omega_j(\zeta) \quad . \tag{43}$$

If $\infty \in \Omega$ the above definitions of $c_n(\zeta)$, $c_{\beta n}(\zeta)$ make sense only for $\zeta \neq \infty$. The correct definitions at $\zeta = \infty$ will be given later (Corollary 3.6, p.45)

c) F is a potential field

In section II we found that the physical quantities of main interest were the vector fields

$$F_\beta(z) = -\text{ (positive constant)}\cdot \overline{c}_{\beta 1}(z) \qquad \text{and} \tag{44}$$

$$F(z) = -\text{ (positive constant)}\cdot \overline{c}_1(z) \tag{45}$$

((2.43) resp. (2.68) ; in (44) we have changed the notation from F_{rest} to F_β). We shall show that these vector fields are potential fields, that is that

$$F = \operatorname{grad} u \tag{46}$$

$$F_\beta = \operatorname{grad} u_\beta \tag{47}$$

for some real functions u and u_β.

Indeed we have

<u>Lemma 3.1</u>: $$c_1(z) = \frac{\partial}{\partial z} c_0(z) \quad , \tag{48}$$

$$c_{\beta 1}(z) = \frac{\partial}{\partial z} c_{\beta 0}(z) \quad , \tag{49}$$

showing that (46), (47) hold with

$$u(z) = -\ (\text{pos. const.})\cdot c_0(z) + \text{const.} \tag{50}$$

$$u_\beta(z) = -\ (\text{pos. const.})\cdot c_{\beta 0}(z) + \text{const.} \tag{51}$$

<u>Proof</u> of the lemma: We prove the first equality, the proof of the second one being identical.

Put

$$h_{10}(z,\zeta) = \frac{\partial}{\partial z} h(z,\zeta) \ , \quad h_{01}(z,\zeta) = \frac{\partial}{\partial \zeta} h(z,\zeta) \quad . \tag{52}$$

Then (41) and (37) together with

$$h(z,\zeta) = \text{Re}\, H(z,\zeta) = \frac{1}{2}\left(H(z,\zeta) + \overline{H(z,\zeta)}\,\right) \tag{53}$$

give

$$c_1(\zeta) = \frac{\partial}{\partial z}\Big|_\zeta H(z,\zeta) = 2\frac{\partial}{\partial z}\Big|_\zeta h(z,\zeta) = 2h_{10}(\zeta,\zeta) =$$
$$= h_{10}(\zeta,\zeta) + h_{01}(\zeta,\zeta) = \frac{\partial}{\partial \zeta} h(\zeta,\zeta) = \frac{\partial}{\partial \zeta} c_0(\zeta) \quad ,$$

as was to be proved.

As to analogues of Lemma 3.1 for the higher order coefficienrs $c_n(\zeta)$ one has for example that

$$c_n(\zeta) = \frac{1}{n}\,\frac{\partial c_{n-1}(\zeta)}{\partial \zeta} + \text{holomorphic function} \tag{54}$$

for all $n \geq 1$. Also, it turns out, $c_n(\zeta)$ is a linear combination, with rational coefficients, of

$$\frac{\partial^n}{\partial \zeta^n} c_0(\zeta)\ , \quad \frac{\partial^{n-2}}{\partial \zeta^{n-2}} c_2(\zeta)\ , \ldots, \frac{\partial}{\partial \zeta} c_{n-1}(\zeta)$$

if n is odd, and of

$$\frac{\partial^n}{\partial\zeta^n} c_0(\zeta) , \quad \frac{\partial^{n-2}}{\partial\zeta^{n-2}} c_2(\zeta) , \ldots , \frac{\partial^2}{\partial\zeta^2} c_{n-2}(\zeta) , \quad h_{mm}(\zeta,\zeta)$$

if n is even. Here h_{mm} is the function

$$h_{mm}(z,\zeta) = \frac{\partial^{2m}}{\partial z^m \partial\zeta^m} h(z,\zeta) , \tag{55}$$

and $m = \frac{n}{2}$. The proofs of these facts are simple but somewhat lengthy, and since we do not need the results we omit the proofs.

d) <u>Boundary behaviour and inequalities for $c_n(\zeta)$, $c_{\beta n}(\zeta)$.</u>

In this section we shall study the behaviour of $c_n(\zeta)$ and $c_{\beta n}(\zeta)$ as $\zeta \to \partial\Omega$. We first derive upper and lower bounds for $c_0(\zeta)$ showing that $c_0(\zeta) \to -\infty$ as $\zeta \to \partial\Omega \setminus \{\infty\}$, and asymptotically

$$c_0(\zeta) \sim \log d(\zeta) , \tag{56}$$

where $d(\zeta)$ denotes the distance to the boundary. From these estimates for $c_0(\zeta)$ (in Proposition 3.3) corresponding estimates for $c_{\beta 0}(\zeta)$ are obtained by

$$c_0(\zeta) \leq c_{\beta 0}(\zeta) \leq c_0(\zeta) + M , \quad M < \infty . \tag{57}$$

(57) follows from (43) and (33), with

$$M = \Sigma \mid a_{kj} \mid . \tag{58}$$

Put ($K_1,\ldots,K_m$ being the components of $\mathbb{P}\setminus\Omega$)

$$d_j(z) = \inf \{|z-\zeta| : \zeta \in K_j \} , \tag{59}$$

$$D_j(z) = \sup \{|z-\zeta| : \zeta \in K_j \} , \tag{60}$$

$$d(z) = \inf \{|z-\zeta| : \zeta \in \mathbb{P}\setminus\Omega \} = \min \{ d_1(z),\ldots,d_m(z) \} , \tag{61}$$

$$\operatorname{diam} K_j = \sup \{|z-\zeta| : z,\zeta \in K_j\} \quad . \tag{62}$$

Then we have

<u>Proposition 3.3</u>: The following inequalities for $c_0(z)$ hold in $\Omega \setminus \{\infty\}$:

$$\log d(z) \leq c_0(z) \leq \min_{j=1,\ldots,m} \log \frac{4d_j(z)}{1 - \frac{d_j(z)}{D_j(z)}} \quad . \tag{63}$$

In particular, if Ω is simply connected and $\infty \notin \Omega$ (so that $D_1(z) \equiv \infty$)

$$\log d(z) \leq c_0(z) \leq \log 4d(z) \; . \tag{64}$$

If moreover Ω is convex, then

$$\log d(z) \leq c_0(z) \leq \log 2d(z) \quad . \tag{65}$$

Specialized to a boundary neighbourhood (63) gives (for arbitrary Ω): For any $A > \log 4$ there is a $\delta > 0$ such that

$$\log d(z) \leq c_0(z) \leq \log d(z) + A \tag{66}$$

whenever $d(z) < \delta$.
Finally, if $\infty \notin \Omega$ (but Ω is otherwise arbitrary) then (66) holds <u>throughout</u> Ω for <u>some</u> $A < \infty$. ($A = \log 4$ of Ω is simply connected, by (64).)

<u>Remark:</u> If $\infty \in \Omega$ the upper bound in (66) does <u>not</u> hold in a neighbourhood of $z = \infty$ (for any A). For it is easy to see from the transformation properties of $c_0(z)$ under conformal mapping given in section III f) that necessarily

$$c_0(z) = \log|z|^2 + O(1) = 2\log|z| + O(1) \tag{67}$$

as $z \to \infty$ if $\infty \in \Omega$. Thus we only have (for some A)

$$c_0(z) \leq 2 \log d(z) + A \tag{68}$$

in a neighbourhood of $z = \infty$ in that case. ((68) also follows directly from (63), since

$$\log \frac{4d_j(z)}{1 - \frac{d_j(z)}{D_j(z)}} = \log 4d_j(z)D_j(z) - \log(D_j(z) - d_j(z)) \tag{69}$$

and $|D_j(z) - d_j(z)| \leq \operatorname{diam} K_j < \infty$ when $\infty \in \Omega$.)

The following little lemma will be needed in the proof of Proposition 3.3.

Lemma 3.2: Suppose $\Omega \subset \tilde{\Omega}$. Then $c_0(\zeta) \leq \tilde{c}_0(\zeta)$ for $\zeta \in \Omega$. *)

Proof: Put $u(z) = \tilde{g}(z,\zeta) - g(z,\zeta)$ for $z \in \Omega$. Then u is harmonic in Ω and ≥ 0 on $\partial\Omega$ (more precisely, $\underline{\lim}_{z\to z_0} u(z) \geq 0$ for each $z_0 \in \partial\Omega$). Thus $u(z) \geq 0$ throughout Ω. But, by (1), $u(z) = \tilde{h}(z,\zeta) - h(z,\zeta)$. Thus $h(z,\zeta) \leq \tilde{h}(z,\zeta)$ for $z \in \Omega$, and in particular $c_0(\zeta) \leq \tilde{c}_0(\zeta)$, as was to be proved.

Proof of Proposition 3.3: To prove the lower bounds, $\log d(z) \leq c_0(z)$, we just apply Lemma 3.2 to the situation $\mathbb{D}(z;d(z)) \subset \Omega$. If $\tilde{g}(\zeta,\zeta_0)$ denotes the Green's function for $\mathbb{D}(z;d(z))$, then

$$\tilde{g}(\zeta,z) = -\log|\zeta - z| + \log d(z) .$$

Thus $\tilde{c}_0(z) = \log d(z)$, and $\log d(z) \leq c_0(z)$ follows.

The proof of the upper bounds in Proposition 3.3 rests on another lemma, in the spirit of Lemma 3.2. Lemma 3.2 says roughly speaking that $c_0(z)$ increases with the domain. Therefore, to get upper bounds for $c_0(z)$ expressed in $d_j(z)$, $D_j(z)$, one naturally looks for some largest domain $\tilde{\Omega}$ which (with respect to the point z) has the same values of $d_j(z), D_j(z)$ as Ω has, and compares $c_0(z)$ with $\tilde{c}_0(z)$ for $\tilde{\Omega}$. Now there is a rather natural candidate (or family of candidates) for such a domain, namely the domain $\tilde{\Omega}$ obtained from Ω by replacing the complement K of it by its circular projection (with center z) onto some radius emanating from z.

*) Here the functions $\tilde{c}_0(\zeta)$, $\tilde{g}_0(z,\zeta), \ldots$ of course refer to the domain $\tilde{\Omega}$.

In other words

$$\tilde{\Omega} = \mathbb{P} \setminus \bigcup_{j=1}^{m} \left[z + e^{i\vartheta} d_j(z), \quad z + e^{i\vartheta} D_j(z) \right] , \tag{70}$$

where $\vartheta \in \mathbb{R}$ is arbitrary and $[a,b]$ denotes the linear segment with endpoints a and b .*)

Unfortunately, Ω is not a subdomain of $\tilde{\Omega}$, so Lemma 3.2 cannot be applied to show $c_0(z) \leq \tilde{c}_0(z)$. Nevertheless, we have

<u>Lemma 3.4</u>: If $z \in \Omega$, $\vartheta \in \mathbb{R}$, and $\tilde{\Omega} = \tilde{\Omega}_{(z,\vartheta)}$ is as described above, then $c_0(z) \leq \tilde{c}_0(z)$. (71)

The proof of this lemma will be given in a later section (III i)) because it fits better into the context there. So we assume Lemma 3.4 for the moment and continue the proof of Proposition 3.3.

According to Lemma 3.4 an upper bound for $c_0(z)$ is obtained by just computing $\tilde{c}_0(z)$ for the domain (70) with, say, $\vartheta = 0$. This computation however becomes complicated if $m > 1$, so we have contented ourselves with the cruder upper bounds obtained by letting an application of Lemma 3.2 follow, yielding the estimate

$$c_0(z) \leq \min_{j=1,..,m} \tilde{c}_0^{(j)}(z) , \tag{72}$$

where $\tilde{c}_0^{(j)}(z)$ refers to the domain

$$\tilde{\Omega}_j = \mathbb{P} \setminus [z+d_j(z), \ z+D_j(z)] \tag{73}$$

(thus $\tilde{\Omega} \subset \tilde{\Omega}_j$) .

Now the upper bound in (63) is just (72). To see this amounts to checking that for

$$\tilde{\Omega} = \mathbb{P} \setminus [a,b] \ , \qquad 0 < a < b \leq \infty , \tag{74}$$

we have

$$\tilde{c}_0(0) = \log \frac{4a}{1 - \frac{a}{b}} \quad . \tag{75}$$

---

*) See fig. 3.13 (p.75), where $m = 1$ and $\vartheta = 0$.

This is a matter of computation. Using that $\zeta \mapsto w = \sqrt{\frac{\zeta-a}{\zeta-b}}$ maps $\tilde{\Omega}$ onto the half-plane $\operatorname{Re} w > 0$ (for one branch of $\sqrt{\ }$) with $\zeta = 0$ mapped on $w = \sqrt{\frac{a}{b}}$, one finds that

$$\tilde{g}(\zeta,0) = -\log\left|\sqrt{\frac{\zeta-a}{\zeta-b}} - \sqrt{\frac{a}{b}}\right| + \log\left|\sqrt{\frac{\zeta-a}{\zeta-b}} + \sqrt{\frac{a}{b}}\right| =$$

$$= -\log\left|\frac{\zeta-a}{\zeta-b} - \frac{a}{b}\right| + 2\log\left|\sqrt{\frac{\zeta-a}{\zeta-b}} + \sqrt{\frac{a}{b}}\right| =$$

$$= -\log\left|\frac{\zeta(b-a)}{(\zeta-b)b}\right| + 2\log\left|\sqrt{\frac{\zeta-a}{\zeta-b}} + \sqrt{\frac{a}{b}}\right| =$$

$$= -\log|\zeta| + \log\left|\frac{(\zeta-b)b}{b-a}\right| + 2\log\left|\sqrt{\frac{\zeta-a}{\zeta-b}} + \sqrt{\frac{a}{b}}\right| .$$

Thus

$$\tilde{c}_0(0) = \log\left|\frac{-b\cdot b}{b-a}\right| + 2\log\left|\sqrt{\frac{-a}{-b}} + \sqrt{\frac{a}{b}}\right| =$$

$$= 2\log b - \log(b-a) + 2\log 2\sqrt{\frac{a}{b}} = \log\frac{4ab}{b-a} = \log\frac{4a}{1-a/b} ,$$

showing (75).

Thus (63) is proved. (64) is an immediate consequence of (63) ($m = 1$, $D_1(z) = \infty$, $d_1(z) = d(z)$).

If Ω is convex then, given $z \in \Omega$, Ω is contained in a halfplane $\tilde{\Omega}$ whose boundary L is at distance $d(z)$ from z (fig. 3.1). If z^* denotes the point obtained from z by reflection in L, the Green's function for $\tilde{\Omega}$ is

$$\tilde{g}(\zeta,z) = -\log|\zeta-z| + \log|\zeta-z^*| ,$$

showing that

$$\tilde{c}_0(z) = \log|z-z^*| = \log 2d(z).$$

By Lemma 3.2 again this proves $c_0(z) \leq \log 2d(z)$, the upper bound in (65).

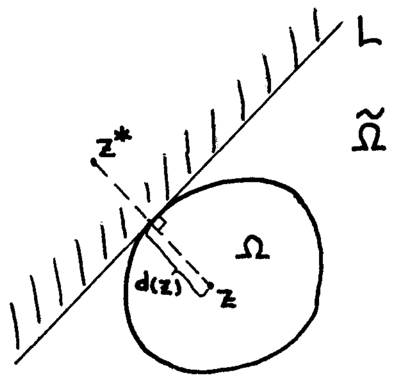

Fig. 3.1

Next, (66) is a simple consequence of (63). For it is clear that given $\epsilon > 0$ we can choose $\delta_j > 0$ so small (for example $\delta_j = \frac{\epsilon}{2} \cdot \operatorname{diam} K_j$) so that

$$\frac{d_j(z)}{D_j(z)} < \epsilon$$

whenever $d_j(z) < \delta_j$. Now if $\epsilon > 0$ is chosen so that $\log \frac{4}{1-\epsilon} < A$, then $\delta = \min\{\delta_1, \ldots, \delta_m\}$ works in (66).

As to the final assertion in Proposition 3.3 we first observe that $\infty \notin \Omega$ means that $\infty \in K_j$ for (exactly) one j, say $j = 1$. This means $D_1(z) \equiv \infty$, so that (63) gives

$$c_0(z) \leq \log 4d_1(z) \quad . \tag{76}$$

Since $\infty \notin K_j$ for $j \neq 1$, $K_2, \ldots, K_m$ are bounded sets. Therefore, given $\delta > 0$ the quotient $d_1(z)/d(z)$ is bounded for $d(z) \geq \delta$, say

$$\frac{d_1(z)}{d(z)} \leq B \qquad (d(z) \geq \delta) .$$

Thus combining with (76) gives

$$c_0(z) \leq \log d(z) + \log 4B \tag{77}$$

for $d(z) \geq \delta$. Since we already have a bound of the kind (77) when $d(z) < \delta$ (if δ is sufficiently small) this proves the final assertion of the proposition.

We turn next to the functions $c_n(z)$, $c_{\beta n}(z)$ for $n \geq 1$, for whose moduli we need upper bounds.

<u>Proposition 3.5</u> In all Ω,

$$|c_n(\zeta)| \leq \frac{1}{n} \frac{1}{d(\zeta)^n} \quad , \quad n \geq 1, \tag{78}$$

and, for some constant A (depending on Ω),

$$|c_{\beta n}(\zeta)| \leq \frac{1}{n} \frac{A}{d(\zeta)^n} \quad , \qquad n \geq 1. \tag{79}$$

Proof: From (35), (37) we have

$$G'(z,\zeta) = -\frac{1}{z-\zeta} + H'(z,\zeta) = -\frac{1}{z-\zeta} + \sum_{n=1}^{\infty} nc_n(\zeta)(z-\zeta)^{n-1} \tag{80}$$

(the slashes denote derivatives with respect to the first variable). Therefore

$$nc_n(\zeta) = \frac{1}{2\pi i}\int_\gamma \frac{G'(z,\zeta)dz}{(z-\zeta)^n} \quad , \tag{81}$$

the path of integration being some small closed loop about the point ζ (taken in the counter-clockwise direction). Actually, since the integrand in (81) is seen to be a holomorphic differential with respect to z in all $\Omega \setminus \{\zeta\}$ (including the point ∞ if $\infty \in \Omega$) the path of integration may be taken to be any cycle of the kind $\gamma = \partial U$, where U is any smoothly bounded open set such that $\zeta \in U \subset \overline{U} \subset \Omega$. In particular, we can (and shall) choose

$$\gamma = \gamma_r = \{ z \in \Omega : g(z,\zeta) = r \}$$

for any $r > 0$ (corresponding to $U = \{ z \in \Omega : g(z,\zeta) > r \}$).

Along γ_r (in the z variable) we have

$$dg(z,\zeta) = 0 \quad ,$$

and hence

$$G'(z,\zeta)dz = i\,{}^*dg(z,\zeta) \; .$$

Thus (81) becomes

$$nc_n(\zeta) = \frac{1}{2\pi}\int_{\gamma_r} \frac{{}^*dg(z,\zeta)}{(z-\zeta)^n} \qquad (\text{any } r > 0). \tag{82}$$

(The corresponding formula for $c_0(\zeta)$ is

$$c_0(\zeta) = r - \frac{1}{2\pi}\int_{\gamma_r} \log|z-\zeta|\,{}^*dg(z,\zeta) \; . \;)$$

The orientation of γ_r is such that the domain

$\{ z \in \Omega : g(z,\zeta) > r \}$ lies to the left of it. Therefore

$${}^*dg(z,\zeta) = \frac{\partial g(z,\zeta)}{\partial n_z}\, ds_z \leq 0$$

along γ_r, which shows that

$$\int_{\gamma r} |{}^*dg(\cdot,\zeta)| = - \int_{\gamma r} {}^*dg(\cdot,\zeta) = 2\pi .$$

This gives

$$|nc_n(\zeta)| \leq \sup_{z\in\gamma r} \frac{1}{|z-\zeta|^n} \cdot \frac{1}{2\pi} \int_{\gamma r} |{}^*dg(\cdot,\zeta)| = \sup_{z\in\gamma r} \frac{1}{|z-\zeta|^n} ,$$

and letting $r \to 0$

$$|nc_n(\zeta)| \leq \sup_{z\in\partial\Omega} \frac{1}{|z-\zeta|^n} = \frac{1}{d(\zeta)^n} ,$$

proving (78).

The above proof does not work for $c_{\beta n}(\zeta)$. However, formula (81) holds with β:s inserted, and since by (15)

$$G'_\beta(z,\zeta) = G'(z,\zeta) + \Sigma\, a_{kj}\, W'_k(z)\omega_j(\zeta) , \tag{83}$$

where $W_k(z)$ denotes the (multiple-valued) analytic completion of the harmonic measure $\omega_k(z)$, it is in order to prove (79), enough to prove that

$$\left| \frac{1}{2\pi i} \int_\gamma \frac{W'_k(z)dz}{(z-\zeta)^n} \right| \leq \frac{B}{d(\zeta)^n} \qquad (k = 1,\ldots,m , \quad n \geq 1) \tag{84}$$

for some comstant B. Since the integrand in the left member of (84) is a holomorphic differential in all $\Omega \setminus \{\zeta\}$ we are free to choose the path of integration γ to be any cycle of the kind $\gamma = \partial U$ with $\zeta \in U \subset \overline{U} \subset \Omega$. We take it to be

$\gamma = \gamma_\epsilon = \partial U$ where

$U = \{ z \in \Omega : \epsilon < \omega_k(z) < 1 - \epsilon \}$

with $\epsilon > 0$ so small so that $\zeta \in U$. Then $d\omega_k(z) = 0$ along γ, and (84) reduces to

$$\left| \frac{1}{2\pi} \int_{\gamma\epsilon} \frac{{}^*d\,\omega_k(z)}{(z-\zeta)^n} \right| \leq \frac{B}{d(\zeta)^n} \tag{85}$$

(to be proved).

Now it is easy to see that

$$\int_{\gamma\epsilon} |{}^{*}d\,\omega_k| \leq C \tag{86}$$

for some constant C independent of ϵ. Indeed, putting

$$\gamma_\epsilon^{+} = \{ z \in \Omega : \omega_k(z) = 1 - \epsilon \}$$

$$\gamma_\epsilon^{-} = \{ z \in \Omega : \omega_k(z) = \epsilon \}$$

(so that $\gamma_\epsilon = \gamma_\epsilon^{+} \cup \gamma_\epsilon^{-}$) we have

$${}^{*}d\,\omega_k \geq 0 \quad \text{along} \quad \gamma_\epsilon^{+}$$

$${}^{*}d\,\omega_k \leq 0 \quad \text{along} \quad \gamma_\epsilon^{-}$$

(γ_ϵ^{+} and γ_ϵ^{-} are assumed to be oriented so that U lies to the left),

$$\int_{\gamma\epsilon} |{}^{*}d\,\omega_k| = \int_{\gamma\epsilon^{+}} {}^{*}d\,\omega_k - \int_{\gamma\epsilon^{-}} {}^{*}d\,\omega_k = 2\int_{\gamma\epsilon^{+}} {}^{*}d\,\omega_k \quad ,$$

which is independent of ϵ (and in fact equal to $2\int_{\Gamma k} {}^{*}d\,\omega_k = 2p_{kk}$, where (p_{kj}) is the matrix defined by (18). This proves (86) (with equality and $C = 2p_{kk}$), and we get for the left member of (85)

$$\left| \frac{1}{2\pi} \int_{\gamma\epsilon} \frac{{}^{*}d\,\omega_k(z)}{(z-\zeta)^n} \right| \leq \sup_{z\in\gamma_\epsilon} \frac{1}{|z-\zeta|^n} \cdot \frac{1}{2\pi} \int_{\gamma\epsilon} |{}^{*}d\,\omega_k| \leq C\cdot \sup_{z\in\gamma\epsilon} \frac{1}{|z-\zeta|^n} \,.$$

Since $\sup\limits_{z\in\gamma\epsilon} \dfrac{1}{|z-\zeta|^n} \to \dfrac{1}{d(\zeta)^n}$ as $\epsilon \to 0$ this proves (85), and the proof of Proposition 3.5 is complete.

As a corollary of the lower bounds in Proposition 3.3 and of Proposition 3.5 we get the correct definitions of $c_n(z)$, $c_{\beta n}(z)$ for $z = \infty$ when $\infty \in \Omega$:

Corollary 3.6: If $\infty \in \Omega$, $c_n(z)$ and $c_{\beta n}(z)$ become continuously extended to all Ω by putting

$$c_0(\infty) = c_{\beta 0}(\infty) = +\infty \quad , \tag{87}$$

$$c_n(\infty) = c_{\beta n}(\infty) = 0 \qquad \text{for} \quad n \geq 1 \; . \tag{88}$$

Proof: If $\infty \in \Omega$, $d(z) \to \infty$ as $z \to \infty$.

It is interesting also to combine Proposition 3.5 with the upper bounds in Proposition 3.3. Thus, assuming that $\infty \notin \Omega$, the last assertion of Proposition 3.3 shows that for some $B < \infty$

$$\frac{1}{d(z)} \leq B \; e^{-c_0(z)} \tag{89}$$

in all Ω. Similarly

$$\frac{1}{d(z)} \leq B'_\beta \; e^{-c_{\beta 0}(z)} \tag{90}$$

for some $B'_\beta < \infty$. Together with (78) and (79) this gives

$$|c_n(z)| \leq \frac{B^n}{n} \cdot e^{-nc_0(z)} \qquad \text{and} \tag{91}$$

$$|c_{\beta n}(z)| \leq \frac{B_\beta^n}{n} \cdot e^{-nc_{\beta 0}(z)} \tag{92}$$

(in the case $\infty \notin \Omega$) . In particular

$$|c_1(z)| \leq B \cdot e^{-c_0(z)} \qquad \text{and} \tag{93}$$

$$|c_{\beta 1}(z)| \leq B_\beta \cdot e^{-c_{\beta 0}(z)} \; . \tag{94}$$

These last inequalities will come into use later.

If Ω is simply connected (64) shows that the constant B in (89) and (91) can be chosen equal to 4. In (89) this is the best possible constant, but not in (91). In fact, when $n = 1$ the best possible choice for B in (91) (i.e. in (93)) is $B = 2$, as we shall see later (p.80).

e) Discussion of the orbits

We shall now look at some consequences of the results hitherto obtained (sections III a) - d)) in terms of the motion of free vortices. These consequences could of course equally well have been formulated for the electrodynamical problem, but since the motion of vortices actually is our principal interest we have preferred to formulate them for the hydrodynamical problem.

As we have already discussed on p. 18 ff. the differential equation

$$\frac{dz}{dt} = - i \alpha \bar{c}_{\beta 1}(z) \tag{95}$$

for a freely moving vortex, together with

$$c_{\beta 1}(z) = \frac{\partial c_{\beta 0}}{\partial z} , \tag{96}$$

shows that a free vortex moves along a level line of $c_{\beta 0}(z)$,

$$\Gamma_\lambda = \{ z \in \Omega : c_{\beta 0}(z) = \text{constant} = \lambda \} . \tag{97}$$

From Proposition 3.3 we now get estimates for the deviation of these level lines from the boundary of Ω. For by (66) equalities of the kind

$$d(z) \leq e^{c_{\beta 0}(z)} \leq B \cdot d(z) , \tag{98}$$

or

$$\frac{1}{B} e^{c_{\beta 0}(z)} \leq d(z) \leq e^{c_{\beta 0}(z)} , \tag{99}$$

hold in Ω, except in a neighbourhood of $z = \infty$ if $\infty \in \Omega$. In the perhaps most interesting case that Ω is simply connected and $\infty \notin \Omega$ B can be chosen equal to 4. (99) shows that

$$\frac{1}{B} e^{\lambda} \leq d(z) \leq e^{\lambda} \tag{100}$$

whenever $z \in \Gamma_\lambda$. Thus for example each Γ_λ is separated from $\partial\Omega$ by a positive distance ($= \frac{1}{B} e^{\lambda}$). (If $\infty \in \Omega$ (100) holds for $\lambda \leq \nu$, for any $\nu < \infty$, but with B depending on ν.)

Let us next discuss a little more closely the relation between the orbits of free vortices and the level sets Γ_λ of $c_{\beta 0}(\zeta)$. We shall then be careful in the use of the word "orbit" and only use it in the precise sense of being a complete trajectory of a vortex, that is the set of points a vortex occupies during its "life-time". In other words an orbit is the image set of a maximal solution of the differential equation

$$\frac{dz}{dt} = - i \alpha \bar{c}_{\beta 1}(z) . \tag{101}$$

Since $c_{\beta 1}(z)$ is a very well-behaved function (it is analytic as a function of the two variables x and y) it follows from general facts about differential equations that through each point of $\Omega \setminus \{\infty\}$ there passes exactly one orbit. It should be observed that with each orbit there is associated a particular direction, namely the direction in which a vortex of, say, positive strength moves along it.

It is from a mathematical standpoint not completely obvious that the domain of definition of a maximal solution of (101) is always the entire real line, that is that a vortex always exists for all time $t \in \mathbb{R}$. This is however actually the case. For it follows easily from the good local properties of (101) in $\Omega \setminus \{\infty\}$ that any orbit which is part of a compact subset of $\Omega \setminus \{\infty\}$ must necessarily be defined for all $t \in \mathbb{R}$ (otherwise it could be extended a little). Now we know that each orbit is at a positive distance (measured in Euclidean plane metric) from $\partial\Omega$. Therefore the only orbits which are not within a compact subset of $\Omega \setminus \{\infty\}$ are those which in some direction approach ∞, This can occur only if $\infty \in \partial\Omega$, since $c_{\beta 0}(z) \to + \infty$ as $z \to \infty$ if $\infty \in \Omega$ (Corollary 3.6). It therefore only remains to check that a vortex on such an orbit cannot reach (or emerge from) infinity in a finite time. But this is an immediate consequence of the inequality (94), which, in the case $\infty \notin \Omega$, shows that the velocity of a vortex is always bounded on each orbit:

$$\left|\frac{dz}{dt}\right| \leq \text{const. } e^{-\lambda} \qquad \text{for } z \in \Gamma_\lambda . \tag{102}$$

(It is easy to see that the velocity is bounded on each orbit also in the case $\infty \in \Omega$.)

Thus each maximal solution of (101) is indeed defined on all $\mathbb{R}$.

We know that each orbit is a subset of a set Γ_λ. If this Γ_λ is disconnected (topologically) then the orbit can of course only be part of one of the components of Γ_λ, and the general case is in fact that an orbit is precisely a component of a Γ_λ. However, there are exceptions.

This is because there may be points where $c_{\beta 0}(z)$ is stationary. Let us call such points singular points*). Thus by (96) a singular point is just a zero for $c_{\beta 1}(z)$, that is a point where a free vortex is at rest. Or: a singular point is an orbit consisting of a single point.

If γ is a component of a set Γ_λ and γ contains no singular points, then γ constitutes one single orbit. This is immediate from the fact that the velocity of a vortex on γ is never zero. "Most" orbits are of this kind. Since such an orbit cannot have any (finite) end point there are only two possibilities for it: either it is unbounded, tending to infinity in both directions, or it is a closed loop. In the latter case the motion of the vortex is periodic, it returning to each position regurarly within a certain time interval T. This time of revolution will be considered later.

Consider next a component γ of some Γ_λ, such that γ contains a singular point. One possibility then is that γ consists of just that singular point. Otherwise γ contains orbits which have the singular point as an endpoint. These orbits are then open arcs which in the other direction lead either to infinity or to some (other, or back to the same) singular point of γ. A typical situation is pictured in fig. 3.2 where we have 2 singular points, one of which is isolated while the other

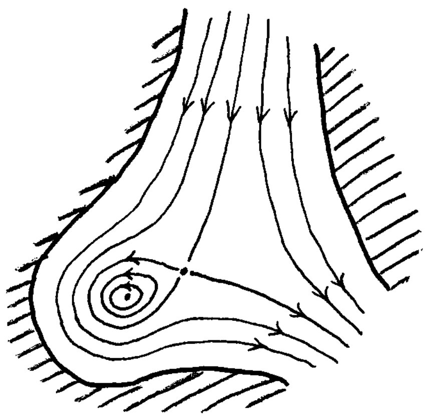

Fig. 3.2

*) In the context of dynamical autonomous systems the word "critical point" is more common. In other parts of this report the word "equilibrium point" is also used.

one is the end point of three distinct orbits. One of these three orbits is bounded, and closed except for the singular point. Since it is a complete orbit a vortex on it will need infinite time to make its short cycle along it. This is rather remarkable and of course quite general (a vortex on an orbit which leads to a singular point will never reach that singular point).

Let us take a closer look at a particular singular point. We can assume *) that it is $z = 0$, and we moreover assume that we have rotated the coordinate system so that the principal axes of the quadratic part of the Taylor expansion of $c_{\beta 0}$ at $z = 0$ are the x- and y-axes. This means that (say)

$$c_{\beta 0}(z) = c_{\beta 0}(0) + ax^2 + by^2 + O(|z|^3) , \qquad (103)$$

at $z = 0$ ($z = x + iy$, $a, b \in \mathbb{R}$). We shall find later (p. 70) that $\Delta c_{\beta 0} < 0$ everywhere ($c_{\beta 0}(z)$ is superharmonic). At $z = 0$ this gives

$$a + b < 0 . \qquad (104)$$

In particular the quadratic form $Q(z) = ax^2 + by^2$ does not vanish identically.

Now we have three cases.

1) a and $b < 0$.

The form $Q(z)$ is negative definite and $c_{\beta 0}(z)$ has a strict local maximum at $z = 0$. The orbits near $z = 0$ form closed loops, approximately ellipses, around $z = 0$ (fig. 3.3). One would think that having a vortex at $z = 0$ its state of rest would be unstable since, $c_{\beta 0}(z)$ representing the energy of the system by (2.45) and (2.53), it is in a state of maximum energy. However a disturbance (even a strong such) of the vortex would only have the effect that it falls into one of the orbits close to $z = 0$ and begins to circulate in it. Thus the vortex is not unstable at $z = 0$.

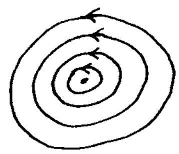

Fig. 3.3

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*) The following discussion do not apply to the point ∞ , which, if it belongs to Ω, should always be considered singular (by Corollary 3.6).

It is natural to ask for the time of revolution in an orbit around $z = 0$. This time depends on the orbit, but it is easy to see that it approaches a finite limit T as the orbits shrink to $z = 0$. In fact a computation shows that

$$T = \frac{2\pi}{\alpha\sqrt{ab}} , \tag{105}$$

or, expressed in a form which is invariant under rotations of the coordinate system,

$$T = \frac{4\pi}{\alpha\cdot\sqrt{\frac{\partial^2 c_0}{\partial x^2}\cdot\frac{\partial^2 c_0}{\partial y^2} - \left(\frac{\partial^2 c_0}{\partial x\partial y}\right)^2}} \tag{106}$$

(the derivatives evaluated at $z = 0$).

2) a and b are of different signs (both $\neq 0$).

$z = 0$ is then a saddlepoint for $c_{\beta 0}(z)$ and the picture is as in fig. 3.4. Thus four open orbit ends meet at $z = 0$ and the neighbouring orbits are approximately branches of hyperbolas (near $z = 0$) . In this case the word unstable for the state of a vortex at $z=0$ is more adequate since a small disturbance of it would cause it to move away along one of the hyperbola-like branches and it would be back in a neighbourhood of $z = 0$ if ever only after a long time (this time tends to infinity as the orbits approach $z = 0$).

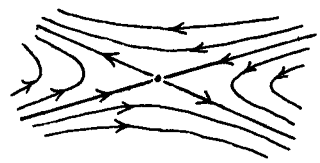


Fig. 3.4

3) $a = 0$ and $b < 0$ or vice versa.

$Q(z)$ is now negative semidefinite and the behaviour of the orbits near $z = 0$ depends on the higher order terms in the Taylor expansion (103) . Let us just mention two types of behaviour.

The first is the one that occurs when the domain Ω is an infinite strip (bounded by two parallel straight lines) or when Ω is a (circular) annulus. Then the singular points constitute a whole line (the symmetry line for a strip for example).(Fig.3.5)

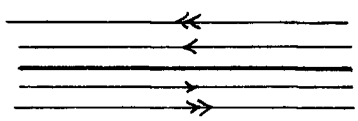


Fig. 3.5

A vortex on that line thus does not move spontaneously. On the other hand no force is needed to move the vortex along the line.

The other type is the "cusp" singularity, represented for example by

$$c_{\beta 0}(z) = x^3 - y^2 \quad .$$

Then the picture is as in fig. 3.6, with two open orbit ends meeting under zero angle at $z = 0$.

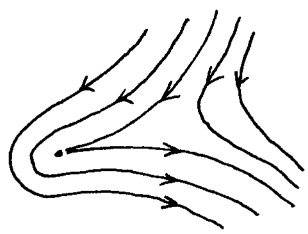


Fig. 3.6

Let us next turn to the question of existence of singular points. If $\infty \in \Omega$, the point ∞ itself is always a singular point, by Corollary 3.6, and this may be the only singular point. If $\infty \notin \Omega$ but Ω is unbounded (in other words, $\infty \in \partial\Omega$) then Ω need not contain any singular point at all. For the upper half plane, for example, we have

$$\begin{cases} c_{\beta 0}(z) = \log 2y \ , \\ c_{\beta 1}(z) = \dfrac{1}{2iy} \qquad\qquad (z = x + iy, \quad y > 0), \end{cases}$$

so that $c_{\beta 1}(z)$ has no zero there.

On the other hand if Ω is bounded it must contain at least one singular point. This is obvious from the fact that then $c_{\beta 0}(z) \to -\infty$ as $z \to \partial\Omega$ without exceptions, so that $c_{\beta 0}(z)$ must have a point of maximum in Ω, and such a point is necessarily singular.

Somewhat more generally, each compact component of a set

$$K_\lambda = \{ z \in \Omega : c_{\beta 0}(z) \geq \lambda \} \tag{107}$$

must contain a point of maximum of $c_{\beta 0}(z)$, and so a singular point. Assuming for simplicity that $\infty \notin \Omega$ we know by Proposition 3.3 that an inequality $\log d(z) \leq c_{\beta 0}(z) \leq \log d(z) + A$ holds throughout Ω. Putting

$$D_\lambda = \{ z \in \Omega : d(z) \geq e^\lambda \} \tag{108}$$

we therefore have

$$D_\lambda \subset K_\lambda \subset D_{\lambda-A} . \qquad (109)$$

This gives information about the singular points directly from the geometry of Ω in certain cases. Namely, whenever a compact (i.e. bounded) component of $D_{\lambda-A}$ contains a non-empty component of D_λ there must be a non-empty compact component of K_λ in between, and hence there must be a singular point in $D_{\lambda-A}$. By this it is for example easy to see that if Ω is a region built up of two discs connected by a sufficiently narrow channel (fig. 3.7), then each disc must contain a singular point. Moreover, there must be a saddle-point for $c_{\beta 0}(z)$ in (or near) the channel. Thus such a region must have at least three singular points.

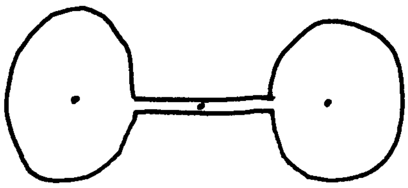


Fig. 3.7

We shall complete our discussion of the orbits by directing our attention to the time variable: can anything general be said about the time needed for a vortex to travel between two different points on an orbit? It turns out it can.

To arrive at that we consider a portion of Ω which is free from singular points. We shall introduce a new coordinate system there, with

$$\begin{cases} t = \text{time} \quad \text{and} \\ \lambda = c_{\beta 0}(z) \end{cases} \qquad (110)$$

as new variables. For this to make sense we have to draw a line γ_0 transversal to the orbits (that is, such that γ_0 is never tangent to an orbit and never intersects an orbit more that once), to be the line of "time zero", $t = 0$. The coordinate transformation is then defined by the map

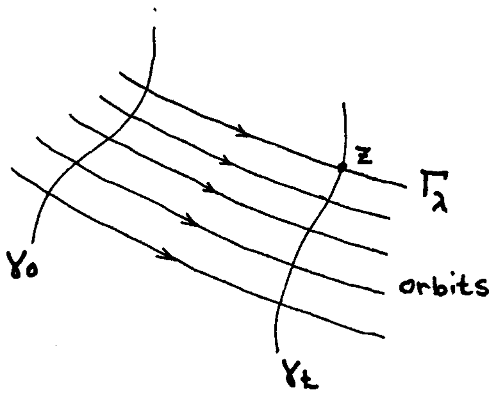
$$(t,\lambda) \mapsto z = x + iy \quad , \qquad (111)$$

where z is the position at time t of a vortex on Γ_λ which at time 0 was on γ_0. If γ_t denotes the line of positions taken at time t by vortices which at time 0 was on γ_0 the coordinate transformation (110) thus can be summarized by

$$\Gamma_\lambda \cap \gamma_t = \{z\} . \tag{112}$$

(See fig. 3.8).

It is easy to see that this actually defines a non-singular change of coordinates if restricted to a sufficiently small open set free from singular points.



$(t,\lambda) \mapsto z$

Fig. 3.8

Although the map (111) depends on the more or less arbitrary choice of the line γ_0 it turns out that its Jacobi-determinant is always the same. We shall compute it. Thus regarding $z = x + iy$ as a function of t and λ by (111) the differential equation (95) for the motion of a vortex shows that (since it moves along a line λ = constant)

$$\frac{\partial z}{\partial t} = - i \alpha \, \bar{c}_{\beta 1}(z) = - i \alpha \frac{\partial}{\partial \bar{z}} c_{\beta 0}(z) \tag{113}$$

that is, since $\lambda = c_{\beta 0}(z)$

$$\frac{\partial z}{\partial t} = - i \alpha \frac{\partial \lambda}{\partial \bar{z}} \quad , \tag{114}$$

or

$$\left| \begin{aligned} \frac{\partial x}{\partial t} &= \frac{\alpha}{2} \frac{\partial \lambda}{\partial y} \\ \frac{\partial y}{\partial t} &= - \frac{\alpha}{2} \frac{\partial \lambda}{\partial x} \end{aligned} \right. \quad , \tag{115}$$

Therefore

$$\begin{vmatrix} \frac{\partial x}{\partial t} & \frac{\partial x}{\partial \lambda} \\ \frac{\partial y}{\partial t} & \frac{\partial y}{\partial \lambda} \end{vmatrix} = \frac{\partial x}{\partial t}\frac{\partial y}{\partial \lambda} - \frac{\partial x}{\partial \lambda}\frac{\partial y}{\partial t} = \frac{\alpha}{2}\left(\frac{\partial \lambda}{\partial y}\frac{\partial y}{\partial \lambda} + \frac{\partial x}{\partial \lambda}\frac{\partial \lambda}{\partial x}\right) = \frac{\alpha}{2} \quad . \tag{116}$$

(116) is more suggestively written

$$dxdy = \frac{\alpha}{2}\, dtd\lambda \quad . \tag{117}$$

Thus we have the remarkable result that the coordinate change (111) is (essentially) area-preserving. This is what we aimed at. (117) shows for example that the time τ needed for a vortex to travel between two points z_1 and z_2 on Γ_λ, say $\{z_j\} = \Gamma_\lambda \cap \gamma_{t_j}$ ($j = 1,2$), where $\tau = t_2 - t_1$, is obtained by

$$\tau\cdot(\lambda - \lambda') = \pm \frac{2}{\alpha}\, |D_{(t_1, t_2, \lambda, \lambda')}| \ , \tag{118}$$

where λ' is any number close to λ, $D_{(t_1,t_2,\lambda,\lambda')}$ is the region bounded by γ_{t_1}, γ_{t_2}, Γ_λ and $\Gamma_{\lambda'}$, and $|\ldots|$ denotes area (see fig. 3.9).

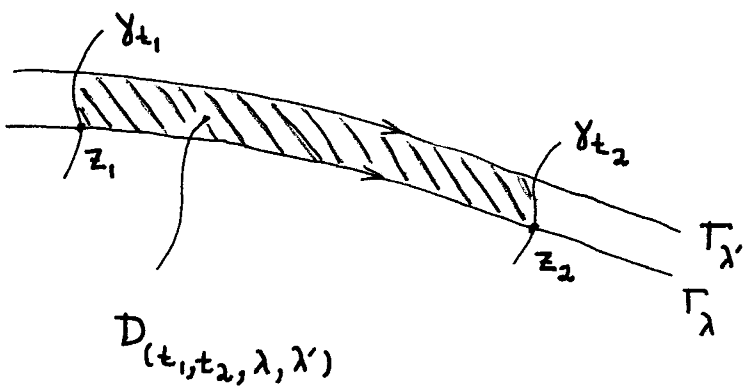


Fig. 3.9

To get another example, put

$$\Omega_\lambda = \{ z \in \Omega : c_{\beta 0}(z) > \lambda \} \quad . \tag{119}$$

Suppose that $|\Omega_\lambda| < \infty$ and that Γ_λ contains no singular points. Then $\Gamma_\lambda = \partial\Omega_\lambda$ (in general only $\partial\Omega_\lambda \subset \Gamma_\lambda$ is true), and Γ_λ consists of one or more closed loops. If T_λ denotes the sum of the times of revolution about these loops, then it is a more or less immediate consquence of (117) that

$$T_\lambda = -\frac{2}{\alpha}\frac{d}{d\lambda} | \Omega_\lambda | \quad . \tag{120}$$

(there must be a minus sign since $| \Omega_\lambda |$ increases as λ decreases.)

Or somewhat more generally, if D is a fixed region in Ω and $T_{D,\lambda}$ denotes the time needed to travel along $\Gamma_\lambda \cap D$ (fig. 3.10), then

$$T_{D,\lambda} = -\frac{2}{\alpha}\frac{d}{d\lambda} | \Omega_\lambda \cap D | \quad . \tag{121}$$

(It is assumed that $| \Omega_\lambda \cap D | < \infty$ and that $\Gamma_\lambda \cap D$ contains no singular points.)

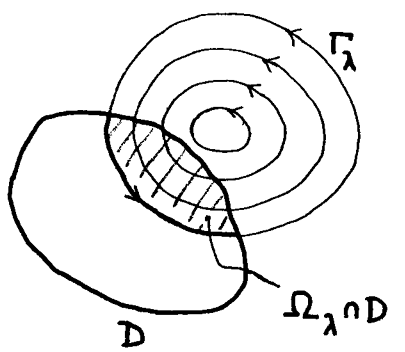


Fig. 3.10

(117) has another very interesting corollary. Namely that <u>the flow of vortex motions</u> (or: the flow associated with the vector field $\mathbf{A} = -i\alpha\, \overline{c}_{\beta 1}(z)$) <u>is area preserving</u>. By this flow is meant the family $\{ \mu_t : t \in \mathbb{R} \}$ of mappings

$$\mu_t : \Omega \to \Omega \tag{122}$$

defined by:

$\mu_t(z)$ = the position at time t of a vortex which at time 0 was at z.

If D is a subregion of Ω $\mu_t(D)$ is defined in an obvious way, and for

$\{ \mu_t \}$ to be area preserving means that

$$|\mu_t(D)| = |D| \tag{123}$$

for all $D \subset \Omega$ and all $t \in \mathbb{R}$. That $\{ \mu_t \}$ actually is area preserving is an immediate consequence of (117), since in the coordinate system (t,λ) μ_τ is just a translation,

$$\mu_\tau : (t,\lambda) \mapsto (t+\tau,\lambda) ,$$

evidently measure preserving (fig. 3.11). (The singular points do not cause any trouble here, as is easily seen.)

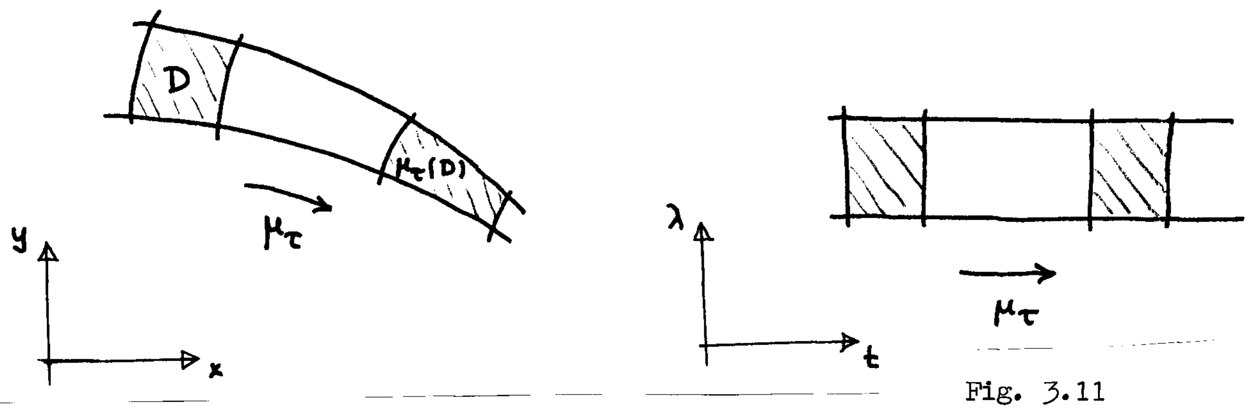


Fig. 3.11

The property of $\{ \mu_t \}$ of being area-preserving resembles very much a theorem in Hamiltonian mechanics, namely "Liouville's theorem" about invariance of the phase density under the group of motions (see for example [L - L1], § 46). Indeed, our vortex problem can be put into the framework of Hamiltonian mechanics in such a way that the invariance of dxdy under $\{ \mu_t \}$ appears just as an instance of Liouville's theorem. To see this, write the differential equation (95) in the form (compare (115))

$$\begin{cases} \dfrac{dx}{dt} = \dfrac{\alpha}{2} \dfrac{\partial}{\partial y} c_{\beta 0}(x + iy) \\ \\ \dfrac{dy}{dt} = - \dfrac{\alpha}{2} \dfrac{\partial}{\partial x} c_{\beta 0}(x + iy) \end{cases} . \tag{124}$$

These differential equations are then identified with Hamilton's canonical equations ([L-- L1], § 40)

$$\begin{cases} \dfrac{dq}{dt} = \dfrac{\partial H}{\partial p} \\ \\ \dfrac{dp}{dt} = - \dfrac{\partial H}{\partial q} \end{cases} \tag{125}$$

by the choice

$$\begin{cases} q = x & \text{(position variable)} \\ p = y & \text{(momentum variable)} \\ H(q, p) = \frac{\alpha}{2}\, c_{\beta 0}(x + iy) & \text{(Hamiltonian)} \quad . \end{cases} \tag{126}$$

This makes Ω appear as phase space (the "(q, p)-space") and the area measure dxdy as phase density dqdp . Thus the Liouville theorem just becomes that dxdy is invariant under $\{ \mu_t \}$, that is (123), as we wanted.

It might be noted that the area-preserving property of $\{ \mu_t \}$ also has an infinitesimal aspect. This is that the infinitesimal generator of $\{ \mu_t \}$, the vector field $\mathbf{A} = - i\, \alpha\, \overline{c}_{\beta 1}(z)$, is divergence-free ,

$$\operatorname{div} \mathbf{A} = 0 \quad . \tag{127}$$

Now $\mathbf{A}$ is essentially just our old vector field $F_\beta = - 2\, \pi\, \rho\, \alpha^2 \cdot \overline{c}_{\beta 1}(z)$ rotated 90° . Therefore, as is easily seen, (127) is the same thing as

$$\operatorname{curl} F_\beta = 0 \,, \tag{128}$$

which in turn is a consquence of $F_\beta = \operatorname{grad} u_\beta$. Thus we have proved again that $\{ \mu_t \}$ is area-preserving.

f) Behaviour of c_n and $c_{\beta n}$ under conformal mapping

We are going to determine the transformation properties of $c_0(z), c_1(z), \ldots$ and $c_{\beta 0}(z)$, $c_{\beta 1}(z), \ldots$ under conformal mappings. Thus we consider two domains, D and Ω, together with a conformal isomorphism

$$f : D \to \Omega \,.$$

The following system for the notations will be used:

	D	Ω
Variables:	$\zeta, \zeta_0, \ldots$	$z, z_0, \ldots$
Domain functions:	$\tilde{g}(\zeta, \zeta_0), \ldots$	$g(z, z_0), \ldots$
	$\tilde{c}_n(\zeta), \ldots$	$c_n(z), \ldots$

Whenever, in the formulas to follow, both z and ζ (for example) occur in the same equation they are assumed to be related by $z = f(\zeta)$. If D and Ω are multiply connected it is also assumed that their boundary components are numbered so that f maps the j :th boundary component of D onto the j:th one of Ω (for all j).

It is well-known that the Green's function transforms according to

$$g(z, z_0) = \tilde{g}(\zeta, \zeta_0) \ . \tag{129}$$

Similarly, we have for the "modified Green's function"

$$g_\beta(z, z_0) = \tilde{g}_\beta(\zeta, \zeta_0) \ , \tag{130}$$

provided that the same period list $\beta = (\beta_1, \ldots, \beta_m)$ is used on both sides. This is easily seen, for example by noticing that the function (of ζ) $g_\beta(f(\zeta), z_0)$ has the **properties** which uniquely characterize $\tilde{g}_\beta(\zeta, \zeta_0)$. One may alternatively look at the construction of $g_\beta(z, z_0)$ from $g(z, z_0)$ on p. 29ff and find that the matrix (p_{kj}) occurring there is a conformal invariant, hence so is (a_{kj}) (for fixed β), so that (130) follows from (129) together with the obvious transformation formula

$$\omega_k(z) = \tilde{\omega}_k(\zeta) \tag{131}$$

for the harmonic measures.

Since $g_\beta(z, z_0)$ thus transforms in the same way as $g(z, z_0)$, it is clear that the transformation properties of the functions $c_{\beta 0}(z), c_{\beta 1}(z), \ldots$ will be identical with those of $c_0(z)$, $c_1(z), \ldots$. Therefore we need only carry out the computations for the latter ones.

From (129) we get

$$G(z,z_0) = \tilde{G}(\zeta,\zeta_0) + i\cdot\gamma(\zeta_0) \quad , \tag{132}$$

where $\gamma(\zeta_0)$ is some real function. Hence

$$H(z,z_0) = \tilde{H}(\zeta,\zeta_0) + \log \frac{z-z_0}{\zeta-\zeta_0} + i\,\gamma(\zeta_0) \quad . \tag{133}$$

Letting $\zeta \to \zeta_0$ gives

$$c_0(z_0) = \tilde{c}_0(\zeta_0) + \log f'(\zeta_0) + i\,\gamma(\zeta_0) \tag{134}$$

and, by taking the real and imaginary parts:

$$\begin{cases} c_0(z_0) = \tilde{c}_0(\zeta_0) + \log|f'(\zeta_0)| \qquad \text{resp.} & (135) \\ \gamma(\zeta_0) = -\arg f'(\zeta_0) \quad . & (136) \end{cases}$$

It is convenient to have (135) written up in several different ways:

$$c_0(z) = \tilde{c}_0(\zeta) + \log|f'(\zeta)| \quad , \tag{137}$$

$$c_0(z) - \log|dz| = \tilde{c}_0(\zeta) - \log|d\zeta| \quad , \tag{138}$$

$$e^{-c_0(z)}\,|dz| = e^{-\tilde{c}_0(\zeta)}\,|d\zeta| \quad . \tag{139}$$

The transformation properties for $c_1, c_2, \ldots$ are obtained by expanding both members of (133) in power series in $\zeta - \zeta_0$ and identifying coefficients. For c_1 it is however easier to employ
$c_1(z) = \dfrac{\partial c_0(z)}{\partial z}$ together with (137). Since

$$\frac{\partial}{\partial\zeta} \log|f'(\zeta)| = \frac{1}{2}\frac{\partial}{\partial\zeta} \log f'(\zeta) = \frac{1}{2}\frac{f''(\zeta)}{f'(\zeta)} \tag{140}$$

applying $\dfrac{\partial}{\partial\zeta}$ to (137) gives

$$c_1(z)f'(\zeta) = \tilde{c}_1(\zeta) + \frac{1}{2}\frac{f''(\zeta)}{f'(\zeta)} \quad , \quad \text{or} \tag{141}$$

$$c_1(z)dz = \tilde{c}_1(\zeta)d\zeta + \frac{1}{2} d \log \frac{dz}{d\zeta} \quad , \quad \text{or} \tag{142}$$

$$c_1(z)dz - \frac{1}{2} d \log dz = \tilde{c}_1(\zeta)d\zeta - \frac{1}{2} d \log d\zeta \, . \tag{143}$$

(138), (139) and (143) mean, loosely speaking, that the quantities

$$\begin{aligned} &c_0(z) - \log |dz| \quad , \\ &e^{-c_0(z)} |dz| \quad , \\ &c_1(z)dz - \frac{1}{2} d \log dz \end{aligned} \tag{144}$$

are conformal invariants.

The transformation formulas for the higher order coefficients $c_2(z)$, $c_3(z)$, ... tend to be very complicated. For example, the one for $c_2(z)$ also involves $c_1(z)$, and turns out to be

$$c_2(z)f'(\zeta)^2 + c_1(z)f'(\zeta) \cdot \frac{f''(\zeta)}{4f'(\zeta)} =$$

$$= \tilde{c}_2(\zeta) - \tilde{c}_1(\zeta) \cdot \frac{f''(\zeta)}{4f'(\zeta)} + \frac{1}{6} \left[\frac{f'''(\zeta)}{f'(\zeta)} - \frac{3}{2} \left(\frac{f''(\zeta)}{f'(\zeta)} \right)^2 \right] , \quad \text{or} \tag{145}$$

$$c_2(z)dz^2 - \frac{1}{4} d \log \frac{d\zeta}{dz} \cdot c_1(z)dz =$$

$$= \tilde{c}_2(\zeta)d\zeta^2 - \frac{1}{4} d \log \frac{dz}{d\zeta} \cdot \tilde{c}_1(\zeta)d\zeta + \frac{1}{6} \{z, \zeta\} \, d\zeta^2 \quad , \tag{146}$$

where

$$\{z, \zeta\} = \{f, \zeta\} = \frac{f'''(\zeta)}{f'(\zeta)} - \frac{3}{2} \left(\frac{f''(\zeta)}{f'(\zeta)} \right)^2 = \left(\frac{f''(\zeta)}{f'(\zeta)} \right)' - \frac{1}{2} \left(\frac{f''(\zeta)}{f'(\zeta)} \right)^2 \tag{147}$$

is the Schwarzian derivate of f.

In the hydrodynamical context of regarding $- i \alpha \, \overline{c}_{\beta 1}(z)$ as the velocity vector of a freely moving vortex ((2.48)) the transformation formula (141) (or perhaps rather (137) combined with $c_1 = \frac{\partial c_0}{\partial z}$) is known as "Routh's theorem". See [M-T], 13 - 50.

It should perhaps also be emphasized that the presence of the term $\log|f'(\zeta)|$ in (137) has the effect that the orbits $c_{\beta 0}$ = constant of a freely moving vortex are <u>not</u> conformal invariants, that is the image of an orbit under a conformal mapping is not an orbit.

g) <u>The metric $ds = e^{-c_0(z)}|dz|$</u>

The transformation properties (137) - (139) of $c_0(z)$, $c_{\beta 0}(z)$ can be expressed by saying that

$$ds = e^{-c_0(z)}|dz| \quad \text{and} \tag{148}$$

$$ds_\beta = e^{-c_{\beta 0}(z)}|dz| \tag{149}$$

define conformally invariant Riemannian metrics on Ω. Among all conformally invariant metrics on domains in $\mathbf{C}$ the Poincaré metric is perhaps the most well-known. The main purpose of this section is to compare the metrics (148), (149) with the Poincaré metric. The result, which will find an application in section IV d), is that our metrics coincide with the Poincaré metric if Ω is simply connected but are strictly smaller otherwise (p.66 f)
On the way we derive a result (Lemma 3.7) of some interest in its own right, and which will be developed further and applied later (sect. V b)). It is a kind of generalization of what is usually called "the invariant form of Schwarz's lemma" and says that analytic functions are locally distance-decreasing with respect to the metric (148), the decrease being everywhere strict except for analytic isomorphisms.

Let us start however with some remarks concerning the Gaussian curvature for (148) and (149). For (148) this is given by the formula

$$\varkappa(z) = e^{2c_0} \cdot \Delta c_0 = 4\, e^{2c_0} \frac{\partial^2 c_0}{\partial z \partial \overline{z}} \quad . \tag{150}$$

(see for example [A], sec. 1-5.)
Similarly for $\varkappa_\beta(z)$ in terms of $c_{\beta 0}(z)$, A direct computation (using (137)) shows that $\varkappa$ and $\varkappa_\beta$ are invariant under conformal mappings, that is, that

$$\varkappa(z) = \tilde{\varkappa}(\zeta) \qquad \text{and} \tag{151}$$

$$\varkappa_\beta(z) = \tilde{\varkappa}_\beta(\zeta) \tag{152}$$

in the notations of section III f) .

As a consequence of (151) we have that $\varkappa$ (and $\varkappa_\beta$) takes one and the same constant value on all simply connected regions; for any two points on any two simply connected regions can be mapped onto each other by a conformal isomorphism (the constant turns out to be -4; see (4.14)).

We shall to some extent be concerned with the question whether $\varkappa$ and $\varkappa_\beta$ are constant also for multiply connected domains. Since $\varkappa$ (and $\varkappa_\beta$) is real it is constant if and only if $\frac{\partial \varkappa}{\partial z} \equiv 0$. By (150) we have

$$\frac{\partial \varkappa}{\partial z} = 4 \cdot \left[2 \frac{\partial c_0}{\partial z} \frac{\partial^2 c_0}{\partial z \partial \bar{z}} \cdot e^{2c_0} + \frac{\partial^3 c_0}{\partial z^2 \partial \bar{z}} \cdot e^{2c_0}\right] \tag{153}$$

$$= 4 \frac{\partial}{\partial \bar{z}} \left(\left(\frac{\partial c_0}{\partial z} \right)^2 + \frac{\partial^2 c_0}{\partial z^2} \right) \cdot e^{2c_0} \quad .$$

Introducing the quantity

$$q(z) = \frac{\partial^2 c_0}{\partial z^2} + \left(\frac{\partial c_0}{\partial z} \right)^2 = (\text{by Lemma } 3.1) = \frac{\partial c_1}{\partial z} + c_1^2 \tag{154}$$

this becomes

$$\frac{\partial \varkappa}{\partial z} = 4 \frac{\partial q}{\partial \bar{z}} \cdot e^{2c_0} \quad . \tag{155}$$

Thus $\varkappa$ is constant if and only if $q(z)$ is a holomorphic function. In particular $q(z)$ is holomorphic if Ω is simply connected, by the above remark.

Some manipulations of formula (141) give

$$\left[\frac{\partial c_1(z)}{\partial z} + c_1(z)^2 \right] \cdot f'(\zeta)^2 = \frac{\partial \tilde{c}_1(\zeta)}{\partial \zeta} + \tilde{c}_1(\zeta)^2 +$$

$$+ \frac{1}{2} \left[\left(\frac{f''(\zeta)}{f'(\zeta)} \right)' - \frac{1}{2} \left(\frac{f''(\zeta)}{f'(\zeta)} \right)^2 \right] . \tag{156}$$

Thus the behaviour of q under conformal mapping is given by

$$q(z)\ f'(\zeta)^2 = \tilde{q}(\zeta) + \frac{1}{2}\{f,\zeta\}\ , \qquad \text{or} \tag{157}$$

$$q(z)dz^2 = \tilde{q}(\zeta)d\zeta^2 + \frac{1}{2}\{z,\zeta\}\ d\zeta^2$$

($\{f,\zeta\}$ denotes the Schwarzian derivative, (147).)

Of course, by putting in an index β everywhere above, we obtain exact parallel definition and formulas for q_β.

Next we turn to the announced generalization of the so-called invariant form of Schwarz's lemma. Our result, Lemma 3.7, only concerns the metric (148) (<u>not</u> (149)), and says that analytic functions are locally distance-decreasing with respect to this metric, and that the decrease is everywhere strict except for analytic isomorphism. This result is closely related to the Lindelöf principle *). Also, Lemma 3.2 can be thought of as a special case of it.

We consider two regions D and Ω with variables and domain functions denoted ζ, $\zeta_0, \dots$, $\tilde{g}(\zeta,\zeta_0)$, $\tilde{c}_0(\zeta), \dots$ for D an $z, z_0, \dots$, $g(z,z_0)$, $c_0(z), \dots$ for Ω. Then

<u>Lemma 3.7</u> Suppose $f : D \to \Omega$ is analytic. Then

$$e^{-c_0(z)}|dz| \le e^{-\tilde{c}_0(\zeta)}|d\zeta|\ , \qquad \text{that is} \tag{159}$$

$$e^{-c_0(z)}|f'(\zeta)| \le e^{-\tilde{c}_0(\zeta)}\ , \qquad \text{or} \tag{160}$$

$$c_0(z) \ge \tilde{c}_0(\zeta) + \log|f'(\zeta)| \tag{161}$$

for each $\zeta \in D$, $z = f(\zeta) \in \Omega$.
If equality holds for some $\zeta \in D$, then equality holds throughout and f is a conformal isomorphism.

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*) see [N] III § 3 (Prinzip vom hyperbolischen Mass) or [G] VIII § 1-3.
Postscript: I have recently found that Lemma 3.7 actually occurs as a part of a "Lindelöf theorem", in [J] (Ch. IV, § 46).

Proof: Given $\zeta_0 \in D$, let $z_0 = f(\zeta_0)$,

$$E = \{ \zeta_0, \zeta_1, \zeta_2, \dots \} = f^{-1}(z_0) \,, \tag{162}$$

and put

$$u(\zeta) = u(\zeta, \zeta_0) = g(f(\zeta), f(\zeta_0)) - \tilde{g}(\zeta, \zeta_0) \,. \tag{163}$$

Clearly u is harmonic in $D \setminus E$. If

$$f(\zeta) - f(\zeta_0) = a \cdot (\zeta - \zeta_0)^{n_0} + O((\zeta - \zeta_0)^{n_0+1}) \,, \quad a \neq 0,$$

at $\zeta = \zeta_0$, u has the singularity

$$u(\zeta) = -\log|\zeta - \zeta_0|^{n_0} - (-\log|\zeta - \zeta_0|) + \text{harmonic} =$$

$$= -(n_0 - 1) \cdot \log|\zeta - \zeta_0| + \text{harmonic} \tag{164}$$

at $\zeta = \zeta_0$. Similarly, at $\zeta = \zeta_j$, $j \geq 1$,

$$u(\zeta) = -n_j \cdot \log|\zeta - \zeta_0| + \text{harmonic} \tag{165}$$

for certain integers $n_j \geq 1$. Thus u is harmonic except for singularities of the kind $-n \cdot \log|\zeta - \zeta_j|$, $n > 0$, and therfore it is superharmonic in all D.

As $\zeta \to \zeta' \in \partial D$ we have $\tilde{g}(\zeta, \zeta_0) \to 0$. Therefore, since $g(f(\zeta), f(\zeta_0)) > 0$,

$$\varliminf_{\zeta \to \zeta'} u(\zeta) \geq 0 \qquad \text{for} \quad \zeta' \in \partial D \tag{166}$$

and so, by the superharmonicity,

$$u(\zeta) \geq 0 \qquad \text{in} \quad D \,. \tag{167}$$

Moreover, if equality in (167) holds for some $\zeta \in D$, then (still by superharmonicity)

$$u(\zeta) \equiv 0 \quad \text{in} \quad D \,. \tag{168}$$

Thus, given $\zeta_0 \in D$ we have either

$$u(\zeta, \zeta_0) > 0 \quad \text{for all} \quad \zeta \in D \,, \quad \text{or} \tag{169}$$

$$u(\zeta, \zeta_0) \equiv 0 \,, \quad \zeta \in D \,. \tag{170}$$

Using that $u(\zeta, \zeta_0) = u(\zeta_0, \zeta)$ this gives:

$$\left.\begin{array}{l} \text{either} \quad u(\zeta, \zeta_0) > 0 \quad \text{for all} \quad \zeta, \zeta_0 \in D \\ \text{or} \quad u(\zeta, \zeta_0) \equiv 0 \,; \quad \zeta, \zeta_0 \in D \,. \end{array}\right. \tag{171}$$

Now however

$$g(f(\zeta), f(\zeta_0)\ ) = -\log|f(\zeta) - f(\zeta_0)| + h(f(\zeta), f(\zeta_0)) =$$

$$= -\log|f'(\zeta_0)\cdot(\zeta - \zeta_0) + O(\zeta - \zeta_0)^2| + h(f(\zeta), f(\zeta_0)\ ) =$$

$$= -\log|\zeta - \zeta_0| - \log|f'(\zeta_0)| + O(\zeta - \zeta_0) + h(f(\zeta), f(\zeta_0)\ ) \,, \tag{172}$$

and

$$\tilde{g}(\zeta, \zeta_0) = -\log|\zeta - \zeta_0| + \tilde{h}(\zeta, \zeta_0) \quad . \tag{173}$$

thus,

$$u(\zeta, \zeta_0) = -\log|f'(\zeta_0)| + h(f(\zeta), f(\zeta_0)\ ) - \tilde{h}(\zeta, \zeta_0) + O(\zeta - \zeta_0) \,, \tag{174}$$

and, letting $\zeta \to \zeta_0$,

$$u(\zeta_0, \zeta_0) = -\log|f'(\zeta_0)| + h(f(\zeta_0), f(\zeta_0)\ ) - \tilde{h}(\zeta_0, \zeta_0) =$$

$$= -\log|f'(\zeta_0)| + c_0(f(\zeta_0)\ ) - \tilde{c}_0(\zeta_0) \,. \tag{175}$$

Therefore ((171)):

$$\left.\begin{array}{ll} \text{either} \quad c_0(f(\zeta)\ ) > \tilde{c}_0(\zeta) + \log|f'(\zeta)| & \text{for all} \quad \zeta \in D \\ \text{or} \qquad c_0(f(\zeta)\ ) \equiv \tilde{c}_0(\zeta) + \log|f'(\zeta)| & , \quad \zeta \in D \,. \end{array}\right. \tag{176}$$

Thus (159) - (161) are proved, and the proof of the lemma is complete as soon as we have proved that the equality case in (176) can hold only if f is a conformal isomorphism.

So suppose we are in the equality case in (176). Then the same is true for (171), that is

$$\tilde{g}(\zeta,\zeta_0) \equiv g(f(\zeta),f(\zeta_0)) \quad , \quad \zeta,\zeta_0 \in D . \tag{177}$$

We have to prove that f is one-to-one and onto. But $\zeta \neq \zeta_0$ implies

$$\infty > \tilde{g}(\zeta,\zeta_0) = g(f(\zeta),f(\zeta_0)) . \tag{178}$$

Thus $f(\zeta) \neq f(\zeta_0)$, and f is one-to-one. To prove that $f(D) = \Omega$, assume on the contrary that $f(D) \subsetneqq \Omega$. Then there is a point $z \in \Omega \cap \partial f(D)$ (∂ denoting boundary). Since $z \in \partial f(D)$ there is a sequence $\{ \zeta_n \}$ in D such that $f(\zeta_n) \to z$, and because f is injective, as we already know, (or actually just because f is an open map) we necessarily have $\zeta_n \to \partial D$. If $\zeta_0 \in D$ is any point it therefore follows that $\tilde{g}(\zeta_n,\zeta_0) \to 0$, while on the other hand $g(f(\zeta_n),f(\zeta_0)) \to g(z,f(\zeta_0)) > 0$. This contradicts (177), so we must actually have $f(D) = \Omega$.

By this the proof of Lemma 3.7 is finished.

Now we shall apply Lemma 3.5 to compare the metrics (148) and (149) with the Poincaré metric. Denoting the Poincaré metric by

$$ds = \rho(z)|dz| \quad , \tag{179}$$

we shall show that

$$e^{-c_{\beta 0}(z)} \leq e^{-c_0(z)} \leq \rho(z) , \tag{180}$$

where the second inequality is everywhere strict unless Ω is simply connected (in which case all three metrics coincide).

The first inequality in (180) is just (33), $\omega_\beta(z,z) \geq 0$ ($c_{\beta 0}(z) = c_0(z) + \omega_\beta(z,z)$, (43)), and the equality cases of it are discussed on page 33 .

To prove the second inequality in (180) we first recall how the Poincare metric is defined. For $\mathbb{D}$ it is by definition *)

$$ds = \frac{|dz|}{1-|z|^2} \quad , \tag{181}$$

and for an arbitrary multiply connected region Ω it is obtained by carrying over (181) to the universal covering surface of Ω and then observing that the metric so obtained is well defined on Ω itself. More precisely, let $\tilde{\Omega}$ denote the universal covering surface of Ω **). Thus $\tilde{\Omega}$ is a simply connected surface which we regard as lying over Ω with several (infinitely many) sheets. Let $\pi : \tilde{\Omega} \to \Omega$ denote the projection map. According to "the Uniformization Theorem" ([A], Ch 10) $\tilde{\Omega}$ is conformally equivalent to $\mathbb{D}$, with mapping function $f_1 : \mathbb{D} \to \tilde{\Omega}$ and inverse $\varphi_1 = f_1^{-1}$ say. Then $f = \pi \circ f_1 : \mathbb{D} \to \Omega$ is the universal covering map, the "inverse" of which, φ , is multiple valued (unless Ω is simply connected in which case $\tilde{\Omega} = \Omega$).
The situation is pictured in fig. 3.12.

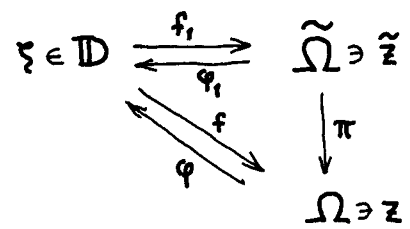

Fig. 3.12

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*) The Poincare metric in [A] however differs from (181) by a factor 2.

**) See for example [A] Ch 10 for a precise definition of universal covering surface

Now, we first get the Poincaré metric on $\tilde{\Omega}$ by pulling back (181) with φ_1 :

$$ds = \frac{|\varphi_1'(\tilde{z})|}{1-|\varphi_1(\tilde{z})|^2} \cdot |d\tilde{z}| \quad , \quad \tilde{z} \in \tilde{\Omega} . \tag{182}$$

Then one convinces oneself that if $\tilde{z}_1$ and $\tilde{z}_2$ are two points on $\tilde{\Omega}$ lying over the same point z on Ω $(\pi(\tilde{z}_1) = \pi(\tilde{z}_2) = z)$ then

$$\frac{|\varphi_1'(\tilde{z}_1)|}{1-|\varphi_1(\tilde{z}_1)|^2} = \frac{|\varphi_2'(\tilde{z}_2)|}{1-|\varphi_1(\tilde{z}_2)|^2} . \tag{183}$$

This is because, regarding $\tilde{z}_2$ as a function of $\tilde{z}_1$ (determined by the relation $\pi(\tilde{z}_2) = \pi(\tilde{z}_1)$) in a neighbourhood of the chosen points, $\varphi_1(\tilde{z}_1)$ and $\varphi_1(\tilde{z}_2) = \varphi_1(\tilde{z}_2(\tilde{z}_1))$ are related by a Möbius transformation mapping $\mathbb{D}$ onto itself, whereby a simple computation shows (183). From this it follows that the expression

$$\rho(z) = \frac{|\varphi'(z)|}{1-|\varphi(z)|^2} \quad , \quad z \in \Omega \tag{184}$$

is well-defined on Ω, despite φ itself being multiple-valued. The Poincaré metric on Ω is then defined by

$$ds = \rho(z)|dz| . \tag{185}$$

To compare $\rho(z)$ with $e^{-c_0(z)}$ we shall use Lemma 3.7, applied to the mapping $\pi : \tilde{\Omega} \to \Omega$. Here we must be a bit careful because $\tilde{\Omega}$ is not a domain in $\mathbb{P}$ and so does not fulfill our general assumptions in section III a) ($\tilde{\Omega}$ is a Riemann surface). However it is easy to see that the definitions of the functions $c_0(\zeta), c_1(\zeta), \ldots$ make sense even for non-schlicht regions, such as $\tilde{\Omega}$, provided that the Green's function exists (which it does for $\tilde{\Omega}$), and provided some care is taken with the function $h(z,\zeta)$ because it will not be globally defined (it will however be defined for z near ζ, which is enough). It is also fairly obvious that the transformation formulas in sect. III f) will still be valid, as well as Lemma 3.7. Thus we can talk about the domain function c_0 for $\tilde{\Omega}$, and it will be denoted $\tilde{c}_0 = \tilde{c}_0(\tilde{z})$.

$\tilde{c}_0(\tilde{z})$ can be expressed in the isomorphism $\varphi_1 : \tilde{\Omega} \to \mathbb{D}$ and borrowing a formula, (4.17), from section IVb) the result is

$$e^{-\tilde{c}_0(\tilde{z})} = \frac{|\varphi_1'(\tilde{z})|}{1-|\varphi_1(\tilde{z})|^2} \quad . \tag{186}$$

Thus, with $z = \pi(\tilde{z})$,

$$e^{-\tilde{c}_0(\tilde{z})} = \frac{|\varphi'(z)|}{1-|\varphi(z)|^2} = \rho(z) \quad , \tag{187}$$

so that Lemma 3.7 gives

$$e^{-c_0(z)} \leq e^{-\tilde{c}_0(\tilde{z})} \cdot \left|\frac{d\tilde{z}}{dz}\right| = \rho(z) \tag{188}$$

(clearly $\left|\frac{d\tilde{z}}{dz}\right| = 1$), with strict inequality holding everywhere unless π is an isomorphism, that is, unless Ω is simply connected. And this is exactly what we wanted to prove ((180)).*

Observe by the way that (187) shows that $\tilde{c}_0(\tilde{z})$ is well-defined as a function of $z = \pi(\tilde{z})$, so we can write $\tilde{c}_0(z)$ instead of $\tilde{c}_0(\tilde{z})$ and regard it as a domain function on Ω itself.

h) c_0, $c_{\beta 0}$ related to the Bergman kernels.

For any region Ω the function

$$K(z,\zeta) = -\frac{2}{\pi} \cdot \frac{\partial^2 g(z,\zeta)}{\partial z \partial \bar{\zeta}} \tag{189}$$

is known as the Bergman kernel for Ω (reference: [B] or [NEH]). It is characterized by the properties that

$$f(\zeta) = \iint_{\Omega} f(z)\overline{K(z,\zeta)}\, dxdy$$

for all f in $L_a^2(\Omega)$, the Hilbert space of all square-integrable (with respect to area measure $dxdy$) analytic functions in Ω, and that $K(\cdot,\zeta)$ itself belongs to $L_a^2(\Omega)$ for all $\zeta \in \Omega$.

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* See p. 109.

Similarly,

$$K_s(z,\zeta) = -\frac{2}{\pi}\cdot\frac{\partial^2 g_\beta(z,\zeta)}{\partial z\partial\overline{\zeta}} \tag{190}$$

is the "reduced Bergman kernel", having the corresponding properties with respect to the subspace $L^2_{as}(\Omega)$ of $L^2_a(\Omega)$, consisting of those functions with single-valued integral in Ω. Notice that the right hand side of (190) does not depend on the choice of β (since the left hand side does not). This may also be seen directly, using the remark on page 32 (formula (32)).

Since $\frac{\partial^2}{\partial z\partial\overline{\zeta}}\log|z-\zeta| = 0$ (189) can also be written

$$K(z,\zeta) = -\frac{2}{\pi}\cdot\frac{\partial^2 h(z,\zeta)}{\partial z\partial\overline{\zeta}} .$$

(Similarly for $K_s(z,\zeta)$.) This gives the following relationship between $c_0(z)$ and $K(z,z)$:

$$\frac{\partial^2}{\partial z\partial\overline{z}} c_0(z) = \frac{\partial^2}{\partial z\partial\overline{z}} h(z,z) = \{ \frac{\partial^2}{\partial z\partial\overline{z}} h(z,\zeta) + \frac{\partial^2}{\partial z\partial\overline{\zeta}} h(z,\zeta) +$$

$$+ \frac{\partial^2}{\partial\zeta\partial\overline{z}} h(z,\zeta) + \frac{\partial^2}{\partial\zeta\partial\overline{\zeta}} h(z,\zeta) \}_{\zeta=z} = 2\{ \frac{\partial^2}{\partial z\partial\overline{\zeta}} h(z,\zeta)\}_{\zeta=z} = -\pi K(z,z).$$

Thus

$$\Delta c_0(z) = -4\pi K(z,z) . \tag{191}$$

In the same way

$$\Delta c_{\beta 0} = -4\pi K_s(z,z) . \tag{192}$$

From the inclusion $L^2_{as}(\Omega) \subset L^2_a(\Omega)$ together with well-known properties of reproducing kernels follows the inequalities $0 < K_s(z,z) \leq K(z,z)$, where the second one is everywhere strict if Ω is not simply connected. Thus besides the inequality (180),

$$c_0(z) \leq c_{\beta 0}(z) , \tag{193}$$

we also have

$$\Delta c_0 \leq \Delta c_{\beta 0} < 0 . \tag{194}$$

The first inequality in (194) is also easily derived directly from (43), using the positive semi-definiteness of the matrix (a_{kj}).

The second inequality in (194) assert that c_0 and $c_{\beta 0}$ are superharmonic. A particular consequence of this is that they cannot have interior minima.

In terms of the potential function u and the vector field F (p.34-35) (191) is

$$\Delta u = (\text{pos. const.}) \cdot K(z,z) \qquad \text{resp.} \tag{195}$$

$$\operatorname{div} F = (\text{pos. const.}) \cdot K(z,z) \tag{196}$$

(similarly for u_β and F_β). This gives the interpretation for $K(z,z)$ of being the "source density" associated with u and F.

By (191) the formula (150) for the Gaussian curvature $\varkappa(z)$ for the metric $ds = e^{-c_0(z)} |dz|$ becomes

$$\varkappa(z) = -4\pi K(z,z) \cdot e^{2c_0(z)} . \tag{197}$$

(Similarly for $\varkappa_\beta$.)* Therefore, having further information about the relationship between $c_0(z)$ and $K(z,z)$ is equivalent to having information about $\varkappa(z)$. For example, for simply connected regions we have $\varkappa \equiv -4$ (p. 62 or formula (4.14)). Thus

$$K(z,z) = \frac{1}{\pi} e^{-2c_0(z)} , \qquad \text{or} \tag{198}$$

$$c_0(z) = -\log \sqrt{\pi K(z,z)} \tag{199}$$

for such regions.

For multiply connected regions (198), (199) are however no longer true. Indeed, we will prove later (p.82 f) that $\varkappa(z)$ is not constant for such regions (as to $\varkappa_\beta(z)$ we do not quite prove that it is not constant, but there is no reason why it should be, and at least it cannot be constant for more than one choice of β, and it can never be constantly equal to -4; see p. 82 f).

Not very much (beyond (191)) seems to be known about the exact relationship between $c_0(z)$ and $K(z,z)$ for multiply connected regions.

---

* (197) also yields the following relationship between $\varkappa(z)$ and $K(z,z)$: $\Delta(\log \varkappa(z)) = \Delta \log K(z,z) - 8\pi K(z\ z)$. (This observation is due to H.S.Shapiro.)

In the book [S-O] the question of such a relation is in fact stated as an "Open Question" ([S-O], Ch VII 5 I and Open Question 7 p.342 $*$)).

From [S-O] we may also extract some positive information. Namely, if $\beta = (\beta_1,\dots,\beta_m)$ is of the form $\beta_j = -2\pi\delta_{kj}$ $(j = 1,\dots,m)$ for some k, then

$$e^{-c_{\beta 0}(\zeta)} \leq \sqrt{\pi K_s(\zeta,\zeta)} \tag{200}$$

([S-O], VII 5 F). Furthermore

$$\sqrt{\pi K_s(\zeta,\zeta)} \leq e^{-c_0(\zeta)} \tag{201}$$

(VII 5 F), and both inequalities are everywhere strict unless the region is simply connected (VII 5 H). In terms of the curvature $\varkappa_\beta$ (200) reads

$$\varkappa_\beta(z) \leq -4 . \tag{202}$$

(200) and (202) can however not be true for arbitrary choices of β; for we know (p. 33) that equality in
$e^{-c_{\beta 0}(\zeta)} \leq e^{-c_0(\zeta)}$ does occur for $\zeta = \zeta_0$ if $\beta = -2\pi(\omega_1(\zeta_0),\dots,\omega_m(\zeta_0))$, so that by (201)

$$\sqrt{\pi K_s(\zeta,\zeta)} < e^{-c_{\beta 0}(\zeta)} , \quad \text{or} \quad K_\beta(\zeta) > -4 \quad \text{for such } \beta \text{ and } \zeta .$$

The book [S-O] contains a lot of other valuable information about the functions $G(z,\zeta)$, $G_\beta(z,\zeta)$, $c_0(\zeta)$, $c_{\beta 0}(\zeta)$, ... , for example characterizations of them by various extremal properties. We do not take up these topics here but just refer to that book. The reader who wants to consult [S-O] will however find that I have chosen extremely bad notations with respect to those in [S-O] (which are the standard ones). Therefore we have, in Appendix 1, given a short "conversion table" between the relevant notations in [S-O] and those used here.

$*$) See Appendix 1 for the notations in [S-O]

1) $c_0(z)$ related to the transfinite diameter

From the equation $F = \operatorname{grad} u$ ((46)) for the force F on the charge in the electrostatic problem one obtains the interpretation for $-u = (\text{pos. const.}) \cdot c_0 + \text{const.}$ ((50)) of being the total energy (up to an additive constant) of the physical system. This energy can also be computed directly in terms of the actual charge configuration on the conductor (= the complement of the region Ω in question). Since charges on a conductor always distribute themselves so that the total energy is minimized this leads to representations for $-u(z)$ and $c_0(z)$ as minima of purely geometric quantities (or actually as limits of such minima). We want to derive such a representation here. The formula we aim at is equivalent to the well-known representation in potential theory for the Robin's constant in terms of the transfinite diameter, and we need only perform a simple variable transformation on this to reach our goal. This formula is then applied to give a simple proof of Lemma 3.4 which was left open in section III d) (p. 39).
[A] Ch 2 and [S-O] Ch VII 6 may serve as references for this section.

The classical situation is this: $\Omega \subset \mathbb{P}$ is a domain with $\infty \in \Omega$ so that $K = \mathbb{P} \setminus \Omega$ is a compact subset of $\mathbb{C}$. The Green's function $g(z, \zeta)$ for Ω has the asymptotic behaviour, for $\zeta = \infty$,

$$g(z,\infty) = \log|z| + \gamma + O(|z|^{-1}) \quad \text{as} \quad z \to \infty . \tag{203}$$

Here, γ is the Robin's constant and

$$\operatorname{Cap} K = e^{-\gamma} \tag{204}$$

the logarithmic capacity of K (Cap K is usually defined otherwise, but this does not matter here). For $n \geq 2$, let

$$\delta_n(K) = \max_{z_1,\ldots,z_n \in K} \Big(\prod_{\substack{i,j=1 \\ i<j}}^{n} |z_i - z_j| \Big)^{\frac{2}{n(n-1)}} , \tag{205}$$

$$I_n(K) = \min_{z_1,\ldots,z_n \in K} \frac{2}{n(n-1)} \sum_{i<j} \log \frac{1}{|z_i - z_j|} \tag{206}$$

Thus $\delta_n = e^{-I_n}$. Apart from a constant factor $I_n(K)$ may be interpreted as the energy of the equilibrium charge configuration on K when K is a perfect conductor and the unit charge is divided into n equal pieces and supplied to K.

One finds that $\delta_2 \geq \delta_3 \geq \dots$ and $I_2 \leq I_3 \leq \dots$. Letting

$$\delta(K) = \inf_n \delta_n = \lim_{n\to\infty} \delta_n \tag{207}$$

$$I(K) = \sup_n I_n = \lim_{n\to\infty} I_n \quad , \tag{208}$$

$\delta(K)$ being the <u>transfinite diameter</u> of K, the central fact then is that

$$\mathrm{Cap}\, K = \delta(K) \quad , \quad \text{or} \tag{209}$$

$$\gamma = I(K) . \tag{210}$$

Now let $\Omega \subset \mathbb{P}$ be arbitrary (i.e. drop the hypothesis $\infty \in \Omega$), $K = \mathbb{P} \setminus \Omega$, and consider any point $a \in \Omega$. If $a \neq \infty$, put

$$\zeta = \frac{1}{z-a} \tag{211}$$

(if $a = \infty$, let $\zeta = z$). Let $\tilde{\Omega}$ and $\tilde{K}$ be the images of Ω respective K in the ζ - plane, and let $g(z,z_0)$ be the Green's function for Ω. Then

$$g(z,a) = -\log|z-a| + c_0(a) + O(|z-a|) = \log|\zeta| + c_0(a) + O(|\zeta|^{-1}) \tag{212}$$

as $z \to a$ and $\zeta \to \infty$. Comparison with (203) together with (210) thus gives

$$c_0(a) = I(\tilde{K}) . \tag{213}$$

Hence

$$c_0(a) = \lim_{n\to\infty} \min_{\zeta_1,\dots,\zeta_n \in \tilde{K}} \frac{2}{n(n-1)} \sum_{i<j} \log \frac{1}{|\zeta_i - \zeta_j|} =$$

$$= \lim_{n\to\infty} \min_{z_1,\dots,z_n \in K} \frac{2}{n(n-1)} \sum_{i<j} \log \frac{|z_i - a|\cdot|z_j - a|}{|z_i - z_j|} \quad , \tag{214}$$

or, replacing a by z,

$$c_0(z) = \lim_{n\to\infty} \min_{z_1,\dots z_n \in K} \frac{2}{n(n-1)} \sum_{i<j} \log \frac{|z-z_i||z-z_j|}{|z_i - z_j|} \quad . \tag{215}$$

This is the formula we wanted to arrive at. The important thing about it is that the right member is a purely geometric quantity.

The physical interpretation of the right member in (215) is obtained by writing

$$\frac{2}{n(n-1)}\sum_{i<j}\log\frac{|z-z_i|\,|z-z_j|}{|z_i-z_j|} = 2\cdot\Big\{\frac{n}{n-1}\sum_{i<j}\frac{1}{n^2}\log\frac{1}{|z_i-z_j|} - \sum_{i=1}^{n}\frac{1}{n}\log\frac{1}{|z-z_i|}\Big\} \qquad (216)$$

Here the factor $\frac{n}{n-1}$ is harmless and may be replaced by 1 without affecting the limit in (215). Having the electrical charge $+1$ at z and the charges $-\frac{1}{n}$ at each of $z_1,\ldots,z_n \in K$ the first sum in the right member of (216) represents the energy stored in the forces between the z_i, while the second sum represents the energy in the forces between z and the z_i. Therefore the expression (216) is (approximately) twice the total energy of that charge configuration, which should make it clear how to interpret (215).

As an application of (215) we can now complete the proof of Proposition 3.3 by proving Lemma 3.4 (p. 39).* Lemma 3.4 says that $c_0(z)$ increases if K is replaced by its circular projection, with center z, onto any radius emanating from z (fig 3.13). This now follows from (215), for if

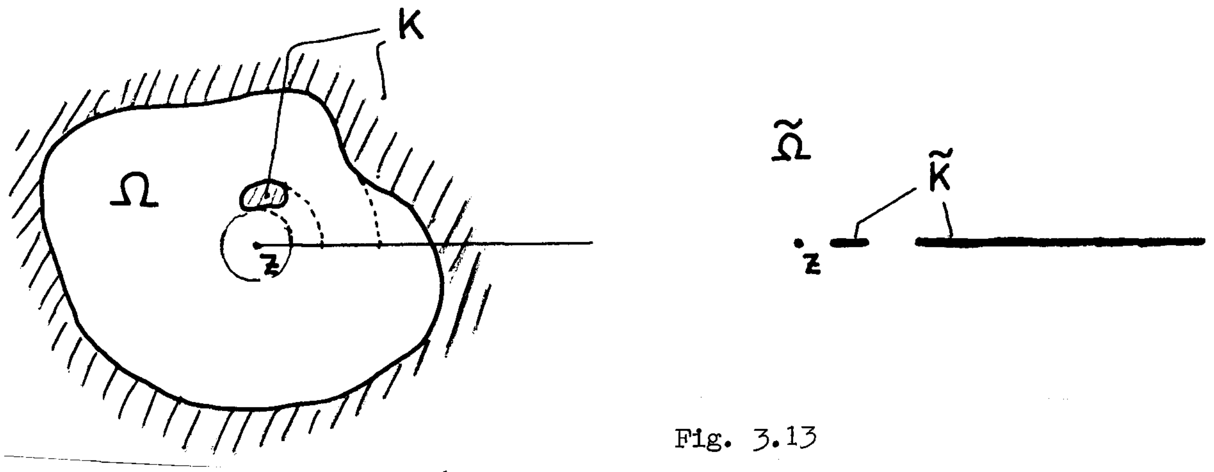

Fig. 3.13

the projection map in question is denoted $\zeta \mapsto \tilde{\zeta}$ and $\tilde{K}$ is the projection of K then in (215)

$$|z-z_i| = |z-\tilde{z}_i| \qquad \text{and}$$

$$|z_i - z_j| \geq |\tilde{z}_i - \tilde{z}_j| \quad ,$$

* This proof of Lemma 3.4 is modelled on the proof of the Koebe one-quarter theorem given in [A], sec 2-3. Lemma 3.4 could also have been proved using this one-quarter theorem in place of the theory of capacity.

and therefore, for each n,

$$\min_{z_1,\dots z_n \in K} \frac{2}{n(n-1)} \Sigma \log \frac{|z-z_i||z-z_j|}{|z_i-z_j|} \le \min_{z_1,\dots,z_n \in K} \frac{2}{n(n-1)} \Sigma \log \frac{|z-\tilde{z}_i||z-\tilde{z}_j|}{|\tilde{z}_i-\tilde{z}_j|} =$$

$$= \min_{\tilde{z}_1,\dots,\tilde{z}_n \in \tilde{K}} \frac{2}{n(n-1)} \Sigma \log \frac{|z-\tilde{z}_i||z-\tilde{z}_j|}{|\tilde{z}_i-\tilde{z}_j|} .$$

Thus $c_0(z) \le \tilde{c}_0(z)$ ($\tilde{c}_0$ referring to $\tilde{\Omega} = \mathbb{P} \setminus \tilde{K}$), which was to be proved.

IV Simply Connected Domains

In this section we specialize to simply connected domains $\Omega \subset \mathbb{P}$. For such domains our two kinds of Green's functions, $g(z,\zeta)$ and $g_\beta(z,\zeta)$, coincide, so there is no need to write out any β:s here *). First we compute "everything" for $\mathbb{D}$ (the open unit disc) and then the formulas in section III f) are applied to get explicit expressions for our domain functions for arbitrary simply connected domains Ω in terms of any Riemann mapping function $f : \mathbb{D} \to \Omega$. The rest of this section is devoted to various kinds of consequences of the so obtained formulas.

a) The unit disc

For the open unit disc $\mathbb{D}$ we have

$$g(z,\zeta) = -\log \left|\frac{z-\zeta}{1-z\bar{\zeta}}\right| . \tag{1}$$

Hence, up to an arbitrary purely imaginary function of ζ

$$G(z,\zeta) = -\log \frac{z-\zeta}{1-z\bar{\zeta}} = -\log(z-\zeta) + \log(1-z\bar{\zeta}) \tag{2}$$

and

$$H(z,\zeta) = \log(1-z\bar{\zeta}) , \tag{3}$$

*) with the exception of a passage at pp 82 - 85 where we make a digression to multiply connected regions again.

which is a single-valued analytic function (with respect to z) in $\mathbb{D}$, with that branch of log chosen which is real for $z = \zeta$. This is seen to be the correct normalization of $H(z,\zeta)$ in the sense that in the expansion

$$H(z,\zeta) = c_0(\zeta) + c_1(\zeta)\cdot(z-\zeta) + c_2(\zeta)\cdot(z-\zeta)^2 + \dots \tag{4}$$

$c_0(\zeta)$ already is real. Thus

$$c_0(\zeta) = \operatorname{Re} H(\zeta,\zeta) = \log(1 - |\zeta|^2) \quad , \quad \text{or} \tag{5}$$

$$e^{-c_0(\zeta)} = \frac{1}{1-|\zeta|^2} \quad , \tag{6}$$

and for $n > 0$

$$c_n(\zeta) = \frac{1}{n!}\frac{\partial^n}{\partial z^n}\bigg|_\zeta H(z,\zeta) = \frac{1}{n!}(-1)\cdot\ldots\cdot(-n+1)\cdot\frac{(-\bar\zeta)^n}{(1-\zeta\cdot\bar\zeta)^n} =$$

$$= -\frac{\bar\zeta^{\,n}}{n(1-|\zeta|^2)^n} = -\frac{1}{n(1/\bar\zeta - \zeta)^n} \quad . \tag{7}$$

In particular

$$c_1(\zeta) = -\frac{\bar\zeta}{1-|\zeta|^2} = \frac{1}{\zeta - 1/\bar\zeta} \quad . \tag{8}$$

The functions $\varkappa(\zeta)$ and $q(\zeta)$ ((3.150),(3.154)), already noted to be constant respectively holomorphic (p. 62) become

$$\varkappa(\zeta) = 4\, e^{2c_0(\zeta)}\,\frac{\partial^2 c_0(\zeta)}{\partial\zeta\partial\bar\zeta} = -4 \quad , \tag{9}$$

$$q(\zeta) = \frac{\partial c_1(\zeta)}{\partial\zeta} + c_1(\zeta)^2 = 0 \quad . \tag{10}$$

b) c_0, c_1, $\varkappa$ and q in terms of Riemann mapping functions

Now let $\Omega \subset \mathbb{P}$ be an arbitrary simply connected domain. Then Ω is conformally equivalent to $\mathbb{D}$ and we get the domain functions $c_0(z)$, $c_1(z)$, $\dots$, $\varkappa(z)$, $q(z)$ for Ω expressed in terms of any Riemann mapping function

$f : \mathbb{D} \to \Omega$

by combining the formulas in III f) with the results in IV a) .

Thus, (3.137) with (5) give

$$c_0(z) = \log(1 - |\zeta|^2) + \log|f'(\zeta)| \quad , \qquad \text{or} \tag{11}$$

$$e^{c_0(z)} = (1 - |\zeta|^2)|f'(\zeta)| \quad , \tag{12}$$

where $\zeta \in \mathbb{D}$, $z = f(\zeta) \in \Omega$. (3.141) with (8) give

$$c_1(z)f'(\zeta) = -\frac{\overline{\zeta}}{1-|\zeta|^2} + \frac{1}{2}\frac{f''(\zeta)}{f'(\zeta)} \, , \tag{13}$$

(3.151) with (9) give

$$\varkappa(z) \equiv -4 \, , \tag{14}$$

and finally, by (3.157) and (10)

$$q(z)f'(\zeta)^2 = \frac{1}{2}\{f, \zeta\} \tag{15}$$

(the Schwarzian derivative $\{f, \zeta\}$ is defined by (3.147)).

It will be convenient to have the formulas for $c_0(z), \ldots, q(z)$ written up also in terms of the inverse mapping function $\varphi = f^{-1}$,

$\varphi : \Omega \to \mathbb{D}$.

Thus

$$c_0(z) = \log(1 - |\varphi(z)|^2) - \log|\varphi'(z)| \quad , \tag{16}$$

$$e^{-c_0(z)} = \frac{|\varphi'(z)|}{1-|\varphi(z)|^2} \quad , \tag{17}$$

$$c_1(z) = -\frac{\overline{\varphi(z)}\cdot\varphi'(z)}{1-|\varphi(z)|^2} - \frac{1}{2}\frac{\varphi''(z)}{\varphi'(z)} \quad , \tag{18}$$

$$\varkappa(z) \equiv -4 \quad , \tag{19}$$

$$q(z) = -\frac{1}{2}\{\varphi, z\} \, . \tag{20}$$

From (13) we see that the zeroes of $c_1(z)$, which we are particularly interested in, are exactly the points $z = f(\zeta)$ for which $\zeta \in \mathbb{D}$ solves

$$\frac{f''(\zeta)}{f'(\zeta)} = \frac{2\bar{\zeta}}{1-|\zeta|^2} \quad . \tag{21}$$

In particular $z = f(0)$ is a zero for $c_1(z)$ if and only if $f''(0) = 0$.

It is interesting to choose $\zeta = 0$ in (12) and (13). Thus with

$$a = f(0) , \tag{22}$$

and (assuming $a \neq \infty$) with the normalization

$$f'(0) > 0 \tag{23}$$

we have

$$e^{c_0(a)} = f'(0) , \tag{24}$$

$$c_1(a) = \frac{1}{2}\frac{f''(0)}{f'(0)} \quad . \tag{25}$$

Therefore the Taylor development of f at the origin begins

$$f(\zeta) = a + e^{c_0(a)} \cdot \zeta + c_1(a) e^{2c_0(a)} \cdot \zeta^2 + \ldots \quad . \tag{26}$$

(Here one can of course compute more terms. The next two turn out to be

$$\left(\frac{3}{2} c_1(a)^2 + c_2(a) \right) e^{3c_0(a)} \cdot \zeta^3 ,$$

$$\left(\frac{8}{3} c_1(a)^3 + 4\, c_1(a)\, c_2(a) + c_3(a) \right) e^{4c_0(a)} \cdot \zeta^4 \quad .)$$

Observe that given Ω and $a \in \Omega \setminus \{\infty\}$ there is a unique mapping function $f = f_a : \mathbb{D} \to \Omega$ satisfying (22) and (23). In [H] the function $e^{c_0(a)}$ (also called the mapping radius) is studied with the development (26) of $f_a(\zeta)$ taken as a starting-point, that is with (24) as a definition of $e^{c_0(a)}$.

If $\infty \notin \Omega$ we can apply the well-known coefficient inequalities for univalent functions to (26). For example the inequality " $|a_2| \leq 2$" (for univalent $F(z) = z + a_2 z^2 + a_3 z^3 + \dots$; see e.g. [A] Theorem 5-2 or [NEH] Chap V, sec 8) shows that

$$|c_1(a)e^{2c_0(a)}| \leq 2e^{c_0(a)} , \tag{27}$$

that is

$$|c_1(a)| \leq 2e^{-c_0(a)} . \tag{28}$$

Thus the constant B in (3.93) can be chosen as $B = 2$ (and no less) as claimed on p. 45 .

c) The differential equation $\Delta u = e^u$ and a differential equation for the inverse mapping functions

Next we shall consider equation (14),

$$\varkappa(z) \equiv -4 . \tag{29}$$

By the definition (3.150) of $\varkappa(z)$ this is a partial differential equation for $c_0(z)$, namely

$$\Delta c_0 = -4e^{-2c_0} . \tag{30}$$

From Proposition 3.3 we get boundary conditions to (30) . Assume for simplicity that $\infty \notin \Omega$. Then (3.64) gives $c_0(z) \to -\infty$ as $z \to \partial\Omega$ in the precise sense that

$$c_0(z) \leq \log 4d(z) , \tag{31}$$

where $d(z)$ denotes the distance to the boundary. (30) looks nicer if we make the substitution

$$u(z) = -2c_0(z) + \log 8 . \tag{32}$$

This u can be regarded as the potential function u in (3.50) having chosen units and constants appropriately. In terms of u (30) and (31) become

$$\Delta u = e^u \quad , \tag{33}$$

$$u(z) \geq - 2\log d(z) - \log 2 \; . \tag{34}$$

To get a proper boundary condition out of (34) we relax it to

$$u(z) \geq - 2 \log d(z) + O(1) \quad , \tag{35}$$

$O(1)$ denoting a bounded function in Ω.

In Appendix 2 we show that the problem

$$\begin{cases} \Delta u = e^u \\ u(z) \geq - 2 \log d(z) + O(1) \end{cases} \tag{36}$$

has no other solutions than the solution (32). Therefore (36) characterizes, and can be used to determine, the function $u(z)$ and, together with (3.46))

$$F = \text{grad } u \quad , \tag{37}$$

the vector field F. The unique solution of (36) may also be characterized as the maximal solution of (33) alone. This also follows from Appendix 2.

Suppose we have determined u, by (36) for example. Then we can compute the function

$$q(z) = \frac{\partial^2 c_0}{\partial z^2} + \left(\frac{\partial c_0}{\partial z} \right)^2 = - \frac{1}{2} \frac{\partial^2 u}{\partial z^2} + \frac{1}{4}\left(\frac{\partial u}{\partial z} \right)^2 \quad , \tag{38}$$

which as we already have remarked (p. 62) is holomorphic (Ω being simply connected). $q(z)$ can on the other hand also be expressed in terms of any inverse Riemann mapping function $\varphi : \Omega \to \mathbb{D}$, namely by

$$q(z) = - \frac{1}{2} \{ \varphi, z \} \; . \tag{39}$$

We can look upon (39) as a differential equation to be solved for $\varphi(z)$ when $q(z)$ is given. Differential equations of this kind are well studied*) and the solutions of (39) are exactly the functions of the form

*) see for example [NEH] Ch V, sec 7 or [HIL]

$$\varphi(z) = \frac{u_1(z)}{u_2(z)} \quad , \tag{40}$$

where $u_1(z)$ and $u_2(z)$ are linearly independent solutions of

$$u''(z) - q(z)u(z) = 0 \quad . \tag{41}$$

It follows that solving (39) we do not only get one mapping function $\varphi : \Omega \to \mathbb{D}$ but also all functions of the form $\frac{a\varphi(z) + b}{c\varphi(z) + d}$, that is we get all functions mapping Ω conformally onto half-planes and discs.

The steps

$$\Omega \mapsto u \mapsto q \mapsto \varphi \tag{42}$$

constitute an interesting programme for computing the inverse Riemann mapping functions for a given domain. When Ω is multiply connected this programme leads to the multiple-valued inverse φ of the universal covering map (see p. 67) and in fact one of the early attempts (proposed by H.A. Schwarz and later brought to an end by Picard, Poincaré and Bieberbach; see [BB]) to prove the famous uniformization theorem ran along these lines.

d) <u>$\Delta u \neq e^u$ for multiply connected domains.</u>

The derivation of (33) depended strongly on Ω being simply connected. In fact, having an equation of the kind

$$\Delta u = \text{const}\cdot e^u \tag{43}$$

is equivalent to the curvature $\varkappa$ being constant, and, as noticed on page 62 this just reflects the property that any two points of Ω can be mapped onto each other by a conformal self-mapping of Ω. Since this property does not hold for multiply connected regions one cannot expect equations of the kind (43) to hold for such regions. It also seems very hard to get any physical interpretation for equation (43).

When Ω is multiply connected the potential that is relevant in the hydrodynamical situation is not u itself, but rather

$$u_\beta(z) = - 2c_{\beta 0}(z) + \log 8 \ . \tag{44}$$

There is however no reason why this function either should satisfy an equation

$$\Delta u_\beta = \text{const.}\, e^{u_\beta} \quad . \tag{45}$$

At least it cannot do so for more than one choice of $\beta = (\beta_1, \cdots, \beta_m)$, since the left member of (45) is found to be independent of β (this is seen directly from (3.192) for example), while the right member depends effectively on β.

Let us be a little more precise concerning the equation (43) for multiply connected Ω. Suppose that u (or u_β) satisfies

$$\Delta u = \alpha \cdot e^u \quad , \tag{46}$$

where α is a constant, (necessarily positive, by (3.194)). Then $u_1 = u - \log \alpha$ (respectively, $u_1 = u_\beta - \log \alpha$) satisfies

$$\Delta u_1 = e^{u_1} \quad , \tag{47}$$

and also the boundary condition

$$u_1(z) \geq -2 \log d(z) + O(1) \ . \tag{48}$$

From Appendix 2, (p.106), it follows that u_1 then must be

$$u_1(z) = \tilde{u}(z) = -2\tilde{c}_0(z) + \log 8 \quad , \tag{49}$$

where $\tilde{u}$ and $\tilde{c}_0$ are defined with respect to the universal covering surface $\tilde{\Omega}$ of Ω. This gives

$$c_0(z) = \tilde{c}_0(z) - \frac{1}{2} \log \alpha \tag{50}$$

(respectively, $c_{\beta 0}(z) = \tilde{c}_0(z) - \frac{1}{2} \log \alpha$).

Now, it is rather easy to see that

$$c_0(z) - \tilde{c}_0(z) \to 0 \quad \text{as} \quad z \to \partial\Omega \ , \tag{51}$$

at least if the boundary $\partial\Omega$ is sufficiently smooth, say consists of analytic curves. Indeed, let $\tilde{g}(\tilde{z},\tilde{\zeta})$ denote the Green's function for $\tilde{\Omega}$, which we, for $\tilde{z}$ and $\tilde{\zeta}$ close to each other on $\tilde{\Omega}$, may regard as a function of their projections z and ζ on Ω. Then we have

$$c_0(z) - \tilde{c}_0(z) = \lim_{\zeta\to z} (g(z,\zeta) - \tilde{g}(z,\zeta)) . \tag{52}$$

If $\partial\Omega$ is analytic then $g(z,\zeta) - \tilde{g}(z,\zeta)$ extends across $\partial\Omega$ to a function which is regular harmonic in z and ζ (close to each other) in a neighbourhood of $\Omega \cup \partial\Omega$. Since $g(z,\zeta)$ and $\tilde{g}(z,\zeta)$ vanish for $z \in \partial\Omega$ it follows that $c_0(z) - \tilde{c}_0(z)$ vanishes on $\partial\Omega$, as asserted.

From (51) we conclude that the constant $\frac{1}{2}\log\alpha$ in (50) must be zero, that is that $c_0(z) = \tilde{c}_0(z)$. This however contradicts the second inequality in

$$c_{\beta 0}(z) \geq c_0(z) > \tilde{c}_0(z) \quad ; \tag{53}$$

(53) is just (3.180) combined with (3.187). (In writing strict inequality in (53) we have assumed that the connectivity of Ω is strictly greater than one.) This contradiction shows that (46) cannot hold if Ω is multiply connected. The circumstance that we carried out the argument only for regions with analytic boundary does not mean any loss of generality since (46) is equivalent to the conformal invariant $\varkappa(z)$ being constant ($\varkappa(z) = -4\alpha$) and all regions we consider are conformally equivalent to such with analytic boundary.

The corresponding reasoning for u_β instead of u, however, does not completely rule out the possibility that u_β satisfies an equation (46) for some choice of β. In place of (51) one obtains

$$c_{\beta 0}(z) - \tilde{c}_0(z) \to a_{jj} \quad \text{as} \quad z \to \Gamma_j \quad (j = 1,\dots,m), \tag{54}$$

where $\Gamma_1,\dots,\Gamma_m$ are the components of $\partial\Omega$ and a_{kj} the matrix elements in (3.15). (54) combined with (50) and (53) yields

$$a_{jj} = -\frac{1}{2}\log\alpha > 0 \quad , \quad j = 1,..,m \quad , \tag{55}$$

which does not give rise to any contradiction in general. For some particular choices of β, namely if $\beta_j = -2\pi\delta_{kj}$ $(j = 1,\dots,m)$ for some k, (55) is however in conflict with (3.21), so at least for such β (46) is disproved. We also see from (55) that if (46) is true for some β

then necessarily $0 < \alpha < 1$. In terms of the curvature $\varkappa_\beta$ this means that if it is constant it must lie in the interval $-4 < \varkappa_\beta < 0$.

Let us finally point out that equation (46) is not quite so special as it looks. What we mean is that if u (or u_β) did satisfy an equation of the type

$$\Delta u = f(u) \tag{56}$$

for every region (or for every region of some conformal type), with the function f possibly depending on the region, then this equation must necessarily be of the kind (46). For (56) can be written

$$\frac{\Delta u}{e^u} = f_1(u) \tag{57}$$

or, switching over to c_0 and $\varkappa$,

$$\varkappa = f_2(c_0) \quad , \tag{58}$$

where f_1 and f_2 are some other functions. (58) means roughly speaking that each level line of the function c_0 is also a level line of $\varkappa$, and it is rather easy to see from the fact that $\varkappa$ and c_0 transform in different ways under conformal mappings (3.137) resp. (3.151) that this cannot be the case for all regions in a whole conformal class unless $\varkappa$ and f_2 are constant. And then we are back to the case of equation (46), as asserted.

e) A bounded starlike domain with several zeroes for $c_1(z)$

As we have seen in section III e) (p. 51) the function $c_1(z)$ (and $c_{\beta\Gamma}(z)$) always has at least one zero if the domain in question is bounded. We also indicated that for regions with certain geometrical properies the number of zeroes must be strictly greater that one. In this section we shall give a more specific example of a region with several zeroes for $c_1(z)$. The region we are going to construct has the further property of being starlike with respect to the origin (every point of it can "be seen from the origin", that is the line segment connecting the origin with any other point in the region is itself entirely contained in the region.) This at once frustrates a conceivable generalization of the result we are going to prove in section V, namely that for convex regions (other than infinite strips) $c_1(z)$ can only have one zero.

Physical intuition suggests one should try regions looking like that in fig. 4.1, with the zeroes for $c_1(z)$ at the three indicated points (the middle one being a saddle point for $c_0(z)$, the other two minima).

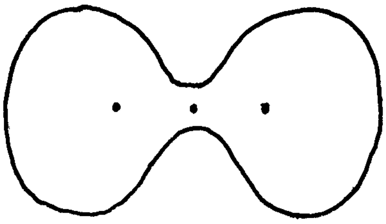

Fig. 4.1

Such a domain is obtained by, for $R > 1$, taking the following mapping function from the unit disc:

$$f(\zeta) = \zeta \cdot \frac{1-R^2}{\zeta^2-R^2} = \frac{1-R^2}{2} \left[\frac{1}{\zeta-R} + \frac{1}{\zeta+R} \right] . \tag{59}$$

One easily checks that f is univalent in $\mathbb{D}$ for all $R > 1$. Moreover, $\Omega \subset \mathbb{D}$ (where $\Omega = f(\mathbb{D})$) and $\pm 1 \in \partial\Omega$ for all $R > 1$. As $R \to \infty$, $f(\zeta) \to \zeta$ uniformly on $\mathbb{D}$. Hence Ω approaches $\mathbb{D}$ as $R \to \infty$.
As $R \to 1$, $f(\zeta) \to 0$. On the other hand, still as $R \to 1$, $\frac{1}{1-R^2} f(\zeta) \to \frac{\zeta}{\zeta^2-1} = \frac{1}{\zeta-1/\zeta}$, and the function $\frac{1}{\zeta-1/\zeta}$ maps $\mathbb{D}$ onto the slit domain $\mathbb{C} \setminus \{ iy : |y| \geq \frac{1}{2} \text{ (y real)} \}$. This gives an idea of how Ω behaves as $R \to 1$ (cf. fig. 4.2)

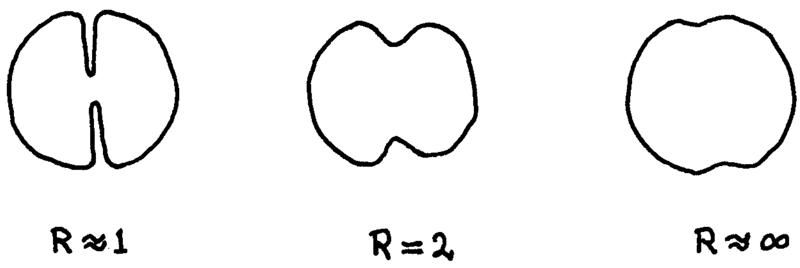

Fig. 4.2

Now, computations give

$$f'(\zeta) = (R^2-1) \frac{\zeta^2+R^2}{(\zeta^2-R^2)^2} = \frac{R^2-1}{2} \left[\frac{1}{(\zeta-R)^2} + \frac{1}{(\zeta+R)^2} \right] , \tag{60}$$

$$\frac{f''(\zeta)}{f'(\zeta)} = -2\zeta \cdot \frac{\zeta^2+3R^2}{\zeta^4-R^4} , \tag{61}$$

hence, by (11), (12) and (13) :

$$c_0(z) = \log(R^2-1) + \log \frac{(1-|\zeta|^2)|R^2+\zeta^2|}{|R^2-\zeta^2|^2} \quad , \tag{62}$$

$$e^{-c_0(z)} = \frac{|R^2-\zeta^2|^2}{(R^2-1)(1-|\zeta|^2)|R^2+\zeta^2|} \quad , \tag{63}$$

and

$$c_1(z)f'(\zeta) = -\frac{\bar{\zeta}}{1-|\zeta|^2} - \frac{\zeta\cdot(\zeta^2+3R^2)}{\zeta^4-R^4} \quad . \tag{64}$$

Thus, the zeroes for $c_1(z)$ are the points $z = f(\zeta)$ for which

$$\frac{\bar{\zeta}}{1-|\zeta|^2} + \frac{\zeta\cdot(\zeta^2+3R^2)}{\zeta^4-R^4} = 0 \quad . \tag{65}$$

One solution of (65) clearly is

$\zeta = 0$.

Considering only real values of ζ (probably there are no non-real solutions of (65)), the other zeroes are obtained from

$$\zeta^4 - R^4 + (1-\zeta^2)(\zeta^2+3R^2) = 0 \quad ,$$

which gives

$$-\zeta^2\cdot(3R^2-1) + 3R^2 - R^4 = 0 \quad ,$$

$$\zeta = \pm\sqrt{\frac{3R^2-R^4}{3R^2-1}} = \pm\sqrt{\frac{3-R^2}{3-1/R^2}} \quad .$$

Thus, for $1 < R < \sqrt{3}$, equation (65) has the three real solutions

$$\zeta = 0,\ \pm\sqrt{\frac{3-R^2}{3-1/R^2}} \quad . \tag{66}$$

The corresponding points $z = f(\zeta)$ in Ω are

$$z = 0,\ \pm\frac{1}{4}\sqrt{(3-R^2)(3-\frac{1}{R^2})} \quad . \tag{67}$$

One finds that (for $1 < R < \sqrt{3}$)

$$0 < \frac{1}{4}\sqrt{(3-R^2)(3-\frac{1}{R^2})} < \frac{1}{2} \quad , \tag{68}$$

and that

$$\frac{1}{4}\sqrt{(3-R^2)(3-\frac{1}{R^2})} \to \begin{cases} 0 & \text{as} \quad R \to \sqrt{3} \,, \\ \frac{1}{2} & \text{as} \quad R \to 1 \,, \end{cases} \tag{69}$$

in accordance with what one expects from the figures 4.1 and 4.2.

Finally, to show that Ω is starlike (with respect to the origin), one only has to check that

$$\operatorname{Re} \frac{\zeta f'(\zeta)}{f(\zeta)} \geq 0 \qquad \text{for} \quad |\zeta| < 1 \,. \tag{70}$$

(See for example [NEH] Chap V, sec. 8.)
We have, by (59) and (60)

$$\frac{\zeta f'(\zeta)}{f(\zeta)} = \frac{R^2+\zeta^2}{R^2-\zeta^2} \,,$$

from which (70) follows immediately.

V <u>Convex Regions</u>

a) <u>Only one zero for $c_1(z)$.</u>

We have seen that the domain function $c_1(z)$ always has at least one zero if the domain Ω in question is bounded, and that the number of zeroes in general is greater than one. The object of this section is to show that for convex regions, however, the number of zeroes is never greater than one, with the exception of infinite strips. This result was proven already 1950 by Hans H. Haegi ([H], Satz 4), in the formulation that the "mapping radius" $e^{c_0(z)}$ has at most one stationary point when the domain is convex and not an infinite strip.* Nevertheless we have preferred to include our own proof of this result, which is quite different from Haegi's proof.

Our proof goes in two steps. First we reformulate the statement of the result (Theorem 5.1) so as to become essentially the statement that a certain anti-analytic mapping has at most one fixed point (Proposition 5.5 or the statment (43)). This reformulation makes use of Schwarz's lemma. Then the reformulated statement is proved, and this proof essentially

* See p. 109.

consists of another application of Schwarz's lemma. In section V b) we discuss in some generality how the reformulated fixed point statement is related to the "invariant form of" Schwarz's lemma. Here also our metric $ds = e^{-c_0(z)} |dz|$ comes in.

Theorem 5.1: Suppose that Ω is convex but not an infinite strip. Then $c_1(z)$ has at most one zero in Ω.

An infinite strip is a domain of the kind

$$\Omega = \{ z \in C : | \operatorname{Im}(az + b)| < 1 \} \quad , \tag{1}$$

where $a,b \in C$, $a \neq 0$. That this case has to be excluded is clear since $c_1(z)$ for symmetry reasons must vanish along the whole symmetry line for such a domain.

A convex domain is necessarily simply connected. To prove Theorem 5.1 we shall need the following well known characterization of convex regions in terms of the (any) Riemann mapping function from the unit disc:

Lemma 5.2: Suppose $f : \mathbb{D} \to \Omega$ is a conformal isomorphism. Then Ω is convex if and only if

$$\operatorname{Re} \left[\zeta \frac{f''(\zeta)}{f'(\zeta)} + 1 \right] \geq 0 \tag{2}$$

for $\zeta \in \mathbb{D}$.

For a proof of this lemma we refer to literature, say [A], section 1-3.

Now, let Ω be simply connected and let

$$f : \mathbb{D} \to \Omega$$

be any conformal isomorphism. By (4.21) the zeroes for $c_1(z)$ in Ω correspond exactly, under $z = f(\zeta)$, to the solutions in $\mathbb{D}$ of the equation

$$\frac{f''(\zeta)}{f'(\zeta)} = \frac{2\bar{\zeta}}{1-|\zeta|^2} \tag{3}$$

Multiplying (3) by ζ, thereby introducing an extra solution $\zeta = 0$, and adding 1, gives the new equation

$$\zeta \frac{f''(\zeta)}{f'(\zeta)} + 1 = \frac{1 + |\zeta|^2}{1 - |\zeta|^2} \quad . \tag{4}$$

Put

$$F(\zeta) = \zeta \frac{f''(\zeta)}{f'(\zeta)} + 1 \quad . \tag{5}$$

Thus $F(\zeta)$ is holomorphic in $\mathbb{D}$ and has the following properties:

1. $F(0) = 1$ (6)
2. Ω is convex $\Leftrightarrow$ Re $F(\zeta) \geq 0$ in $\mathbb{D}$ (7)
3. the zeroes of $c_1(z)$ in Ω correspond exactly, under $z = f(\zeta)$, to the solutions, besides the trivial solution $\zeta = 0$, of the equation

$$F(\zeta) = \frac{1+|\zeta|^2}{1-|\zeta|^2} \tag{8}$$

in $\mathbb{D}$.

Conversely, starting from any function $F(\zeta)$ holomorphic in $\mathbb{D}$ and satisfying $F(0) = 1$, the equation (5) can be solved for $f(\zeta)$:

$$\frac{f''(\zeta)}{f'(\zeta)} = \frac{F(\zeta)-1}{\zeta} \ , \tag{9}$$

$$\log f'(\zeta) = \int_0^\zeta \frac{F(z)-1}{z}\, dz + \log A \ , \qquad A \neq 0 \ , \tag{10}$$

$$f(\zeta) = A \cdot \int_0^\zeta \exp \left\{ \int_0^t \frac{F(z)-1}{z}\, dz \right\} dt + B. \tag{11}$$

One sees that all solutions, $f(\zeta)$, are locally univalent (i.e. $f'(\zeta) \neq 0$) in $\mathbb{D}$, hence determine, possibly non-schlicht, domains in $\mathbb{C}$. The significance of the integration constants A and B is that "the" solution domain is determined only up to a rigid transformation in C (arg A and B) and a homothetic scale change ($|A|$).

In order to carry over the hypotheses in Theorem 5.1 to the function $F(\zeta)$, we must investigate the case when Ω is an infinite strip. Since the most general function mapping $\mathbb{D}$ onto the right half-plane Re $w > 0$ is

$$\zeta \mapsto w = \frac{a\zeta+\bar{a}}{-b\zeta+\bar{b}} \ , \qquad a\bar{b} + \bar{a}b > 0 \ , \tag{12}$$

and $\log w$ maps the right half-plane onto an infinite strip (namely $|\text{Im } z| < \frac{\pi}{2}$), the most general function mapping $\mathbb{D}$ conformally onto an infinite strip is

$$f(\zeta) = c\cdot\log\frac{a\zeta+\bar a}{-b\zeta+\bar b} + d \quad , \tag{13}$$

with $a\bar b + \bar a b > 0$, $c \neq 0$.
Some computations show that (13) yields

$$F(\zeta) = \frac{\bar a\bar b + ab\cdot\zeta^2}{\bar a\bar b + (a\bar b - \bar a b)\zeta - ab\cdot\zeta^2} \quad . \tag{14}$$

Conversely, if $F(\zeta)$ is of the form (14) all integrals $f(\zeta)$ of (5) are of the form (13), hence map $\mathbb{D}$ onto infinite strips. Therefore

4 $F(\zeta)$ is of the form (14) $\Leftrightarrow$ Ω is an infinite strip.

Now it follows from 1 - 4 that Theorem 5.1 is a consequence of

<u>Proposition 5.3</u>:

Suppose $F(\zeta)$ is holomorphic in $\mathbb{D}$,
$F(0) = 1$,
$\operatorname{Re} F(\zeta) \geq 0$, $\zeta \in \mathbb{D}$
$F(\zeta)$ is not of the form (14) .

Then the equation

$$F(\zeta) = \frac{1+|\zeta|^2}{1-|\zeta|^2}$$

has at most one solution in $\mathbb{D}$, besides the trivial solution $\zeta = 0$.

It will be convenient to perform a Möbius-transformation on Proposition 5.3. Namely, define

$$S(\zeta) = \frac{F(\zeta)-1}{F(\zeta)+1} = \frac{\zeta f''(\zeta)}{\zeta f''(\zeta) + 2f'(\zeta)} \tag{16}$$

Then

$$F(\zeta) = \frac{1+S(\zeta)}{1-S(\zeta)} \quad , \tag{17}$$

$$F(0) = 1 \Leftrightarrow S(0) = 0 \; , \tag{18}$$

$$F(\zeta) = \frac{1+|\zeta|^2}{1-|\zeta|^2} \Leftrightarrow S(\zeta) = |\zeta|^2 \; . \tag{19}$$

Moreover, since $F \mapsto S = \frac{F-1}{F+1}$ maps the half-plane $\operatorname{Re} F \geq 0$ onto the unit disc $|S| \leq 1$:

$$\operatorname{Re} F(\zeta) \geq 0 \iff |S(\zeta)| \leq 1 \quad . \tag{20}$$

Finally, (13) becomes

$$S(\zeta) = \frac{2ab\cdot\zeta^2 - (a\bar{b} - \bar{a}b)\zeta}{2\overline{ab} + (a\bar{b} - \bar{a}b)\zeta} \quad . \tag{21}$$

Thus Proposition 5.3 is equivalent to

Proposition 5.4:

Suppose $S(\zeta)$ is holomorphic in $\mathbb{D}$
$S(0) = 0$
$|S(\zeta)| \leq 1 \quad , \quad \zeta \in \mathbb{D}$
$S(\zeta)$ is not of the form (21) .

Then the equation

$$S(\zeta) = |\zeta|^2$$

has at most one solution in $\mathbb{D}$, besides the trivial solution $\zeta = 0$.

We shall rewrite the problem one last time before solving it. Namely, put

$$T(\zeta) = \frac{S(\zeta)}{\zeta} \, , \quad \zeta \in \mathbb{D} \, . \tag{22}$$

Then, by Schwarz's lemma, the first three hypotheses in Proposition 5.4 are equivalent to

$$\left.\begin{array}{l} T(\zeta) \text{ is holomorphic in } \mathbb{D} \text{ and} \\ |T(\zeta)| \leq 1 \quad . \end{array}\right. \tag{23}$$

(21) becomes

$$T(\zeta) = \frac{2ab\cdot\zeta - (a\bar{b} - \bar{a}b)}{(a\bar{b} - \bar{a}b)\cdot\zeta + 2\overline{ab}} \tag{24}$$

or

$$T(\zeta) = \frac{A\cdot\zeta + B}{\bar{B}\cdot\zeta + \bar{A}} \quad , \tag{25}$$

where

$$\begin{cases} A = 2ab \\ B = \bar{a}b - a\bar{b} \quad , \end{cases} \tag{26}$$

$$|A|^2 - |B|^2 = (a\bar{b} + \bar{a}b)^2 > 0 \quad . \tag{27}$$

(25) means that T is a Möbius-transformation, mapping $\mathbb{D}$ onto itself.

In the equation $S(\zeta) = |\zeta|^2$ we have divided out the trivial solution $\zeta = 0$, and the equation becomes

$$T(\zeta) = \bar{\zeta} \quad . \tag{28}$$

Therefore, the following proposition implies Proposition 5.4.

Proposition 5.5:

Suppose $T(\zeta)$ is holomorphic in $\mathbb{D}$ with

$$|T(\zeta)| \leq 1 \quad . \tag{29}$$

Then the equation

$$T(\zeta) = \bar{\zeta} \tag{30}$$

has at most one solution in $\mathbb{D}$, unless T is a Möbius-transformatiom (mapping $\mathbb{D}$ onto itself).

When T is a Möbius-transformation on $\mathbb{D}$, that is when

$$T(\zeta) = \frac{A\zeta + B}{\bar{B}\zeta + \bar{A}} \;, \quad |A|^2 - |B|^2 > 0 \;, \tag{31}$$

the solutions of (28) either consists of just two points on $\partial\mathbb{D}$ or else consists of a whole circular arc C (fig. 5.3), intersecting $\partial\mathbb{D}$ at right angles. The latter case occurs exactly when

$$\operatorname{Re} B = 0 \;, \tag{32}$$

and it is not hard to see that these are exactly the Möbius-transformations which arise from our process $f \mapsto F \mapsto S \mapsto T$, where f maps $\mathbb{D}$ onto an infinite strip, Ω. In this case $\mathbb{D} \cap C$ corresponds to the symmetry-line of Ω.

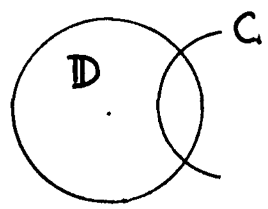

Fig. 5.3

Proof of Proposition 5.5:

Suppose (30) has two solutions, ζ_1 and ζ_2, $\zeta_1 \neq \zeta_2$. Then we must prove that T is a Möbius-transformation.

Consider first the case that $\zeta_1 = 0$. This means that $T(0) = 0$. Then Schwarz's lemma says that

$$|T(\zeta)| \leq |\zeta| \qquad \text{for all } \zeta \in \mathbb{D}, \tag{33}$$

and that equality in (33) can hold at a point in $\mathbb{D} \setminus \{0\}$ (if and) only if

$$T(\zeta) \equiv \alpha \cdot \zeta, \qquad |\alpha| = 1. \tag{34}$$

But ζ_2 solves (30) means that equality in (33) holds at $\zeta = \zeta_2 \in \mathbb{D} \setminus \{0\}$. Thus T is a Möbius-transformation, (34).

If $\zeta_1 \neq 0$, let U be a Möbius-transformation (mapping $\mathbb{D}$ onto itself) with $U(0) = \zeta_1$. For example

$$U(\zeta) = \frac{\zeta + \zeta_1}{1 + \bar{\zeta}_1 \cdot \zeta}. \tag{35}$$

Then (30) is equivalent to (with $\zeta = U(z)$)

$$\overline{TU(z)} = U(z), \tag{36}$$

or

$$T_1(z) = \bar{z}, \tag{37}$$

where

$$T_1(z) = \overline{U^{-1}(\overline{TU(z)})}. \tag{38}$$

Now, (37) has the two solutions $z_1 = U^{-1}(\zeta_1) = 0$ and $z_2 = U^{-1}(\zeta_2) \neq 0$, and since clearly $|T_1(z)| \leq 1$ in $\mathbb{D}$, the previous argument (Schwarz's lemma) shows that T_1 is a Möbius-transformation. Thus also

$$T(\zeta) = \overline{U(\overline{T_1 U^{-1}(\zeta)})} \tag{39}$$

is a Möbius-transformation, which proves the Proposition.

By this also Proposition 5.4, Proposition 5.3 and Theorem 5.1 are proved.

One might ask if the conclusion of Theorem 5.1 holds true for some wider class of domains that just the convex ones. It seems however hard to find any very natural such class, since the perhaps most natural candidate, the starlike domains, has turned out to fail to have the desired property (section IV e)).

There is however another kind of generalization of Theorem 5.1 which "should be" true, although I have not been able to prove it. Namely, one expects that all level lines $c_0(z) = \text{constant}$, that is all orbits of a freely moving vortex, are convex curves *) if Ω is convex.
Since $c_0(z)$ has the same level lines as the function $u(z) = -2c_0(z) + \log 8$ ((4.32)), and this function is characterized as the unique solution of

$$\begin{aligned} \Delta u &= e^u \quad \text{in } \Omega \\ u &= +\infty \quad \text{on } \partial\Omega \end{aligned} \tag{40}$$

(more precisely stated in (4.36)), this conjecture could also be stated: Are all level lines of the unique solution of (40) convex if Ω is convex? In this form the conjecture makes sense in $\mathbb{R}^n$ for arbitrary $n \geq 2$. This conjecture is suggested by Harold S. Shapiro.

b) <u>A fixed point lemma.</u>

There is another very elegant way to look at (and prove) Proposition 5.5 based on the so called invariant form of Schwarz's lemma or, better, on Lemma 3.7. Namely, observe first that the assumption (29) without serious loss of generality can be replaced by

$$|T(\zeta)| < 1 , \quad \zeta \in \mathbb{D} , \tag{41}$$

since the Proposition is trivial if $|T(\zeta)| = 1$ for some $\zeta \in \mathbb{D}$.
Rewriting (30) as

$$\overline{T(\zeta)} = \zeta \tag{42}$$

Proposition 5.5 therefore can be formulated

> An anti-analytic mapping
>
> $$\overline{T} : \mathbb{D} \to \mathbb{D} \tag{43}$$
>
> can have at most one fixed point unless it is an anti-Möbius-transformation

This is a special case of

<u>Lemma 5.6</u> ("fixed point lemma"): Let Ω be a domain in $\mathbb{P}$, and let

---

*) By this is meant that the sets $\{ z \in \Omega : c_0(z) > \lambda \}$ are convex.

$$f : \Omega \to \Omega$$

be an analytic or anti-analytic mapping. Then f can have at most one fixed point unless it is a conformal, or an anti-conformal, isomorphism.

To prove this, consider first the more general situation that f is an analytic or anti-analytic mapping between two different domains D and Ω,

$$f : D \to \Omega .$$

Letting ζ, $\tilde{c}_0(\zeta), \ldots$ refer to D and z, $c_0(z), \ldots$ to Ω, we have the metrics

$$d\tilde{s} = e^{-\tilde{c}_0(\zeta)} |d\zeta| \quad \text{and} \tag{44}$$

$$ds = e^{-c_0(z)} |dz| \tag{45}$$

on D and Ω respectively. Recall that Lemma 3.7 says that f is locally distance-decreasing with respect to these metrics *), the decrease being everywhere strict if f is not an isomorphism.
What we need is a global version of this.

Therefore, consider two points $\zeta_1, \zeta_2 \in D$ and let $z_j = f(\zeta_j) \in \Omega$, $j = 1,2$. The distance between ζ_1 and ζ_2 in the metric (44) is by definition

$$\tilde{\delta}(\zeta_1, \zeta_2) = \inf_{\tilde{\gamma} \in \tilde{\Gamma}} \int_{\tilde{\gamma}} d\tilde{s} = \inf_{\tilde{\gamma} \in \tilde{\Gamma}} \int_{\tilde{\gamma}} e^{-\tilde{c}_0(\zeta)} |d\zeta| \ , \tag{46}$$

where $\tilde{\Gamma}$ denotes the class of all rectifiable curves in D, connecting ζ_1 and ζ_2. $\delta(z_1, z_2)$ and Γ are defined similarly. If $\tilde{\gamma} \in \tilde{\Gamma}$ then $f(\tilde{\gamma}) \in \Gamma$ **). Thus $f(\tilde{\Gamma}) \subset \Gamma$. By Lemma 3.7 we have

$$e^{-c_0(z)} |dz| \leq e^{-\tilde{c}_0(\zeta)} |d\zeta| \ , \quad z = f(\zeta) \ , \tag{47}$$

---

*) Actually, Lemma 3.7 states this only for analytic mappings, but it is easy to see that it holds also for anti-analytic mappings, by applying it to $\bar{f} : D \to \bar{\Omega}$ if f is anti-analytic. Here the bars denote complex conjugation.

**) We think of $\tilde{\gamma}, \gamma$ as parametrized curves rather than subsets of D, Ω. Thus $\tilde{\gamma}$ is a function from some parameter interval to D, and $f(\tilde{\gamma})$ is by definition the composed function $f \circ \tilde{\gamma}$.

and, integrating along $\tilde{\gamma}$,

$$\int_{f(\tilde{\gamma})} e^{-c_0(z)} |dz| \le \int_{\tilde{\gamma}} e^{-\tilde{c}_0(\zeta)} |d\zeta| \quad . \tag{48}$$

Taking infima therefore gives

$$\delta(z_1,z_2) = \inf_{\gamma \in \Gamma} \int_\gamma ds \le \inf_{\gamma \in f(\tilde{\Gamma})} \int_\gamma ds = \inf_{\tilde{\gamma} \in \tilde{\Gamma}} \int_{f(\tilde{\gamma})} ds \le$$

$$\le \inf_{\tilde{\gamma} \in \tilde{\Gamma}} \int_{\tilde{\gamma}} d\tilde{s} = \tilde{\delta}(\zeta_1,\zeta_2) \quad . \tag{49}$$

Lemma 3.7 moreover says that unless f is an isomorphism, the inequality (47) is everywhere strict. In that case also the inequality (49) must be strict, as is easily seen. Therefore, the following global counterpart of Lemma 3.7 is proved.

<u>Lemma 5.7:</u> Suppose $f : D \to \Omega$ is analytic (or anti-analytic). Then, with the above notations,

$$\delta(z_1,z_2) \le \tilde{\delta}(\zeta_1,\zeta_2) \tag{50}$$

for all $\zeta_1, \zeta_2 \in D$, $z_j = f(\zeta_j)$.

If equality holds for some pair ζ_1, ζ_2, $\zeta_1 \neq \zeta_2$, then f is an isomorphism (and equality holds for all ζ_1, ζ_2).

Now, having Lemma 5.7 the proof of Lemma 5.6 just consists of the observation that if z_1 and z_2 are two fixed points for f, then $\delta(f(z_1),f(z_2)) = \delta(z_1,z_2)$, and the conclusion follows immediately from Lemma 5.7.

If D and Ω are simply connected the metrics (44) and (45) coincide with the Poincaré metrics for D and Ω, and Lemma 5.7 (and also Lemma3.7) become the same as what is called the invariant form of Schwarz's lemma. When D, Ω are allowed to be multiply connected the inequalities in Lemma 3.7 and 5.7 are true also for the Poincaré metrics, but the assertions about the cases of equality are false. For example if Ω is multiply connected, $D = \mathbb{D}$ and $f : \mathbb{D} \to \Omega$ is the universal covering map, then the definition of the Poincaré metric on Ω (see p.67f) just amounts to saying that f is locally an isometry, despite that it is not an isomorphism. Therefore, Lemma 5.6 cannot be proved by using the Poincare metric in the multiply connected case instead of the metric (45).

c) <u>The zeroes for c_1 from a function-theoretic point of view.</u>

We shall finish this section by giving some function-theoretic versions of the results that $c_1(z)$ has at least one zero if Ω is bounded and at most one if Ω is convex, not an infinite strip (p.51 resp. Theorem 5.1). Namely, let Ω be a given simply connected domain and let $f : \mathbb{D} \to \Omega$ be any Riemann mapping function. By the formula ((4.13))

$$c_1(z)f'(\zeta) = -\frac{\bar{\zeta}}{1-|\zeta|^2} + \frac{1}{2}\cdot\frac{f''(\zeta)}{f'(\zeta)}\ , \quad z = f(\zeta)\ , \tag{51}$$

we see that $z = f(0)$ is a zero for $c_1(z)$ if and only if $f''(0) = 0$. Since, when Ω is given, the quantities $z = f(0)$ and $\arg f'(0)$ can be prescribed arbitrarily and uniquely determine f, we have the following:

> Suppose Ω is bounded (and simply connected). Then the Riemann mapping function $f : \mathbb{D} \to \Omega$ can be chosen so that $f''(0) = 0$.
>
> Or:
>
> Suppose f is univalent and bounded in $\mathbb{D}$. Then there exists a Möbius-transformation $U : \mathbb{D} \to \mathbb{D}$ such that $(f \circ U)''(0) = 0$.

Reformulation of Theorem 5.1:

> Suppose Ω is convex, but not an infinite strip. Then there is at most one choice of the mapping function $f : \mathbb{D} \to \Omega$ for which
>
> $$\begin{cases} f'(0) > 0 \\ f''(0) = 0\ . \end{cases}$$
>
> Or, calling a univalent function convex univalent if $f(\mathbb{D})$ is a convex domain:
>
> Suppose f is convex univalent and not of the form (13). Then there is at most one Möbius-transformation $U : \mathbb{D} \to \mathbb{D}$ such that
>
> $$\begin{cases} (f \circ U)'(0) > 0 \\ (f \circ U)''(0) = 0\ . \end{cases}$$

VI Appendices

Appendix 1 : Notations in [S-O].

In the book [S-O] so called "capacity functions" are used to study various kinds of degeneracy questions for boundary components of Riemann surfaces. An example of a capacity function is the function

$$p(z,\zeta) = c_{\beta 0}(\zeta) - g_{\beta}(z,\zeta) \quad , \tag{1}$$

with say $\beta = (-2\pi, 0, \ldots, 0)$. It is characterized by being harmonic in z with the singularity

$$p(z,\zeta) = \log|z-\zeta| + O(|z-\zeta|) \tag{2}$$

at $z = \zeta$, and by being constant (as a function of z) on each boundary component, with the constants so chosen so that

$$\int_{\gamma} {}^{*}dp(\cdot,\zeta) = \begin{cases} 2\pi & \text{for} \quad \gamma = \Gamma_1 \\ 0 & \text{for} \quad \gamma = \Gamma_j , \quad j \neq 1 \end{cases} \tag{3}$$

(Γ_1 is the boundary component which correspond to the -2π in β). Since $g_{\beta}(z,\zeta) = 0$ for $z \in \Gamma_1$ (this follows from (3.11) and (3.13)) the boundary constant for $p(z,\zeta)$ on Γ_1 is $c_{\beta 0}(\zeta)$, that is

$$c_{\beta 0}(\zeta) = \lim_{z \to \Gamma_1} p(z,\zeta) \ . \tag{4}$$

Now, although the function $g_{\beta}(z,\zeta)$ exists only for a rather restricted class of domains, the combination $p(z,\zeta)$ of $g_{\beta}(z,\zeta)$ and $c_{\beta 0}(\zeta)$ can be defined on an arbitrary Riemann surface. If then (4) is taken as a definition of $c_{\beta 0}(\zeta)$ it can happen that $c_{\beta 0}(\zeta) \equiv +\infty$, or equivalently that the "capacity" $e^{-c_{\beta 0}(\zeta)}$ vanishes identically. Such a phenomenon indicates that the boundary component Γ_1 of the surface is degenerate in some sense (it is "weak"). This is the kind of phenomena studied in [S-O].

Thus our various domain functions $c_0(\zeta)$, $c_{\beta 0}(\zeta), \ldots$ are studied in [S-O], but with other applications than ours in mind. Despite that, [S-O] contains a lot of material which is of interest also in our context, for example extremal properties possessed by $c_0(\zeta)$, $c_{\beta 0}(\zeta), \ldots$ Since our notations disagree very badly with those in [S-O] (which are the standard ones) we have on the following few pages given a short "conversion table" between the relevant notations in [S-O] and those used here, to be of help if [S-O] is consulted. In that table all functions are supposed to refer to a region Ω which fulfils our regularity assumptions (p.27), with boundary components denoted $\Gamma_1, \ldots, \Gamma_m$.

Notation in [S-O]	Notation here	Comment
$p_\beta(z,\zeta)$	$c_0(\zeta) - g(z,\zeta)$	capacity function (β stands for the ideal boundary of Ω)
$p_{1\gamma}(z,\zeta)$	$c_{\beta 0}(\zeta) - g_\beta(z,\zeta)$	capacity function; γ is a component of $\partial\Omega$; if $\gamma=\Gamma_k$ then $\beta_j=-2\pi\delta_{kj}$ $(j=1,\dots,m)$
$k_\beta(\zeta)$	$c_0(\zeta)$	Robins constant; $\gamma(\zeta)$ is another common notation for it.
$k_{1\gamma}(\zeta)$	$c_{\beta 0}(\zeta)$	γ and β are related is above
$c_\beta(\zeta)=e^{-k_\beta(\zeta)}$	$e^{-c_0(\zeta)}$	capacity
$c_{1\gamma}(\zeta)=e^{-k_{1\gamma}(\zeta)}$	$e^{-c_{\beta 0}(\zeta)}$	capacity
$c_1(\zeta)=\max\limits_{\substack{\gamma=\Gamma_k \\ k=1,\dots,m}} c_{1\gamma}(\zeta)$		
$F_\beta(z,\zeta)=e^{p_\beta+ip^*_\beta}$	$e^{c_0(\zeta)}\cdot e^{-G(z,\zeta)}$	multiple-valued analytic (with respect to z)
$P_{1\gamma}(z,\zeta)=e^{p_{1\gamma}+ip^*_{1\gamma}}$	$e^{c_{\beta 0}(\zeta)}\cdot e^{-G_\beta(z,\zeta)}$	univalent (as a function of z), mapping Ω onto a circular slit disc centered at 0 and with radius $e^{c_{\beta 0}(\zeta)}$ (ζ is mapped omto 0 and γ onto the circumference of the disc). The quantity $e^{c_{\beta 0}(\zeta)}$ is therefore often called the mapping radius with respect to Ω, at least if ζ is simply connected (then $e^{c_{\beta 0}(\zeta)}=e^{c_0(\zeta)}$).

Notation in [S-O]	Notation here	Comment
$K(z,\zeta)$	$K(z,\zeta)$	the Bergman kernel
$\tilde{K}(z,\zeta)$	$K_s(z,\zeta)$	reduced Bergman kernel
$c_D(\zeta)=\sqrt{\pi\tilde{K}(\zeta,\zeta)}$	$\sqrt{\pi K_s(\zeta,\zeta)}$	

Appendix 2 : Uniqueness questions for $\Delta u = e^u$.

The problem (4.36),

$$\begin{cases} \Delta u = e^u \\ u(z) \geq - 2 \log d(z) + O(1) \end{cases} \tag{1}$$

in Ω has the solution ((4.32))

$$u(z) = - 2c_0(z) + \log 8 \tag{2}$$

if Ω is simply connected. We shall prove that (1) has no other solutions than (2) and also that (2) is the maximal function satisfying $\Delta u = e^u$. It will also be proved that whether or not Ω is simply connected, (1) has at most one solution, and when it has, it is at the same time the maximal solution of $\Delta u = e^u$.

We first consider the case $\Omega = \mathbb{D}$ and then pass to the general case by conformal mapping. The following lemma, which we need, is essentially the same as Lemma 1-1 in [A].

Lemma 7.1: Suppose u and v are real, twice continously differentiable functions in $\mathbb{D}$ satisfying

$$\begin{cases} \Delta u = e^u & (3) \\ u(z) \to + \infty \quad \text{as} \quad z \to \partial\mathbb{D} \, , ^{*)} & (4) \end{cases}$$

$$\Delta v \geq e^v \, . \tag{5}$$

Then $u \geq v$ in $\mathbb{D}$.

Proof: Choose a number $0 < r < 1$ and consider the function

$$w(z) = u(z) - v(rz) - \log r^2 \, , \qquad z \in \mathbb{D} \, .$$

Clearly w is twice continously differentiable in $\mathbb{D}$, and by (4)

$$w(z) \to + \infty \qquad \text{as} \quad z \to \partial\mathbb{D} \, .$$

---

*) This is to be interpreted: for each $M < + \infty$ there is a compact $K \subset \mathbb{D}$ such that $u > M$ outside K .

Therefore w must have a minimum point in $\mathbb{D}$, say z_0. Then

$$\Delta w(z_0) \geq 0 \quad . \tag{7}$$

On the other hand

$$\Delta w(z\ ) = \Delta\,[u(z) - v(rz) - \log r^2] \leq e^{u(z)} - e^{v(rz\ )+\log r^2} \ , \tag{8}$$

by (3) and (5), so that (7) gives

$$e^{u(z_0)} \geq e^{v(rz_0)+\log r^2} \ , \qquad \text{or} \tag{9}$$

$$w(z_0) \geq 0 \quad . \tag{10}$$

Thus $w(z) \geq 0$ for all $z \in \mathbb{D}$, that is

$$u(z) \geq v(rz) + \log r^2 \ , \qquad z \in \mathbb{D} \tag{11}$$

Since this holds for all $0 < r < 1$ we get

$$u(z) \geq v(z) \ , \qquad z \in \mathbb{D} \ , \tag{12}$$

as was to be proved.

One solution of the problem (3) - (4) is

$$u(z) = -\,2\log(1 - |z|^2) + \log 8 \tag{13}$$

(obtained from (2) and (4.5)). It follows immediately from the lemma that this is the only solution. It also follows that the function (13) is maximal among all functions satisfying $\Delta u = e^u$.

Now we want to carry over Lemma 7.1 to an arbitrary simply connected domain Ω. Let $f : \mathbb{D} \to \Omega$ be any Riemann mapping function and $\varphi = f^{-1}$ its inverse. We use the variable ζ in $\mathbb{D}$ and z in Ω. It is easy to see that the correspondence

$$\tilde{u} \leftrightarrow u \tag{14}$$

between functions $\tilde{u}$ in $\mathbb{D}$ and functions u in Ω, defined by

$$\tilde{u}(\zeta) = u(f(\zeta)) + \log|f'(\zeta)|^2 = u(z) - \log|\varphi'(z)|^2 \ , \quad z = f(\zeta) \tag{15}$$

is a one-to-one correspondence between solutions of the equalities/inequalities

$\Delta\tilde{u} \gtreqless e^{\tilde{u}}$ in $\mathbb{D}$ and solutions of the corresponding equalities/inequalities $\Delta u \gtreqless e^{u}$ in Ω (in fact $e^{-\tilde{u}}\Delta_\zeta \tilde{u} = e^{-u}\Delta_z u$).
Moreover, the correspondence is order-preserving (i.e. $\tilde{u} \leq \tilde{v}$ in $\mathbb{D}$ $\Leftrightarrow$ $u \leq v$ in Ω). From this it follows that Lemma 7.1 carries over verbatim to Ω if only (4) is replaced by the corresponding boundary condition in Ω. By (15) this boundary condition is

$$u(z) - \log|\varphi'(z)|^2 \to +\infty \quad \text{as} \quad z \to \partial\Omega . \tag{16}$$

In order to get a boundary condition which is more geometrical than (16) we shall use the inequality

$$d(z) \leq \frac{1 - |\varphi(z)|^2}{|\varphi'(z)|} , \tag{17}$$

where $d(z)$ is the distance from z to the boundary. (17) is simply the inequality $d(z) \leq e^{c_0(z)}$ in Proposition 3.3, combined with (4.17). From (17) there follows

$$2 \log d(z) \leq - 2 \log|\varphi'(z)| + 2 \log(1 - |\varphi(z)|^2) , \tag{18}$$

and

$$u(z) + 2 \log d(z) \leq u(z) - \log|\varphi'(z)|^2 + 2 \log(1 - |\varphi(z)|^2). \tag{19}$$

Since the last term in (19) tends to $-\infty$ as $z \to \partial\Omega$, we conclude that the boundary condition

$$u(z) + 2 \log d(z) \geq O(1) , \quad z \in \Omega , \tag{20}$$

is stronger than (i.e. implies) (16). Now, Lemma 7.1 clearly remains valid if (4) is replaced by a stronger condition. Therefore, we have proved

<u>Lemma 7.2:</u> Suppose u and v are real, twice continously differentiable functions in a simply connected domain Ω, satisfying

$$\begin{cases} \Delta u = e^u & (21) \\ u(z) \geq - 2 \log d(z) + O(1) & (22) \end{cases}$$

$$\Delta v \geq e^v . \tag{23}$$

Then $u \geq v$ in Ω.

As an immediate consequence of Lemma 7.2 we have

Theorem 7.3: Suppose Ω is simply connected. Then :

a) Among all twice continously differentiable functions in Ω satisfying

$$\Delta u = e^u \quad , \tag{24}$$

there is a unique **maximal** one, namely

$$u(z) = - 2c_0(z) + \log 8 = 2 \log \frac{|\varphi'(z)|}{1-|\varphi(z)|^2} + \log 8 \; . \tag{25}$$

where φ is any conformal isomorphism of Ω onto $\mathbb{D}$.

b) This function is also the unique solution to the problem

$$\begin{cases} \Delta u = e^u & (26) \\ u(z) \geq - 2 \log d(z) + O(1) & (27) \end{cases}$$

(In the last member of (25) we have used (4.17).)

Let us finally consider the case that Ω is multiply connected. If f now denotes any universal covering map $\mathbb{D} \to \Omega$ and φ its multiple-valued inverse, we can proceed exactly as in the simply connected case, the only difference being that the correspondence (14) now is a one-to-one correspondence only between a certain subclass of functions $\tilde{u}$ in $\mathbb{D}$ *) and functions u in Ω . The inequality (17) is still true because it is the inequality $d(z) \leq e^{\tilde{c}_0(z)}$, where $\tilde{c}_0(z)$ refers to the universal covering surface $\tilde{\Omega}$ of Ω (it is easy to see that the part of Proposition 3.3 which concerns this inequality is valid also for nonschlicht regions, such as $\tilde{\Omega}$). The multiple-valuedness of φ will never cause any problems because φ always occurs in the single-valued combination

$$\frac{1 - |\varphi(z)|^2}{|\varphi'(z)|} \quad (= e^{\tilde{c}_0(z)} \; ; \; \text{cf p. } 68).$$

---

*) To be exact: to $f : \mathbb{D} \to \Omega$ there corresponds a certain group G of Möbius-transformation on $\mathbb{D}$ (the covering transformations), and the sub-class in question consists of those functions $\tilde{u}$ in $\mathbb{D}$ which satisfy

$$\tilde{u}(\varphi(z)) + \log|\varphi'(z)|^2 = \tilde{u}(T\varphi(z)) + \log|(T\varphi)'(z)|^2 \quad (= u(z) \;) \quad \text{for all} \quad T \in G$$

(having chosen a branch of φ in a neighbourhood of a given point).

It follows that Lemma 7.2 remains true with the hypothesis of simple connectedness dropped.

Passing to Theorem 7.3, it is seen that a) remains true (without the hypothesis of simple connectedness) if (25) is replaced by

$$u_1(z) = -2\,\tilde{c}_0(z) + \log 8 = 2\log\frac{|\varphi'(z)|}{1-|\varphi(z)|^2} + \log 8 \tag{28}$$

since this is the function on Ω which under (14) corresponds to the maximal solution (13) of $\Delta u = e^u$ in $\mathbb{D}$).
As to b), the problem arises that it is not obvious (perhaps not even true) that the function (28) really satisfies the boundary condition (27).* (That the function (25) does satisfy (27) is due to the upper bounds for $c_0(z)$ given in Proposition 3.3, and those upper bounds do not hold for $\tilde{c}_0(z)$ since their proofs rest on Lemma 3.4 whose proof uses the theory of capacity and transfinite diameters (section III i)) and therefore depends in an essential way on the domain being embedded in $\mathbb{P}$.) In any case, leaving this question open, there remains the following part of b) (being a direct consequence of Lemma 7.2):

The problem (26) - (27) has at most one solution, and when the solution exists it is (28), the maximal solution of $\Delta u = e^u$.

This statement will be enough for our purposes (p.83).

---

* (27) is indeed not true in complete generality, i.e. with our regularity assumptions on p. 27 dropped. For example, for $\Omega = \mathbb{C}\setminus\{0,1\}$ one can show that

$$\tilde{c}_0(z) = \log|z| + \log\log\frac{1}{|z|} + O(1) \qquad \text{as} \quad z \to 0 .$$

(See [A], 1 - 8 (Theorem 1 - 12).)
Thus

$$u_1(z) = -2\log|z| - 2\log\log\frac{1}{|z|} + O(1) \quad \text{as} \quad z \to 0$$

which does not fit into (27).

VII References

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Addendum

Two footnotes

p. 69: The reasoning on p. 67-69 to show that the metric (148) is shorter than the Poincaré metric can be replaced by the following more transparent argument, inspired by [K] (outline):
The universal covering map $f : \mathbb{D} \to \Omega$ is locally distance-preserving for the Poincaré metric (by the definition of this metric), but is distance-decreasing for the metric (148) (by Lemma 3.7). Since these two metrics are found to coincide on $\mathbb{D}$ the desired conclusion follows.

p. 88: [H] also shows (Satz 5) that if Ω is m-symmetric (and convex) then this stationary point coincides with the symmetry point. (A region Ω is m-symmetric if there is a symmetry point z_0 such that Ω is left invariant by rotations of angles $k \cdot \frac{2\pi}{m}$ (k integer) about z_0.) Moreover, [H] obtains sharp upper bounds for the values of $e^{c_0(z)}$ at these stationary (maximum) points (Satz 6, Satz 7).