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Quadrature Identities and the Schottky Double

by

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Contents

0. Introduction

This report originates from work done by Dov Aharonov and Harold S. Shapiro ([AS 2]) concerning so called "quadrature domains". A quadrature domain is a domain $\Omega \subset \mathbb{C}$ with the property that an identity of the kind

$$(*) \quad \int_{\Omega} f dxdy = \sum_{j=1}^{m} a_j f^{(n_j)}(z_j) \qquad (z_j \in \Omega)$$

holds for all $f \in L_a^1(\Omega)$, the class of integrable, analytic functions on Ω (or for some suitable subclass thereof).

In [AS 2], among other things, precise versions of the following statements are proved:

1) Ω is a quadrature domain if and only if there exists a meromorphic function $h(z)$ on Ω such that

$$(**) \quad h(z) = \bar{z} \quad \text{on} \quad \partial\Omega$$

([AS 2, Lemma 2.3]; see also [D1, Ch 14]).

2) a simply connected domain Ω is a quadrature domain if and only if Ω is the conformal image of $\mathbb{D}$ (the open unit disk) under a rational function (with the poles off $\bar{\mathbb{D}}$). In particular there exist plenty of simply connected quadrature domains. ([AS 2, Theorem 1] and [D1, Ch 14].)

3) if Ω is a quadrature domain, $\partial\Omega$ is part of an algebraic curve ([AS 2, Theorem 3]).

In all three cases the a priori assumption about Ω is that

$$(***) \quad \int_{\Omega} \frac{dxdy}{|z|} < \infty \qquad (z = x + iy) ,$$

and the test class of functions is $L_a^1(\Omega)$.

Among the questions left open in [AS 2] are:

4) the question of existence of multiply connected quadrature domains

5) uniqueness questions: to what extent can different domains have the same quadrature formula?

In this report question 4) is settled: for any (bounded) domain W, bounded by finitely many analytic Jordan curves, there are quadrature domains Ω, arbitrary close to W and conformally equivalent to W (Theorem 3.3).

As to the uniqueness question 5), it turns out that, in the multiply connected case, there in general are whole families of domains satisfying the same quadrature identity (Theorem 7.1, 7.2, Suggestion 7.3).

However, the hardest uniqueness question remains open: can two different simply connected (or, more generally, two conformally equivalent) domains have the same quadrature formula for the test-class L_a^1?

Further, point 3) above is worked out a little: we show that the boundary of a quadrature domain must be a <u>whole</u> algebraic curve (Theorem 3.4), and the explicit relation between the coefficients of the polynomial function of that curve and the datas (z_j, n_j, a_j in (*)) in the quadrature formula is obtained (Theorem 5.1).

The general idea, underlying most results in this report, is that of completing a plane domain Ω with a "backside" $\tilde{\Omega}$, so that a compact Riemann surface

$$\hat{\Omega} = \Omega \cup \partial\Omega \cup \tilde{\Omega} ,$$

the <u>Schottky double</u> of Ω, is obtained ([SS 1, Ch 2.2]).

From this point of view, the relation (**) simply means that the pair (h(z), z) defines a meromorphic function on $\hat{\Omega}$, namely the function which = h(z) on Ω, = $\bar{z}$ on $\tilde{\Omega}$, extending continuously over $\partial\Omega$ by (**).

Thereby we get an analogue of property 2) for the multiply connected case:

2') Let W be a standard domain (bounded by analytic Jordan curves, say) representing a certain conformal type. Then all quadrature domains Ω, conformally equivalent to W, are obtained as conformal images of W under functions meromorphic on the Schottky double $\hat{W} = W \cup \partial W \cup \tilde{W}$. (Theorem 3.1.) (Note that, in 2, the rational functions are just the meromorphic functions on $\hat{\mathbb{D}} \cong$ the Riemann sphere.) Although the classical theory of compact Riemann surfaces guarantees a good supply of meromorphic functions on $\hat{W}$, we must have functions on W which moreover are univalent on W in order to produce quadrature domains. The existence of such functions is proved by approximating some explicit function, defined and univalent in some neighbourhood of $W \cup \partial W$ in $\hat{W}$, with functions meromorphic on $\hat{W}$, using a Runge approximation theorem for compact Riemann surfaces.

This is the way the existence of multiply connected quadrature domains is proved.

It should be remarked that we mostly work with a somewhat more general type of quadrature formula than (*), namely quadrature formulas also involving line integrals:

$$(*') \qquad \int_{\Omega} f dxdy = \sum_{j=1}^{m} a_j f^{(n_j)}(z_j) + \sum_{j=1}^{n} b_j \int_{\gamma_j} f dz .$$

Here $\gamma_1, \ldots, \gamma_n$ are closed or non-closed curves in Ω.

If all the γ_j are closed, and Ω has finite connectivity, a quadrature formula like (*') holds for all $f \in L^1_a(\Omega)$, if and only if a quadrature formula of type (*) holds for all $f \in L^1_{as}(\Omega)$, the subclass of $L^1_a(\Omega)$ consisting of functions with single-valued integral in Ω.

A limitation in our method of doubling plane domains Ω, is that it requires stronger a priori assumptions on Ω than that (condition (***)) used in [AS 2], namely that Ω has finite area and is bounded by finitely many continua (Lemma 1.1).

Moreover, with our method the test class $L_a^2(\Omega)$ turns out to be more natural than $L_a^1(\Omega)$. However, the assumption of finite area implies that $L_a^2(\Omega) \subset L_a^1(\Omega)$, and it can be shown (according to [AS 2, 1.3]), that the assumption that Ω is bounded by finitely many continua implies that $L_a^2(\Omega)$ is dense $L_a^1(\Omega)$, so the difference is not significant.

The disposition of the material is as follows:

Section 1 contains a lemma which characterizes those plane domains which can be doubled (in a certain technical sense).

In section 2 we prove a kind of abstract quadrature formula on symmetric Riemann surfaces (Proposition 2.1). This formula is fundamental, and by conformal mapping it leads to the basic theorems about quadrature domains in section 3 (Theorems 3.1 and 3.2). Theorem 3.2 is essentially the characteristic property 1) (on p. 0.1) of quadrature domains, which is proved in [AS 2] in quite another way.

Application of a Runge approximation theorem to Theorem 3.1 leads to the main theorem about existence of quadrature domains, Theorem 3.3.

Our Schottky double point of view throws some new light on known results about quadrature identities of certain specified types, and also enables us to generalize some of them. This is in section 4.

Section 5 deals with the relation between the coefficients of the polynomial function of the algebraic boundary curve of a quadrature domain and the data in the quadrature formula for it.

In section 6 we illustrate the general theory by working out the details a little for quadrature formulae of the type:

$$\int_\Omega f dxdy = c_0 f(z_0) + c_1 f'(z_0) + \ldots + c_{n-1} f^{(n-1)}(z_0) \quad (z_0 \in \Omega) .$$

Section 7, finally, deals with questions about uniqueness and multitude of the quadrature domains associated with a fixed quadrature functional. This section is somewhat sketchy and very incomplete.

Lastly, I wish to thank Professor Harold S. Shapiro for having proposed this problem, for many stimulating discussions about it and for all the great interest and encouragement he has shown during the work on this manuscript.

List of Notations:

$\mathbb{D}(a;r) = \{z \in \mathbb{C} : |z - a| < r\}$.

$\mathbb{D} = \mathbb{D}(0;1)$.

$\mathbb{C}^+ = \{z \in \mathbb{C} : \operatorname{Im} z > 0\}$, $\mathbb{C}^- = \{z \in \mathbb{C} : \operatorname{Im} z < 0\}$.

$\mathbb{P}_n$ = the n-dimensional complex projective space.

$\mathbb{P} = \mathbb{P}_1 \cong$ the Riemann sphere.

$H(\Omega)$: the space of holomorphic functions on Ω, provided with the topology of uniform convergence on compacts (Ω any Riemann surface).

$M(\Omega)$: the meromorphic functions on Ω.

$L^p_a(\Omega)$: the Banach space of holomorphic functions f on $\Omega \subset \mathbb{C}$ with $||f||_p = \{\int_\Omega |f|^p dxdy\}^{1/p} < \infty$, $1 \leq p \leq \infty$.

$L^p_{as}(\Omega)$: the subspace of $L^p_a(\Omega)$ consisting of those f which have single-valued integrals.

$\Gamma_a(W)$: the Hilbert-space of analytic square-integrable differentials on W with inner product:

$$\langle a(z)dz, b(z)dz \rangle = -\frac{1}{2i} \int_W a(z)\overline{b(z)}dzd\bar{z} .$$

$\Gamma_{ae}(W)$: the subspace of $\Gamma_a(W)$ consisting of exact differentials (i.e. $a(z)dz = df(z)$ for some single valued analytic function f).

$L_a^2(\Omega)$ and $L_{as}^2(\Omega)$ are naturally identified with $\Gamma_a(\Omega)$ resp $\Gamma_{ae}(\Omega)$ via the isometric isomorphism:

$$L_a^2(\Omega) \ni f(z) \overset{\approx}{\leftrightarrow} f(z)dz \in \Gamma_a(\Omega)$$

$\hat{W}$: the Schottky double of a bordered Riemann surface, $W \cup \Gamma$. $\hat{W} = W \cup \Gamma \cup \tilde{W}$, where $\tilde{W}$ is the "back-side".

$\tilde{\xi}$: the conjugate (opposite) point of a point $\xi \in \hat{W}$ ($\xi \in W \Longleftrightarrow \tilde{\xi} \in \tilde{W}$ and so on).

ϕ: $\hat{W} \to \hat{W}$ the involution, i.e. the anticonformal automorphism of order 2 ($\phi \circ \phi$ = identity) which exchanges conjugate points ($\because$ $\tilde{\xi} = \phi(\xi)$).

df: this kind of notation for analytic differentials will often be used even if the integral $f(\xi) = \int^{\xi} df$ is not single-valued.

$\tilde{f}$, f^*, $d\tilde{f}$, df^*: if f is a function on $\hat{W}$, then:

$$\begin{cases} \tilde{f} = f \circ \phi & \text{(i.e. } \tilde{f}(\xi) = f(\tilde{\xi}) \\ f^* = (f \circ \phi)^- & \phantom{\text{(i.e. }} f^*(\xi) = \overline{f(\tilde{\xi})} \text{)} \end{cases}$$

f analytic $\Rightarrow$ $\tilde{f}$ anti-analytic, f^* analytic.

If df is an analytic differential then $d\tilde{f}$, df^* are well-defined by:

$$\begin{cases} d\tilde{f} = d(\tilde{f}) \\ df^* = d(f^*) \ . \end{cases}$$

<u>Warning</u>: the notations f^* and df^* have nothing to do with harmonic conjugates (the harmonic conjugates of f and df are $-i \cdot f$ resp $-i \cdot df$).

$\zeta(z)$, $S(z)$: if γ is a (regular) analytic arc, $\zeta(z)$ is the anticonformal reflection in γ and $S(z)$ the so called Schwarz function of γ, i.e.:

$S(z) = \overline{\zeta(z)}$ for $z \in$ some neighbourhood of γ.

$z = \zeta(z) = \overline{S(z)}$ for $z \in \gamma$. (p. 1.3 f)

(regular) analytic arc: continuous arc, γ, which locally can be parametrized by analytic functions

$f : I \to \mathbb{C}$

where $I \subset \mathbb{R}$ are open intervals and $f'(t) \neq 0$ for $t \in I$. The condition $f'(t) \neq 0$ for $t \in I$ can be replaced by the condition that f shall be univalent in some neighbourhood U of I (by breaking up γ into sufficiently small parametric arcs).

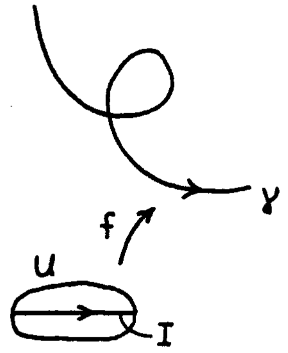

fig 0.1

quasi-regular analytic arc: oriented continuous arc, γ, which locally can be parametrized by analytic functions

$f : I \to \mathbb{C}$

($I \subset \mathbb{R}$ open intervals) such that:

i) f preserves the orientation (I oriented in the obvious way)

ii) f is univalent in $U^+ = U \cap \{z : \operatorname{Im} z > 0\}$ for some neighbourhood U of I.

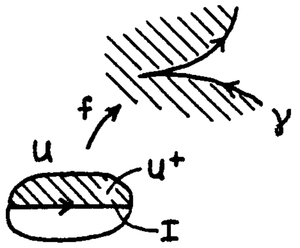

fig 0.2

1. A Lemma about doubles of plane domains

This first section is devoted to some generalities about doubles of plane domains which will be needed in the following. For the precise definition and basic properties of the Schottky double see for example [AS 1, II3E] and [SS 1, Ch 2.2].

Lemma 1.1: A domain $\Omega \subset \mathbb{P}$ is conformally equivalent to one half, W, of a compact symmetric Riemann surface $\hat{W} = W \cup \Gamma \cup \tilde{W}$ if and only if

a) Ω has finite connectivity

b) no component of $\mathbb{P} \setminus \Omega$ consists of a single point.

Moreover, if $f : W \to \Omega$ is any such conformal equivalence:

i) Ω has finite area $\Longleftrightarrow$ $df \in \Gamma_{ae}(W)$

ii) $\partial\Omega$ is a finite union of disjoint Jordan curves $\Longleftrightarrow$ f extends to a homeomorphism: $f : W \cup \Gamma \xrightarrow{\approx} \Omega \cup \partial\Omega$

iii) $\partial\Omega$ is a finite union of quasi-regular analytic curves, positively oriented with respect to Ω (private terminology, to be explained in the proof) $\Longleftrightarrow$ f extends to a meromorphic function on a neighbourhood of $W \cup \Gamma$

iv) $\partial\Omega$ is part of an algebraic curve. If $\Omega \cup \partial\Omega \subsetneq \mathbb{P}$ $\partial\Omega$ is even a whole algebraic curve. $\Longleftarrow$ f extends to a meromorphic function on $\hat{W}$

Remarks:

① The conditions a) and b) may be summarized: $\partial\Omega \subset \mathbb{P}$ consists of a finite number of continua (closed connected sets consisting of more than one point). Domains with this property are the only ones to be considered in this report.

② The hypothesis that $\Omega \cup \partial\Omega \subsetneqq \mathbb{P}$ in (iv) is necessary for the conclusion that $\partial\Omega$ shall be a whole algebraic curve. A counter-example is:

$$\begin{cases} W = \{z : \operatorname{Im} z > 0\} , \\ \hat{W} = \mathbb{P} , \\ f(z) = z^2 . \end{cases}$$

One finds that f is univalent on W, meromorphic on $\hat{W}$

$$\Omega = f(W) = \mathbb{C} \setminus \bar{\mathbb{R}}_+ \quad (\bar{\mathbb{R}}_+ = \{x \in \mathbb{R} : x \geq 0\}) ,$$

and

$$\partial\Omega = \bar{\mathbb{R}}_+ \cup \{\infty\} ,$$

which is not a whole algebraic curve.

③ In (iv) one clearly wants to have an implication in the direction ➡ . In order to have that it is necessary to strengthen the left hand side by some condition of combinatorial nature which guarantees that the branches of algebraic functions related to the curve $\partial\Omega$ fit together in the right way. A sufficient condition would be that the Schwarz function $S(z)$ of $\partial\Omega$ extends to a meromorphic function in Ω. With this condition, however, (iv) almost reduces to a tautology.

Proof of lemma 1.1: If Ω satisfies a) and b) then Ω can be mapped conformally onto a domain $W \subset \mathbb{P}$ bounded by analytic curves (by repeated use of the Riemann mapping theorem in a well-known manner), and this domain can be doubled in the usual way.

Conversely, if Ω is conformally equivalent to W where $\hat{W} = W \cup \Gamma \cup \tilde{W}$ is compact symmetric, then Ω has finite connectivity since $\hat{W}$ has finite genus (connectivity (Ω) = genus $(\hat{W})$ + 1). This proves a). To prove b), suppose $\mathbb{P} \setminus \Omega$ had a component consisting of a single point z_0.

If $f : W \overset{\approx}{\to} \Omega$ is conformal, the inverse image under f of

$$\{0 < |z - z_0| \leq \varepsilon\} \subset \Omega$$

is doubly connected and closed in W, but not compact and hence not closed in $\hat{W}$. Therefore its closure in $\hat{W}$ must contain a (unique) component Γ_0 of Γ. (fig 1.1). Since Γ_0 is a whole continuum it follows that f maps a ring domain $(f^{-1}(\{0 < |z - z_0| < \varepsilon\}$ conformally onto a disc with a point deleted, and this is known to be impossible, proving b).

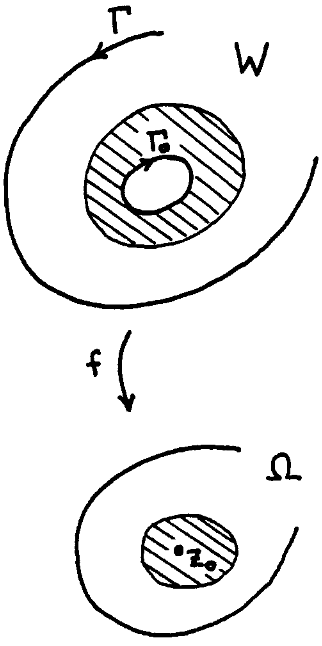

fig. 1.1

(i) follows immediately from:

$$\operatorname{area}(\Omega) = -\frac{1}{2i}\int_{\Omega} dz d\bar{z} = -\frac{1}{2i}\int_{W} df \wedge d\bar{f} = \pi \cdot \|df\|^2 .$$

(ii) ⇐ is obvious, and ⇒ is proved in the same way as the corresponding statement for plane domains, W. Grunsky 1.1.7
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(iii) The terminology: First, by a (regular) analytic arc we mean an arc, γ, which locally can be parametrized by analytic functions

$$f : I \to \mathbb{C}$$

where $I \subset \mathbb{R}$ are open intervals and $f'(t) \neq 0$ for $t \in I$ (fig 1.2).

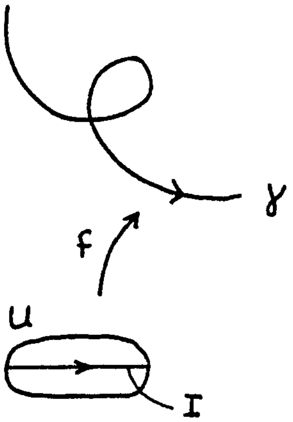


fig 1.2

When γ is oriented only orientation-preserving parametrizations (with the obvious orientations on the intervals $I \subset \mathbb{R}$) are allowed.

The functions $f : I \to \mathbb{C}$ are holomorphic in neighbourhoods $U \subset \mathbb{C}$ of I. The condition $f'(t) \neq 0$, $t \in I$

implies that f can be supposed to be univalent in U (by partitioning γ into sufficiently small parts and taking the U:s sufficiently thin).

Now, we define quasi-regular analytic arcs by relaxing the above condition in the following way:

An oriented arc, γ, is a quasi-regular analytic arc if it can be parametrized by analytic functions

$$f : I \to \mathbb{C}$$

which are univalent in $U^+ = U \cap \mathbb{C}^+$ for sufficiently small neighbourhoods $U \subset \mathbb{C}$ of the open intervals $I \subset \mathbb{R}$.

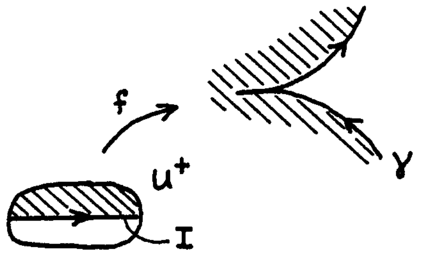


fig 1.3

Examples of quasi-regular analytic arcs which are not regular are the oriented boundaries of the various kinds of slit domains, where in neighbourhoods of the end-points of the slits quasi-regular parametrizations such as

$$f : (-1, 1) \to \mathbb{C} , \quad f(t) = t^2$$

have to be used (fig 1.4).

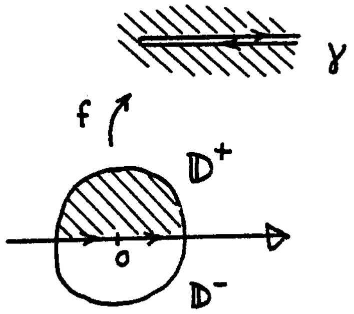


fig 1.4

Suppose $f : (-\varepsilon, \varepsilon) \to \mathbb{C}$ has the power series at the origin

$$f(t) = a_0 + a_1 t + a_2 t^2 + a_3 t^3 + \dots \quad .$$

Then f defines a regular analytic arc for some $\varepsilon > 0$ if $a_1 \neq 0$. If $a_1 = 0$ but $\operatorname{Im} a_2 \neq 0$, then it is easily seen that the arc defined by f is quasi-regular for some $\varepsilon > 0$ if and only if $\operatorname{Im} a_2 > 0$.

If $a_1 = a_2 = 0$ then the arc is not quasi-regular (it is supposed that the zero of f' at $t = 0$ is of minimal order among all parametrizations of the same arc).

When γ is an (regular) analytic arc there is the anti-analytic reflection mapping, ζ, defined in a strip Ω around γ, and locally given by

$$\zeta(z) = f(\overline{f^{-1}(z)}) , \quad z \in f(U) \subset \Omega$$

where $f : I \to \mathbb{C}$, $I \subset U \subset \mathbb{C}$ is a parametrization as usual. $\zeta \circ \zeta$ = identity and γ is the set of fix points of ζ (fig 1.5).

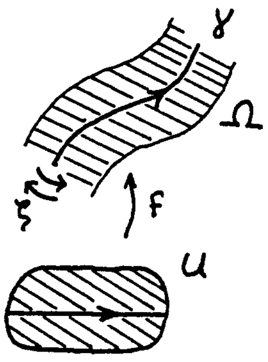


fig 1.5

When γ is only quasi-regular we have a one-sided reflection, ζ. Its domain of definition is a strip, Ω^+, to the left of γ (fig 1.6). Locally

$$\zeta(z) = f(\overline{f^{-1}(z)}) \quad , \quad z \in f(U^+) \subset \Omega^+$$

with notations as before. Trying to extend ζ over γ in a neighbourhood of a singular point on γ leads to many-valued function.

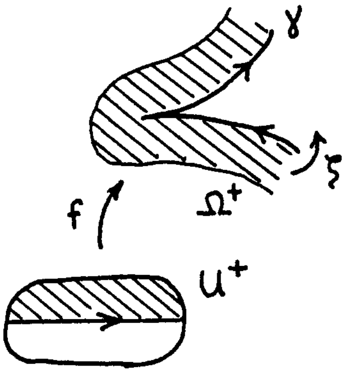


fig 1.6

The existence of ζ leads to the following reflection principle for quasi-analytic arcs:

Suppose: $h: \Omega_1 \to \Omega_2$ is holomorphic ($\Omega_1, \Omega_2 \subset \mathbb{C}$),

$\gamma_1 \subset \partial\Omega_1$ an (regular) analytic arc,

$\gamma_2 \subset \partial\Omega_2$ quasi-regular and positively oriented with respect to Ω_2,

$h(z) \to \gamma_2$ as $z \to \gamma_1$

Then: h extends to a holomorphic function in a neighbourhood of $\Omega_1 \cup \gamma_1$.

If ζ_1 and ζ_2 are the reflections in γ_1 and γ_2 respectively, ζ_1 defined in the strip $U_1 \supset \gamma_1$ and ζ_2 in the half-strip $U_2^+ \subset \Omega_2$, then the extension is given by

$$h(z) = \zeta_2(h(\zeta_1(z)))$$

for $z \in U_1 \smallsetminus (\Omega_1 \cup \gamma_1)$.

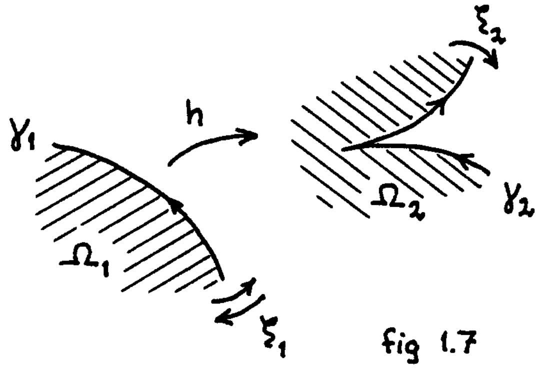


fig 1.7

The proof consists of applications of the ordinary reflection principle to functions of the type $f_2^{-1} \circ h \circ f_1$ where f_1 and f_2 are local parametrizations of the arcs γ_1 and γ_2 (the details are left to the reader).

Having established the definition and elementary properties of quasi-regular analytic arcs, part (iii) of the lemma is now almost a triviality:

$\Rightarrow$ follows immediately from the above reflection principle, and to prove $\Leftarrow$ one only has to notice that the function $f : W \to \Omega$ itself, expressed in local coordinates around γ, serves as parametrization of $\partial\Omega$ as quasi-regular analytic curve (observe, in the $\Leftarrow$ - part, that $f(\Gamma) = \partial\Omega$ since, first $f(\Gamma) \subset \Omega \cup \partial\Omega$ by the continuity of f, then $f(\Gamma) \subset \partial\Omega$ by univalency of f in W, and finally $f(\Gamma) = \partial\Omega$ by a compactness argument: $z_j \to z \in \partial\Omega$, $\{z_j\} \subset \Omega \Rightarrow f^{-1}(\{z_j\}) \subset W$ contains a convergent subsequence, $\{\xi_j\}$, so $f(\xi) = z$ where $\xi = \lim \xi_j \in \Gamma$).

(iv) If f is <u>any</u> meromorphic function on $\hat{W}$, then $\gamma = f(\Gamma)$ is a subset of an algebraic curve. In fact, put

$$g(\xi) = f^*(\xi) = \overline{f(\tilde{\xi})} \ , \qquad \xi \in \hat{W} \ .$$

Then f and g are two meromorphic functions of the same order, say m, on $\hat{W}$. Hence, there is a non-trivial, irreducible polynomial:

$$P(z, w) = \Sigma \, a_{kl} z^k w^l$$

of degree $\leq$ m in each of z and w separately, such that:

$$P(f(\xi), g(\xi)) \equiv 0 , \quad \xi \in \hat{W} .$$

Since $g(\xi) = \overline{f(\xi)}$ for $\xi \in \Gamma$, it follows that:

$$P(z, \bar{z}) = 0 \quad \text{for} \quad z \in \gamma ,$$

proving the assertion.

Moreover, P is uniquely determined up to multiplication by a constant factor $\neq 0$, and it is not hard to see that this factor can be chosen so that

$$a_{kl} = \bar{a}_{lk} , \quad k, l = 0, 1, \ldots, m .$$

We shall call a polynomial with this property <u>self-conjugate</u>. It is equivalent to the polynomial

$$Q(x, y) = P(x + iy , \ x - iy)$$

having real coefficients.

Now, put

$$V = \{z \in \mathbb{P} : P(z, \bar{z}) = 0\}$$

This is an algebraic set which in general consists of a finite number of curves plus a finite number of isolated points. Put

$$V = V_0 \cup V_1$$

where V_n is the n-dimensional part of V i.e.

V_0 = {isolated points of V}

V_1 = {points of V belonging to some curve of V} =
= {limit points in V}

By the <u>algebraic curve</u> of P we mean the part V_1.

We have just proved that $\gamma \subset V$ ($\gamma = f(\Gamma)$), which of course implies that $\gamma \subset V_1$ since, otherwise, f would have to map a whole component of Γ into a single point $\in V_0$, which is impossible.

We have until now only assumed that f is meromorhpic on $\hat{W}$. If f moreover is univalent on W it is straight forward to check that $\gamma = \partial\Omega$ (= the topological boundary of $\Omega = f(W)$ in $\mathbb{P}$) (see remark at the end of the proof of (iii)).

Hence $\partial\Omega \subset V_1$ which proves the first part of the assertion in (iv). The second statement in (iv) is that in fact $\partial\Omega = V_1$ if $\Omega \cup \partial\Omega \subsetneqq \mathbb{P}$.

To prove this it clearly suffices to prove the following two statements (f meromorphic on $\hat{W}$, notations as above):

① If f and $g = f^*$ generate the field of meromorphic functions on $\hat{W}$, then $\gamma = V_1$.

② If f is univalent on W and $\Omega \cup \partial\Omega \subsetneqq \mathbb{P}$, then f and $g = f^*$ <u>do</u> generate the function-field on $\hat{W}$.

<u>Proof of ①</u>: it only remains to prove that $\gamma \supset V_1$. Take any point $z_0 \in V_1$. We have to find a point $\xi_0 \in \Gamma \subset \hat{W}$ with $f(\xi_0) = z_0$. Let $\gamma_0 \subset V_1$ be an arc which passes through z_0. Since
$P(z, \bar{z}) = 0$

for $z \in \gamma_0$, there is for each $z \in \gamma_0$ at least one $\xi \in \hat{W}$ with

$$\begin{cases} f(\xi) = z \\ g(\xi) = \bar{z} \end{cases}$$

($(f(\xi), g(\xi))$ runs through the <u>whole</u> complex curve $\{(z, w) \in \mathbb{P}_2 : P(z, w) = 0\}$ as ξ runs through $\hat{W}$).

It follows that there is an arc $\Gamma_0 \subset \hat{W}$ such that for each $z \in \gamma_0$ there is exactly one $\xi \in \Gamma_0$ with

$$\begin{cases} f(\xi) = z \\ g(\xi) = \bar{z} \end{cases}$$

(so that f maps Γ_0 homeomorphically onto γ_0) (fig 1.8).

Now we have:

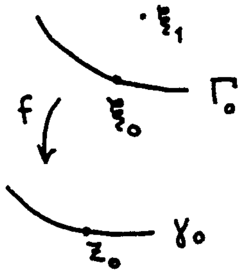


fig 1.8

$$\begin{cases} f(\tilde{\xi}) = \overline{g(\xi)} = z = f(\xi) \\ g(\tilde{\xi}) = \overline{f(\xi)} = \bar{z} = g(\xi) \end{cases}$$

for $\xi \in \Gamma_0$.

Since f and g generate the function-field on $\hat{W}$ this implies that

$$h(\xi) = h(\tilde{\xi}) \ , \quad \xi \in \Gamma_0$$

for <u>every</u> meromorphic function, h, on $\hat{W}$. In fact, h can be written

$$h(\xi) = R_0(f(\xi)) + R_1(f(\xi))g(\xi) + \ldots + R_{m-1}(f(\xi))g(\xi)^{m-1} \ ,$$

$$\xi \in \hat{W} \ ,$$

where $R_0, \ldots, R_{m-1}$ are rational functions.

Hence $h(\xi) = h(\tilde{\xi})$ for $\xi \in \Gamma_0 \setminus$ (a finite set), where the finite set consists of those points at which the right-hand-member of the equation above contains undetermined expressions such as $\frac{0}{0}$, $\frac{\infty}{\infty}$, $\infty \pm \infty$.

This implies, by continuity, $h(\xi) = h(\tilde{\xi})$ for all $\xi \in \Gamma_0$.

But now, if all meromorphic functions take equal values at two points, ξ and $\tilde{\xi}$, these points must coincide, $\xi = \tilde{\xi}$. For example it is a well-known consequence of the Riemann - Roch theorem that if $\xi \neq \tilde{\xi}$, there is a meromorphic function with a pole at ξ and regular at $\tilde{\xi}$.

Hence $\xi = \tilde{\xi}$ for all $\xi \in \Gamma_0$, i.e. $\Gamma_0 \subset \Gamma$. In particular $\xi_0 \in \Gamma$, where ξ_0 is the point on Γ_0 which corresponds to $z_0 \in \gamma_0$ ($f(\xi_0) = z_0$, $g(\xi_0) = \bar{z}_0$).

This proves statement ①.

Remark: It may happen that there are points $\xi_1 \in \hat{W} \setminus \Gamma$ with

$$\begin{cases} f(\xi_1) = z_o \\ g(\xi_1) = \bar{z}_o \end{cases}$$

but they are isolated, and so cannot make up a whole arc.

Proof of ② : It is well-known (it is a consequence of the discussion in [AS 1, Ch V 25 D and 25 F]) that for f and g to generate the function-field it is sufficient (and necessary) that there is a point $z \in \mathbb{P}$ such that g takes distinct values at $\xi_1, \ldots, \xi_m \in \hat{W}$, where

$$\{\xi_1, \ldots, \xi_m\} = f^{-1}(\{z\})$$

and m is the order of f.

In our case we simply take any

$$z \in \mathbb{P} \setminus (\Omega \cup \partial\Omega) = \mathbb{P} \setminus f(W \cup \Gamma)$$

such that $\xi_1, \ldots, \xi_m$ are distinct ($\{\xi_1, \ldots, \xi_m\} = f^{-1}(z)$).

This is clearly possible since $\mathbb{P} \setminus (\Omega \cup \partial\Omega)$ is open and non-empty, and $\xi_1, \ldots, \xi_m$ are distinct for all but finitely many $z \in \mathbb{P}$.

But now $z \notin f(W \cup \Gamma)$ implies that

$$\xi_1, \ldots, \xi_m \in \tilde{W} ,$$

and since f is univalent on W, g is univalent on $\tilde{W}$. Hence

$$g(\xi_1), \ldots, g(\xi_m)$$

are distinct, proving statement ②.

2. Some basic properties of quadrature formulas

The following proposition contains the "abstract" quadrature formula which is the origin of all more concrete quadrature identities for plane domains.

Proposition 2.1:

Let: $\hat{W} = W \cup \Gamma \cup \tilde{W}$ be a compact symmetric Riemann surface of genus = p (W conformally equivalent to a plane domain);

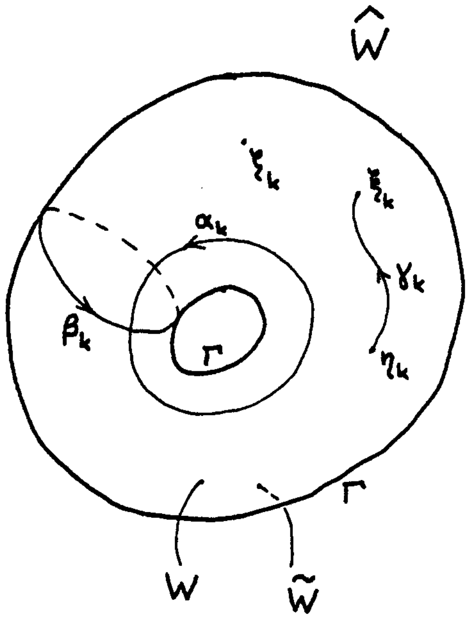


fig 2.1

$\alpha_1, \beta_1, \ldots, \alpha_p, \beta_p$ a canonical homology basis of $\hat{W}$ as indicated in fig 2.1;

$\zeta_1, \ldots$ a finite number of points $\in W$;

$n_1, \ldots$ integers ≥ 0 associated with $\zeta_1, \ldots$;

$\gamma_1, \ldots$ a finite number of arcs $\subset W$, supposed not to intersect any of $\alpha_1, \ldots, \beta_p$;

$\partial\gamma_k = (\xi_k) - (\eta_k)$ (formal difference; $\xi_k, \eta_k \in W$).

Suppose: dg is a meromorphic differential on $\hat{W}$ with:

i) residue-free poles of orders $n_k + 1$ at the points $\tilde{\zeta}_k \in \tilde{W}$,

ii) simple poles at $\tilde{\xi}_k$, $\tilde{\eta}_k$ with $\operatorname{res}_{\tilde{\xi}_k} dg + \operatorname{res}_{\tilde{\eta}_k} dg = 0$,

iii) no other singularities, hence $dg \in \Gamma_a(W)$.

Then: the following formula holds for all $df \in \Gamma_a(W)$:

$$\partial\gamma = \sum \operatorname{res}_{\xi_k} dg^* \cdot (\xi_k)$$

$$\boxed{\int_W df \wedge d\bar{g} = 2\pi i \sum_k \operatorname{res}_{\zeta_k}(fdg^*) + 2\pi i \sum_k \operatorname{res}_{\xi_k} dg^* \cdot \int_{\gamma_k} df - \sum_k \int_{\beta_k} dg^* \cdot \int_{\alpha_k} df} \tag{2.1}$$

Remarks:

① Obviously the differentials dg allowed in the Proposition are exactly those meromorphic differentials on $\hat{W}$ that have all their singularities on $\tilde{W}$, since $\Sigma_{\hat{W}}$ res dg = 0 implies that the residue-points can be coupled into pairs fulfilling (ii) (it is allowed that $\xi_k = \eta_j$, $\xi_k = \zeta_j$ etc for certain k, j). 1)

② Suppose dg* has the singular parts (expressed in suitable local variables ζ about $\zeta_1, \ldots, \xi_1, \ldots, \eta_1, \ldots$):

$$dg^*(\zeta) = \frac{a_{k,n_k} d\zeta}{(\zeta - \zeta_k)^{n_k+1}} + \ldots + \frac{a_{k,1} d\zeta}{(\zeta - \zeta_k)^2} + \text{regular terms at } \zeta = \zeta_k,$$

$$dg^*(\zeta) = \frac{b_k d\zeta}{\zeta - \xi_k} + \text{regular terms at } \zeta = \xi_k \ ,$$

$$dg^*(\zeta) = - \frac{b_k d\zeta}{\zeta - \eta_k} + \text{regular terms at } \zeta = \eta_k$$

(b_k does not depend on the local variable).

Also put:

$$c_k = - \frac{1}{2\pi i} \int_{\beta_k} dg^* \ .$$

Since f has the developments:

$$f(\zeta) = \text{integration constant} + \sum_{j=1}^{\infty} \frac{1}{j!} f^{(j)}(\zeta_k)(\zeta - \zeta_k)^j \text{ at } \zeta = \zeta_k$$

the (quadrature) formula then becomes:

$$(2.2)\quad \boxed{\frac{1}{2\pi i}\int_W df\wedge d\bar g = \sum_k \sum_{j=1}^{n_k} \frac{a_{k,j}}{j!}\, f^{(j)}(\zeta_k) + \sum_k b_k\cdot\int_{\gamma_k} df + \sum_k c_k\cdot\int_{\alpha_k} df}$$

Proof of the Proposition:

Put $W' = W \setminus \bigcup_{k=1}^{p} \beta_k$ and let β_k^+, β_k^- be the boundaries of W' at the cuts β_k oriented as indicated in fig 2.2. Then W' is simply connected and:

$$\partial W' = \Gamma + \sum_k \beta_k^+ - \sum_k \beta_k^- .$$

For $\zeta \in W \cap \beta_k$, let $\zeta^+ \in \beta_k^+$, $\zeta^- \in \beta_k^-$ denote the two boundary points of W' arising from ζ.

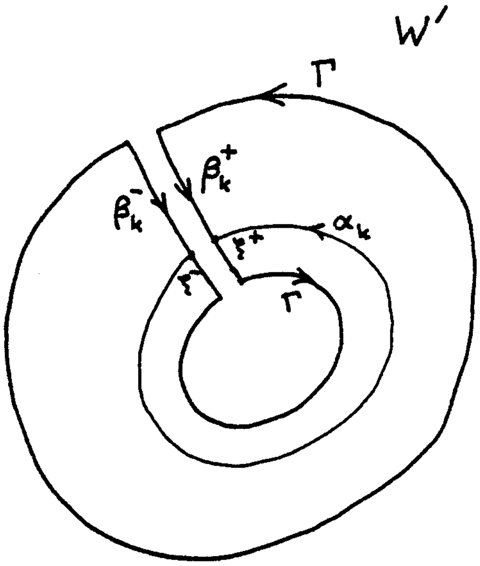


fig 2.2

Thus:

$$(2.3)\quad f(\zeta^+) = f(\zeta^-) + \int_{\alpha_k} df$$

for $df \in \Gamma_a(W)$, where f is any integral of df in W'.

The following, easily verified, formula will also be needed in the computations to come:

$$(2.4)\quad \int_{\beta_k} dg^* = \int_{\beta_{\bar k}} dg^* - \int_{\beta_{\bar k}} d\bar g .$$

Now, suppose (to begin with) that $df \in \Gamma_a(W)$ is continuous on $W \cup \Gamma$. Then, equations (2.3), (2.4) and the fact that $d\bar g = dg^*$ along Γ give:

$$\int_W df \wedge d\bar{g} = \int_{W'} df \wedge d\bar{g} = \int_{\partial W'} fd\bar{g} =$$

$$= \int_\Gamma fd\bar{g} + \sum_k \left(\int_{\beta_k^+} fd\bar{g} - \int_{\beta_k^-} fd\bar{g}\right) =$$

$$= \int_\Gamma fd\bar{g} + \sum_k \int_{\alpha_k} df \cdot \int_{\beta_k^-} d\bar{g}$$

$$= \int_\Gamma fdg^* + \sum_k \int_{\alpha_k} df \cdot \int_{\beta_k^-} d\bar{g}$$

$$= \int_{\partial W'} fdg^* - \sum_k \left(\int_{\beta_k^+} fdg^* - \int_{\beta_k^-} fdg^*\right) + \sum_k \int_{\alpha_k} df \cdot \int_{\beta_k^-} d\bar{g}$$

$$= \int_{\partial W'} fdg^* - \sum_k \int_{\alpha_k} df \cdot \int_{\beta_k^-} dg^* + \sum_k \int_{\alpha_k} df \cdot \int_{\beta_k^-} d\bar{g}$$

$$= 2\pi i \sum_W \mathrm{res}(fdg^*) - \sum_k \int_{\alpha_k} df \cdot \int_{\beta_k} dg^*$$

$$= 2\pi i \sum_k \mathrm{res}_{\zeta_k}(fdg^*) + 2\pi i \sum \left[\mathrm{res}_{\xi_k}(fdg^*) + \mathrm{res}_{\eta_k}(fdg^*)\right] - \sum_k \int_{\alpha_k} df \cdot \int_{\beta_k} dg^*$$

$$= 2\pi i \sum_k \mathrm{res}_{\zeta_k}(fdg^*) + 2\pi i \sum_k \left[f(\xi_k)\mathrm{res}_{\xi_k} dg^* + f(\eta_k)\mathrm{res}_{\eta_k} dg^*\right] - \sum_k \int_{\alpha_k} df \cdot \int_{\beta_k} dg^*$$

$$= 2\pi i \sum_k \mathrm{res}_{\zeta_k}(fdg^*) + 2\pi i \sum_k \mathrm{res}_{\xi_k} dg^* \cdot \int_{\gamma_k} df - \sum_k \int_{\beta_k} dg^* \cdot \int_{\alpha_k} df ,$$

proving formula (2.1) in the case df is continuous on $W \cup \Gamma$.

To prove the formula for general $df \in \Gamma_a(W)$ one has to carry out an 1) approximation procedure. For this purpose, choose a sequence $\Omega_1, \Omega_2 ..$ of domains $\subset W$ such that:

1) $\bar{\Omega}_n \subset \Omega_{n+1}$, $n = 1, 2, ...$

2) $\partial\Omega_n$ is suitably nice, $n = 1, 2, ...$

3) Ω_1 contains all $\zeta_1, ..., \xi_1, ..., \eta_1, ..., \gamma_1, ..., \alpha_1, ...$,

4) $\bigcup_{n=1}^{\infty} \Omega_n = W$

Also, put:

$$\begin{cases} \Omega'_n = \Omega_n \cap W' \\ \beta_{k,n} = \beta_k \cap (\Omega_n \cup \tilde{\Omega}_n) \\ \beta^{\pm}_{k,n} = \beta^{\pm}_k \cap \Omega_n \end{cases}$$

Hence:

$$\partial\Omega'_n = \partial\Omega_n + \sum_k \beta^+_{k,n} - \sum_k \beta^-_{k,n} .$$

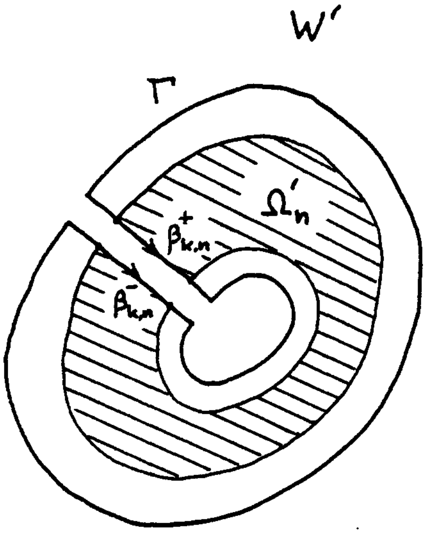


fig 2.3

Of the computations on p. 2.4 the following can be saved (with Ω_n in place of W):

$$(2.5) \quad \int_{\Omega_n} df \wedge d\bar{g} = \int_{\partial\Omega_n} fd\bar{g} + \sum_k \int_{\alpha_k} df \cdot \int_{\beta^-_{k,n}} d\bar{g} ,$$

and (starting from the end and going backwards):

$$(2.6) \quad 2\pi i \sum_k \operatorname{res}_{\zeta_k}(fdg^*) + 2\pi i \sum_k \operatorname{res}_{\xi_k} dg^* \cdot \int_{\gamma_k} df - \sum_k \int_{\beta_k} dg^* \cdot \int_{\alpha_k} df =$$

$$= 2\pi i \sum_{\Omega_n} \operatorname{res}(fdg^*) - \sum_k \int_{\beta_k} dg^* \cdot \int_{\alpha_k} df$$

$$= \int_{\partial\Omega'_n} fdg^* - \sum_k \int_{\beta_k} dg^* \cdot \int_{\alpha_k} df$$

$$(2.6) \quad = \int_{\partial\Omega_n} fdg^* + \sum_k \int_{\alpha_k} df \cdot \int_{\beta^-_{k,n}} dg^* - \sum_k \int_{\alpha_k} df \cdot \int_{\beta_k} dg^* .$$

Since, as $n \to \infty$:

$$\int_{\Omega_n} df \wedge d\bar{g} \to \int_W df \wedge d\bar{g}$$

and

$$\int_{\beta^-_{k,n}} dg^* - \int_{\beta^-_{k,n}} d\bar{g} \to \int_{\beta_k} dg^* \qquad \text{(by (2.4))} ,$$

comparing equations (2.5) and (2.6) gives, that in order to prove formula (2.1), it is enough to prove that:

$$(2.7) \qquad \int_{\partial\Omega_n} f(d\bar{g} - dg^*) \to 0 \quad \text{as} \quad n \to \infty .$$

To see that (2.7) holds we use a partition of unity argument:

Let $\varphi_1, \ldots, \varphi_N$ be C^∞ functions on $\hat{W}$ with compact supports contained in coordinate neighbourhoods $U_1, \ldots, U_N$ and such that:

$$\sum_{k=1}^{N} \varphi_k = 1 \text{ in a neighbourhood of } \Gamma.$$

Then we must prove:

$$\int_{\partial\Omega_n} \varphi_k \cdot f(d\bar{g} - dg^*) \to 0$$

as $n \to \infty$, $k = 1, \ldots, N$.

Let φ denote any of $\varphi_1, \ldots, \varphi_N$, and let $z = x + iy$ be a local parameter $z : U \to Q$, where

$$\begin{cases} U \supset \operatorname{supp} \varphi , \\ Q = \{x + iy \in \mathbb{C} : |x| < 1, |y| < 1\} \end{cases}$$

and such that:

$$\begin{cases} z(U \cap W) = Q \cap \mathbb{C}^+ , \\ z(U \cap \Gamma) = Q \cap \mathbb{R} . \end{cases}$$

(It is clearly enough to consider those φ_j for which $\operatorname{supp} \varphi_j \cap \Gamma \neq \phi$.)

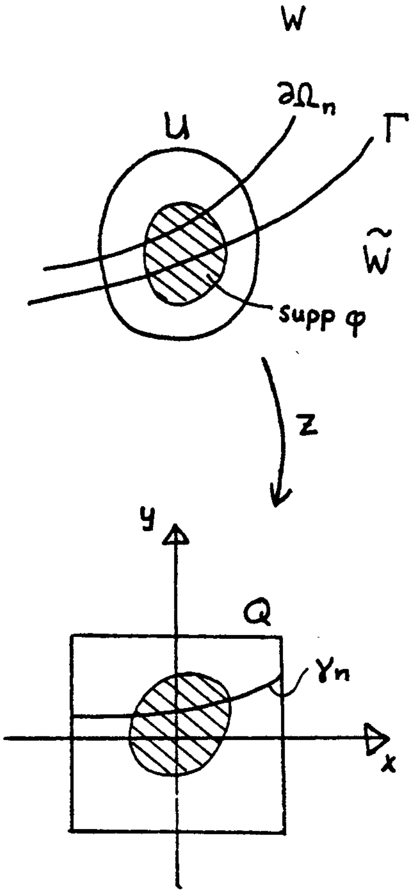


fig 2.4

Put:

$$\begin{cases} \gamma_n = z(\partial\Omega_n \cap U) , & \\ a(x, y)dx + b(x, y)dy = d\bar{g} - dg^*, & (x, y) \in Q , \\ h(x, y) = \varphi \cdot f , & (x, y) \in Q \cap \mathbb{C}^+ . \end{cases}$$

Observe that adx + bdy is continuous on Q and that a = 0 on $\mathbb{R}$ (since $d\bar{g} = dg^*$ along Γ).

Clearly, we can assume that the domains Ω_n were chosen so that γ_n have equations:

$$y = y_n(x) \qquad n = 1, 2, \dots$$

with

$$\max_{|x|\leq 1} \left|\frac{dy_n(x)}{dx}\right| \to 0 \quad \text{as } n \to \infty .$$

Now, we have to prove that:

$$\int_{\gamma_n} h \cdot (adx + bdy) \to 0 \quad \text{as } n \to \infty$$

i.e. that:

$$\int_{-1}^{1} h(x, y_n(x))(a(x, y_n(x)) + b(x, y_n(x)) \frac{dy_n(x)}{dx})dx \to 0 .$$

Since

$$\max_{|x|\leq 1} \left|a(x, y_n(x)) + b(x, y_n(x)) \frac{dy_n(x)}{dx}\right| \to 0$$

by the above remarks, it is enough to prove that:

$$\int_{-1}^{1} |h(x, y_n(x))|dx \leq M < \infty$$

for some constant M.

But, now $df \in \Gamma_a(W)$ implies that $h = \varphi \cdot f$ has finite Dirichlet-integral (over $Q \cap \mathbb{C}^+$):

$$D(h) = \int_{-1}^{1}\int_{0}^{1} \left(\left|\frac{\partial h}{\partial x}\right|^2 + \left|\frac{\partial h}{\partial y}\right|^2\right) dxdy < \infty .$$

In particular

$$\int_{-1}^{1}\int_{0}^{1} \left|\frac{\partial h}{\partial y}\right|^2 dxdy < \infty .$$

Hence

$$\int_{-1}^{1}\int_{0}^{1} \left|\frac{\partial h}{\partial y}\right| dxdy \leq \left(\int_{-1}^{1}\int_{0}^{1} \left|\frac{\partial h}{\partial y}\right|^2 dxdy\right)^{1/2} \cdot \left(\int_{-1}^{1}\int_{0}^{1} 1^2 \; dxdy\right)^{1/2} < \infty .$$

On the other hand (using that $h(x, 1) = 0$):

$$|h(x, y_n(x))| = \left|\int_{y_n(x)}^{1} \frac{\partial h(x, y)}{\partial y} dy\right| \leq \int_{y_n(x)}^{1} \left|\frac{\partial h}{\partial y}\right| dy$$

so that, finally:

$$\int_{-1}^{1} |h(x, y_n(x))| dx \leq \int_{-1}^{1}\int_{y_n(x)}^{1} \left|\frac{\partial h}{\partial y}\right| dydx \leq \int_{-1}^{1}\int_{0}^{1} \left|\frac{\partial h}{\partial y}\right| dxdy < \infty ,$$

as was to be proved.

Quadrature data

To simplify the formulations of the forthcoming theorems we introduce the symbols Q and A to denote totalities of quadrature data as follows:

Given $\hat{W} = W \cup \Gamma \cup \tilde{W}$ of genus p, W (conformally equivalent to) a plane domain, a data set Q shall consist of:

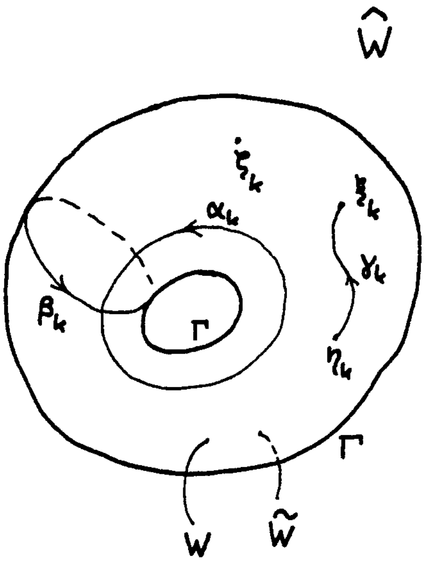


fig 2.5

① a finite number of points $\zeta_1, \ldots \in W$ together with local variables at these points,

② integers $n_1, \ldots \geq 0$ associated with $\zeta_1, \ldots$,

③ a finite number of arcs $\gamma_1, \ldots \subset W$, supposed not to intersect any of $\alpha_1, \ldots, \beta_p$ (canonical homology basis),

④ a subset $I \subset \{1, 2, \ldots, p\}$.

and a <u>data set A</u> associated with Q (the coefficients) shall consist of:

① coefficients $a_{k,1}, \ldots, a_{k,n_k}$ for each ζ_k, $k = 1, \ldots$,

② coefficients $b_1, \ldots$ for each $\gamma_1, \ldots$,

③ coefficients c_k for each $k \in I$.

Given Q we introduce two classes of differentials:

D_Q: $df \in D_Q$ if $df \in \Gamma_a(W)$ and

$$\begin{cases} f'(\zeta_k) = \ldots = f^{(n_k)}(\zeta_k) = 0 \ , & k = 1, \ldots \\ \int_{\gamma_k} df = 0 \ , & k = 1, \ldots \\ \int_{\alpha_k} df = 0 \ , & k \in I \end{cases}$$

E_Q: $dg \in E_Q$ if dg is meromorphic on $\hat{W}$ with (and having no other singularities than)

$$\begin{cases} \text{residue-free poles of orders} \leq n_k + 1 \text{ at } \tilde{\zeta}_k \ , & k = 1, \ldots \\ \text{simple poles at } \tilde{\xi}_k \text{ and } \tilde{\eta}_k \text{ with } \operatorname*{res}_{\tilde{\xi}_k} dg + \operatorname*{res}_{\tilde{\eta}_k} dg = 0 \ , & k = 1, .. \\ \int_{\beta_k} dg = 0 \ , & k \notin I \end{cases}$$

Given Q and A we define the functional $L_{Q,A}$ on $\Gamma_a(W)$:

$$L_{Q,A}(df) = \sum_k \sum_{j=1}^{n_k} a_{kj} f^{(j)}(\zeta_k) + \sum_k b_k \int_{\gamma_k} df + \sum_{k \in I} c_k \int_{\alpha_k} df .$$

Remark: $f^{(j)}(\zeta_k)$ means of course the j:th derivative of $f(\zeta) = \int^{\zeta} df$ with respect to the local variable chosen at ζ_k.

Now we can state one of the main theorems.

Theorem 2.2: Suppose $dg \in \Gamma_a(W)$ and quadrature data Q are given. Then the following are equivalent:

a) dg extends to a meromorphic differential dG on $\hat{W}$ with $dG \in E_Q$.

b) there exist coefficient data A such that

$$\frac{1}{2\pi i} \int_W df \wedge \overline{dg} = L_{Q,A}(df) \quad \text{for all } df \in \Gamma_a(W) .$$

c) $$\frac{1}{2\pi i} \int_W df \wedge \overline{dg} = 0 \quad \text{for all } df \in D_Q .$$

Comments:

① $D_Q = \left\{ df \in \Gamma_a(W) : L_{Q,A}(df) = 0 \text{ for all A (associated with Q)} \right\}$,

i.e. D_Q is the (pre-)annihilator of the finite-dimensional space of functionals

$$\left\{ L_{Q,A} \in \Gamma_a(W)^* : \text{all A associated with Q} \right\}.$$

By elementary functional analysis this in fact proves: b) ⟺ c). 1)

② In b) dg is the element $\in \Gamma_a(W)$ which represents the functional $L_{Q,A}$ in the Hilbert space $\Gamma_a(W)$. Hence, taking Q to consist of a single point ζ_1 (and $a_{11} = -\frac{1}{\pi}$) dg becomes the Bergman kernel for the class $\Gamma_a(W)$:

$$dg(z) = k(z, \zeta_1)dz , \quad z \in W .$$

The extension of dg to $\hat{W}$ given by a) is the so called <u>adjoint kernel</u> $h(z, \zeta_1)dz$:

$$dg^*|_W(z) = h(z, \zeta_1)dz$$

③ For $I \subset \{1, \ldots, p\}$ there are the intermediate Hilbert spaces $\Gamma_{a,I}(W)$ defined by:

$$\Gamma_{a,I}(W) = \left\{ df \in \Gamma_a(W) : \int_{\alpha_k} df = 0 \quad \text{for} \quad k \in I \right\} .$$

Clearly:

$$I = \phi \Rightarrow \Gamma_{a,I} = \Gamma_a$$

$$I = \{1, \ldots, p\} \Rightarrow \Gamma_{a,I} = \Gamma_{ae}$$

and $\Gamma_{ae} \subset \Gamma_{a,I} \subset \Gamma_a$ in general.

The theorem can be adapted for these spaces as follows:

<u>Theorem 2.3</u>: Let $dg \in \Gamma_{a,I}(W)$ and quadrature data Q be given, where it is supposed that I is the subset of $\{1, \ldots, p\}$ which occurs in Q. Then the following are equivalent:

a) dg extends to a meromorphic differential dG on $\hat{W}$ with $dG \in E_Q$.

b) there are coefficient data A with $c_k = 0$ for all $k \in I$ such that

$$\frac{1}{2\pi i} \int_W df \wedge \overline{dg} = L_{Q,A}(df) \quad \text{for all} \quad df \in \Gamma_{a,I}(W) .$$

c) $$\frac{1}{2\pi i} \int_W df \wedge \overline{dg} = 0 \quad \text{for all} \quad df \in D_Q \subset \Gamma_{a,I}(W) .$$

④ Taking $I = \{1, \ldots, p\}$ and Q to consist of a single point ζ_1 ($a_{11} = -\frac{1}{\pi}$) together with I in the theorem, $dg|_W$ and $dg^*|_W$ become the Bergman kernel and adjoint kernel respectively for the class $\Gamma_{a,I}(W) = \Gamma_{ae}(W)$.

Moreover, $dg|_W$ and $dg^*|_W$ have single-valued integrals, g and g* on W, and the functions

$$\begin{cases} g_1 = g^* + g \\ g_2 = g^* - g \end{cases}$$

map W univalently onto horizontal resp. vertical slit domains in $\mathbb{P}$. Even more, for all λ, $\mu > 0$ the function:

$$\lambda \cdot g_1 + \mu \cdot g_2 = (\lambda + \mu)g^* - (\lambda - \mu)g$$

is univalent on W and maps W onto a domain in $\mathbb{P}$ the complement of which consists of convex sets bounded by analytic Jordan curves. In particular $\lambda = \mu = 1/2$ gives that g^* is univalent on W. This has a consequence which will be used later, namely that for $p > 0$, g is <u>not univalent</u> on W. In fact, since dg has a pole of order 2 at $\tilde{\zeta}_1$ as only singularity it follows that dg has altogether

$$(2p - 2) + 2 = 2p$$

zeroes on $\hat{W}$. But since g^* is univalent on W, dg^* has no zeroes on W or even on $W \cup \Gamma$, since Γ is mapped onto regular analytic Jordan curves. Hence dg has no zeroes on $\tilde{W} \cup \Gamma$, and so must have all its 2p zeroes on W. And if $p > 0$ this implies that g cannot be univalent on W.

(If $p = 0$, g <u>is</u> univalent on W, since $\hat{W} = \mathbb{P}$, W = a disc or a half-plane and g a meromorphic function of order 1, i.e. a Möbius-transformation.)

The fact that dg has 2p zeroes on W is interesting enough to be stated explicitly:

<u>Theorem 2.4</u>: if W is a plane domain bounded by $p + 1$ continua, the reproducing kernel (Bergman kernel), $k(z, \zeta)dz$, for the class $\Gamma_{ae}(W)$ has exactly 2p zeroes in W, for every choice of $\zeta \in W$.

<u>Proof of theorem 2.2</u>: b) ⟺ c) is already proved. (Comment ① .)

a) ⟹ b) follows directly from the formula (2.2).

b) → a): it is a standard fact that on $\hat{W}$ (or any compact Riemann surface) there exists a unique meromorphic differential dG with:

1) poles with arbitrary prescribed singular parts, subject to the only condition that the sum of the residue be 0.

2) the periods $\int_{\beta_k} dG$, $k = 1, \ldots, p$ prescribed.

Hence it follows from the formula (2.2) that there exists a unique $dG \in E_Q$ such that

$$\frac{1}{2\pi i} \int_W df \wedge \overline{dG} = L_{Q,A}(df) \quad \text{for all} \quad df \in \Gamma_a(W)$$

But clearly $dG|_W \in \Gamma_a(W)$ so the uniqueness for the representative of a functional on a Hilbert space gives that $dG|_W = dg$. Hence $dG \in E_Q$ is the extension of dg, and the theorem is proved.

3. Behaviour under Conformal Mapping and Existence Theorems

Suppose $\Phi : W \rightarrow W'$ is a conformal equivalence between the plane domains W and W'. Then Φ induces an isometric isomorphism

$$\Phi^* : \Gamma_a(W') \rightarrow \Gamma_a(W)$$

by "pull-back of differentials". That is:

$$\Phi^*(df) = d(f \circ \phi) \quad \text{for} \quad df \in \Gamma_a(W') \ ,$$

or, more explicitly, if z and ζ are local variables at corresponding points on W resp W', and

$$df = \frac{df}{d\zeta}\, d\zeta \ ,$$

then

$$\Phi^*(df) = df(\Phi(z)) = \frac{df}{d\zeta} \cdot \frac{d\zeta}{dz} \cdot dz \ .$$

That Φ^* is isometric is an immediate consequence of the "change of variable formula":

$$\int_{W'} df \wedge \overline{dg} = \int_{\Phi(W)} df \wedge \overline{dg} = \int_{W} d(f \circ \Phi) \wedge d(g \circ \Phi) =$$

$$= \int_{W} \Phi^*(df) \wedge \overline{\Phi^*(dg)} \ , \quad df,\ dg \in \Gamma_a(W') \ .$$

Loosely speaking, the inner product on $\Gamma_a(W)$ is invariantly defined.

If Q is a set of quadrature data on W, Φ maps Q onto a dataset Q' on W' defined in an obvious way

(if Q consists of $\zeta_1, \ldots ; n_1, \ldots ; \gamma_1, \ldots ; I$, Q' consists of $\Phi(\zeta_1), \ldots ; n_1, \ldots ; \Phi(\gamma_1), \ldots ; I$, where it is assumed that $I \subset \{1, \ldots, p\}$ refers to canonical homology basises $\alpha_1, \ldots, \beta_p$ on W resp $\Phi(\alpha_1), \ldots, \Phi(\beta_p)$ on W').

Hence there are the subspaces

$$D_Q \subset \Gamma_a(W)$$

and

$$D_{Q'} \subset \Gamma_a(W') ,$$

and it is easily seen that Φ^* maps $D_{Q'}$ onto D_Q.

From these considerations it follows, that given:

$$\begin{cases} dg \in \Gamma_a(W') \\ dG = \Phi^*(dg) \in \Gamma_a(W) , \end{cases}$$

$$\frac{1}{2\pi i} \int_{W'} df \wedge \overline{dg} = 0 \quad \text{for all} \quad df \in D_{Q'}$$

if and only if

$$\frac{1}{2\pi i} \int_{W} dF \wedge \overline{dG} = 0 \quad \text{for all} \quad dF \in D_Q .$$

By the equivalence b) $\Longleftrightarrow$ c) in Theorem 2.2 we therefore get (with Q, Q', dg, dG given as above):

there exist coefficients A' such that

$$\frac{1}{2\pi i} \int_{W'} df \wedge \overline{dg} = L_{Q',A'}(df) \quad \text{for all} \quad df \in \Gamma_a(W')$$

if and only if there exist coefficients A such that

$$\frac{1}{2\pi i} \int_{W} dF \wedge \overline{dG} = L_{Q,A}(dF) \quad \text{for all} \quad dF \in \Gamma_a(W)$$

Now, we apply this in the following situation:

Let $W' = \Omega \subset \mathbb{C}$ be a domain of finite area bounded by $p + 1$ continua*. By Lemma 1.1 there exists a compact symmetric Riemann surface $\hat{W} = W \cup \Gamma \cup \tilde{W}$ and a conformal equivalence

$$G : W \to \Omega .$$

---

* a continuum is a closed connected set consisting of more than one point

If z denotes the usual coordinate variable in $\mathbb{C}$, we have:

$$\begin{cases} dz \in \Gamma_a(\Omega) \\ G^*(dz) = d(z \circ G) = dG \in \Gamma_a(W) \ . \end{cases}$$

Suppose quadrature datas Q' are given on Ω, and let Q be the corresponding datas on W.

Observing that

$$\frac{1}{2\pi i} \int_\Omega df \wedge \overline{dz} = \frac{1}{2\pi i} \int_\Omega \frac{df}{dz} dz d\bar{z} = - \frac{1}{\pi} \int_\Omega f' dxdy \ ,$$

Theorem 2.2 then gives:

Theorem 3.1: If $\Omega \subset \mathbb{C}$, $W = \hat{W} \cup \Gamma \cup \tilde{W}$, $G : W \to \Omega$, Q and Q' are as above, the following are equivalent:

a) dG extends to a meromorphic differential on $\hat{W}$ with $dG \in E_Q$.

b) there exists coefficients A' such that

$$\int_\Omega fdxdy = L_{Q',A'}(fdz) \quad \text{for all} \quad f \in L^2_a(\Omega) \ (\text{i.e. } fdz \in \Gamma_a(\Omega)).$$

c) $\int_\Omega fdxdy = 0$ for all $fdz \in D_{Q'}$.

A more suggestive version of the theorem is obtained if one identifies W with Ω via G. The differential dG is then identified with dz.

Suppose first that dG extends to $\hat{W}$. Then its values on the "backside" $\tilde{W}$ may be represented by $dG^*|_W$ $(dG^*(\zeta) = dG(\tilde{\zeta}))$.

Observe that

$$dG|_W = \overline{dG^*_{|W}} \quad \text{on } \Gamma$$

and that $dG^*|_W$ is the only meromorphic differential on W with this property. Hence there is a meromorphic differential $dH(z) = h(z)dz$ on Ω with

$$(3.1) \quad dz = \overline{h(z)dz} \quad \text{on} \quad \partial\Omega .$$

Indeed

$$dH = dG^*|_W \circ G^{-1} .$$

Observe also that (when dG is meromorphic on $\hat{W}$) $\partial\Omega$ consists of analytic curves by Lemma 1.1, and that $h(z)dz$ becomes continuous on $\partial\Omega$, so that (3.1) can be interpreted literally. (3.1) can also be written in integrated form:

$$(3.2) \quad z = \overline{H(z)} + \text{local constant} , \quad z \in \partial\Omega.$$

Here "local constant" means: constant on each component of $\partial\Omega$; in general the constant takes different values on different components of $\partial\Omega$ because dG may have periods on $\hat{W}$. The integral $H(z)$ is single-valued in a neighbourhood of $\partial\Omega$; the only periods of $H(z)$ are those arising from logarithmic singularities corresponding to the residue poles of dG^*.

Suppose conversely that we do not know a priori that $G : W \rightarrow \Omega$ extends to $\hat{W}$, but that we have a meromorphic differential $dH(z) = h(z)dz$ on Ω which satisfies (3.1) - (3.2) in the sense that:

$$(3.3) \quad z - \overline{H(z)} \rightarrow c_j$$

as $z \rightarrow$ the j:th component of $\partial\Omega$ for some constants $c_1, \dots, c_{p+1}$ ($p + 1$ = connectivity of Ω).

Then:

$$G(\zeta) - \overline{H(G(\zeta))} \rightarrow c_j \quad \text{as} \quad \zeta \rightarrow \Gamma_j \quad (\zeta \in W) ,$$

where $\Gamma_1, \dots, \Gamma_{p+1}$ are the components of Γ, appropriately numbered.

It follows that the harmonic function:

$$u(\zeta) = G(\zeta) - \overline{H(G(\zeta))} , \quad \zeta \in W$$

extends over Γ_j by reflection:

$$u(\tilde{\zeta}) - c_j := - (u(\zeta) - c_j) \quad \text{for} \quad \zeta \in W \text{ near } \Gamma_j .$$

This makes du into a (complex-valued) harmonic differential on $\hat{W}$, with the properties:

$$\begin{cases} du(\zeta) = dG(\zeta) - d\overline{H(G(\zeta))} & \text{for} \quad \zeta \in W , \\ du(\zeta) = \overline{dH(G(\tilde{\zeta}))} - dG(\tilde{\zeta}) & \text{for} \quad \zeta \in \tilde{W} . \end{cases}$$

In other words, the analytic part of $du(\zeta)$ is $dG(\zeta)$ on W and $dH(G(\tilde{\zeta}))$ on $\tilde{W}$, and hence dG itself extends to a meromorphic differential on $\hat{W}$ by:

$$dG(\zeta) := \overline{dH(G(\tilde{\zeta}))} \quad \text{for} \quad \zeta \in \tilde{W} .$$

Hence the condition (3.3) forces dG to extend meromorphically to $\hat{W}$, and therefore also $\partial\Omega$ to consist of analytic curves and H(z) to extend analytically over $\partial\Omega$ (the analytic curves may be only quasi-regular in the sense of Lemma 1.1, and the extension of H(z) may be multiple-valued with branch points on $\partial\Omega$).

Combining this discussion with Theorem 3.1 now gives (denoting the items in Q' in that theorem by $\zeta_1, \ldots; n_1, \ldots; \gamma_1, \ldots; I$):

Theorem 3.2: *Suppose* $\Omega \subset \mathbb{C}$ has finite area and is bounded by p + 1 continua ($p < \infty$). *Then* Ω admits a quadrature formula of the type

$$\int_\Omega f dxdy = \sum_k \sum_{j=o}^{n_k-1} a_{kj} f^{(j)}(\zeta_k) + \sum_k b_k \int_{\gamma_k} fdz + \sum_{k \in I} c_k \int_{\alpha_k} fdz .$$

for all $f \in L_a^2(\Omega)$

if and only if the differential dz extends to a meromorphic differential on the double $\hat{\Omega}$, in the sense that there exists a meromorphic differential

$$dH(z) = h(z)dz \quad \text{on} \quad \Omega$$

with

$$dz = \overline{h(z)dz} \quad \text{on } \partial\Omega \ ,$$

interpreted according to (3.3).

When this occurs the singularities of dz on the "back-side", i.e. the singularities of $dH(z) = h(z)dz$, correspond to the data $\zeta_1, \ldots; n_1, \ldots; \gamma_1, \ldots, I$ in the following way:

the integral $H(z)$ has:

① poles of orders $\leq n_k$ at $z = \zeta_k$, $k = 1, \ldots$,

② logarithmic singularities of opposite sign at the end-points ξ_k and η_k of γ_k, $k = 1, \ldots$,

③ and if two boundary components of $\partial\Omega$ can be connected without intersecting any α_k with $k \in I$, then the local constant in

$$z = \overline{H(z)} + \text{local constant on } \partial\Omega$$

takes the same value on these two components.

The last point, ③ , is the interpretation of the part of the statement $dG \in E_Q$ in Theorem 3.1 which says that

$$\int_{\beta_k} dG = 0 \quad \text{for} \quad k \notin I \ .$$

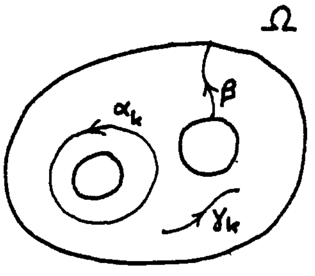


fig 3.1

In fact, if $\beta \subset \Omega$ connects two components of $\partial\Omega$ without intersecting any α_k with $k \in I$ (fig. 3.1), then, if $\tilde{\beta}$ is the corresponding arc on the back-side $\tilde{\Omega}$, $\beta \cup (-\tilde{\beta})$ is a closed curve on $\hat{\Omega}$, and the integral of the extension of dz to $\hat{\Omega}$

along this curve shall be = 0 (we can clearly choose β so that it avoids all the γ_1, ...). But this means:

$$0 = \int_{\beta \cup (-\tilde{\beta})} dz = \int_{\beta} dz - \int_{\beta} \overline{h(z)dz} = \int_{\beta} d(z - \overline{H(z)}),$$

which proves ③ .

Remarks:

① The function $H(z) + c_j$ in (3.3) is the Schwartz function for the j:th component of $\partial\Omega$. For a treatment of quadrature formulas from the Schwarz-function-approach see [D 1, Ch 14] and [D 2].

② In this report we always assume a priori that the domains Ω in consideration have finite area and finite connectivity. However, if a domain $\Omega \subset \mathbb{C}$ is subject only to the condition:

$$\int_{\Omega} \frac{dxdy}{|z|} < \infty ,$$

and admits a quadrature formula of the type:

$$\int_{\Omega} fdxdy = \sum_{k} \sum_{j=o}^{n_k - 1} a_{kj} f^{(j)}(\zeta_k)$$

for the class $L_a^1(\Omega)$, then it <u>must</u> be of finite connectivity. For a proof of this and other aspects of quadrature identities, see [AS 2].

Now, we return to Theorem 3.1. It implies that if:

$$\begin{cases} \Omega \subset \mathbb{C} \text{ has finite area and is bounded by } p + 1 \text{ continua,} \\ \hat{W} = W \cup \Gamma \cup \tilde{W} \text{ is a compact symmetric Riemann surface of genus} = p, \\ G : W \to \Omega \text{ is a conformal equivalence,} \end{cases}$$

then Ω admits a quadrature formula for the class $L_a^2(\Omega)$ if and only if dG extends to a meromorphic differential on $\hat{W}$ (if dG extends to $\hat{W}$ then one can clearly choose data Q so that $dG \in E_Q$).

This shows how to produce domains $\Omega \subset \mathbb{C}$ of arbitrary conformal type (with the usual restrictions) admitting quadrature formulas:

Start with any compact symmetric $\hat{W} = W \cup \Gamma \cup \tilde{W}$, such that W is of the requested conformal type. If we can find a meromorphic differential dG on $\hat{W}$ such that $dG|_W \in \Gamma_{ae}(W)$ and its integral G is univalent on W, then G maps W conformally onto a quadrature domain $\Omega \subset \mathbb{C}$.

The condition $dG|_W \in \Gamma_{ae}(W)$ means exactly that dG shall have all its singularities on the back-side $\tilde{W}$ and that G is single-valued on W. These conditions are easily satisfied. In fact, the poles $\in \tilde{W}$ and singular parts (with sum of residues = 0) can be arbitrarily prescribed. The problem is to get G univalent on W.

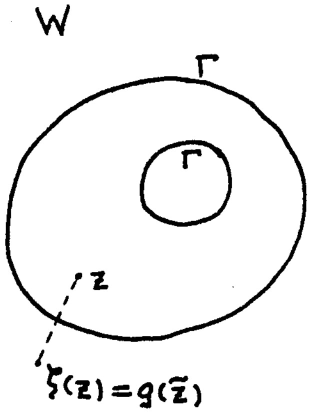


fig 3.2

This problem may be solved by an approximation argument as follows: Realize $\hat{W} = W \cup \Gamma \cup \tilde{W}$ as the Schottky double of a bounded plane domain $W \subset \mathbb{C}$ bounded by regular analytic curves, Γ. Then the identity function

$$z : W \to \mathbb{C}$$

extends by reflection in Γ to a holomorphic function g defined in some neighbourhood of $W \cup \Gamma$ in $\hat{W}$. In fact, if $\zeta(z)$ is the reflection mapping in Γ, g is defined by

$$\begin{cases} g(z) = z & \text{for } z \in W \cup \Gamma \\ g(\tilde{z}) = \zeta(z) & \text{for } \tilde{z} \in \tilde{W} \cap \text{(some neighbourhood of } \Gamma)\,(\tilde{z} \in \tilde{W} \longleftrightarrow z \in W) \end{cases}$$

Since z is univalent in $W \cup \Gamma$, g will be univalent in some neighbourhood of $W \cup \Gamma$.

The next step is to approximate g uniformly in some neighbourhood of $W \cup \Gamma$ by meromorphic functions on $\hat{W}$. To do that one needs a Runge approximation theorem for compact Riemann surfaces. Such theorems are very hard to find in the literature. In fact, the only reference I have found is the paper [G 2] where a stronger theorem, a Mergelyan theorem for compact Riemann surfaces-, is proved. The proof, however, depends on references to other papers, and is not very perspicuous. These circumstances motivated the report [G 3], which contains a proof of the following:

The Runge Approximation Theorem:

Let:

W' be a compact Riemann surface,

$\Omega \subset W'$ an open subset,

$E \subset W' \setminus \Omega$ a set which intersects each component of $W' \setminus \Omega$.

Then:

$M(W') \cap H(W' \setminus E)$ is dense in $H(\Omega)$.

In particular:

$M(W') \cap H(\Omega)$ is dense in $H(\Omega)$.

Here:

$H(\Omega) = \{$holomorphic functions on Ω, with the topology of uniform convergence on compacts$\}$

and

$M(W') = \{$meromorphic functions on $W'\}$.

In the application of this theorem we take:

$W' = \hat{W}$,

Ω = a neighbourhood of $W \cup \Gamma$ on which g is defined ,

E = any closed subset of $\tilde{W}$, for example consisting of a single point.

(E is chosen before Ω, and Ω is then chosen such that $\hat{W} \setminus \Omega$ is connected and $E \subset \hat{W} \setminus \Omega$.)

Thus g is approximated uniformly in some neighbourhood of $W \cup \Gamma$ by a function $G \in M(\hat{W}) \cap H(\hat{W} \setminus E)$, and making the approximation sufficiently fine we achieve that G is univalent on some neighbourhood of $W \cup \Gamma$. Moreover, since G is single-valued on the whole $\hat{W}$, the quadrature formula for $\Omega = G(W)$ will not contain line integrals. This proves:

Theorem 3.3 (existence of quadrature domains):

Suppose $W \subset \mathbb{C}$ is bounded, and bounded by regular analytic curves. Then there is a slightly perturbed domain Ω, conformally equivalent to W, which has a quadrature formula of the type:

$$\int_{\Omega} f dx dy = \sum_{k=1}^{m} \sum_{j=o}^{n_k - 1} a_{kj} f^{(j)}(z_k) \quad \text{for} \quad f \in L_a^2(\Omega) \ .$$

The number, m, and the relative conformal locations (i.e. the preimages in W) of the points $z_1, \ldots, z_m \in \Omega$ can be prescribed at will.

Remark: One easily obtains versions of the approximation theorem which guarantee that all n_k can be chosen = 1 in the quadrature formula. In that case one of course loses the control of the number m.

Another consequence of Theorem 3.1 is that a quadrature domain must have nice boundary:

Theorem 3.4: Suppose $\Omega \subset \mathbb{C}$ has finite area, is bounded by a finite number of continua and admits a quadrature formula:

$$\int_{\Omega} f dx dy = \sum_{k} \sum_{j=o}^{n_k - 1} a_{kj} f^{(j)}(\zeta_k) + \sum_{k} b_k \int_{\gamma_k} f dz + \sum_{k} c_k \int_{\alpha_k} f dz$$

for $f \in L_a^2(\Omega)$.

Then $\partial\Omega$ consists of quasi-regular analytic curves. Moreover, if all b_k and $c_k = 0$, $\partial\Omega$ is a whole algebraic curve.

Proof: Combine Theorem 3.1 with Lemma 1.1 iii) - iv).

When $\partial\Omega$ is an algebraic curve (b_k, $c_k = 0$) the relations between the coefficients in the polynomial of the curve and the data in the quadrature formula may be stated explicitly. This topic will be taken up in section 5.

The fact that $\partial\Omega$ is an algebraic curve when b_k, $c_k = 0$ was first proven in [AS 2] (Theorem 3 and Theorem 6), where also results about $\partial\Omega$ when only $b_k = 0$ can be found (Theorems 8 - 10).

4. Non-existence of Quadrature formulas of certain kinds

In section 3 we have discussed the relation between existence of quadrature domains and the existence of meromorphic differentials of certain kinds on compact symmetric Riemann surfaces. To recapitulate: If $\hat{W} = W \cup \Gamma \cup \tilde{W}$ is a compact symmetric Riemann surface, and if df is a meromorphic differential on $\hat{W}$ with all its singularities on $\tilde{W}$ and such that $f = \int df$ is a univalent function on W, then $\Omega = f(W)$ is a quadrature domain, and all quadrature domains of finite area and bounded by finitely many continua arise in this way (Theorem 3.1).

As a consequence, theorems about meromorphic differentials on compact Riemann surfaces may have corollaries about quadrature domain. This section contains three corollaries (4.2, 4.3 and 4.5) of this kind. The domains $\Omega \subset \mathbb{C}$ appearing in these corollaries are always assumed a priori to be of finite area and bounded by finitely many continua. Each corollary follows from the theorem which precedes it by an argument of the type: take a compact symmetric Riemann surface $\hat{W} = W \cup \Gamma \cup \tilde{W}$ and a conformal mapping $f : W \overset{\approx}{\to} \Omega$ (by Lemma 1.1). Then f (or df) extends meromorphically to $\hat{W}$ with a certain pole and period configuration according to Theorem 3.1. Finally, apply the theorem (belonging to the corollary) to this f.

Corollary 4.2 is a rather classical kind of converse of the mean-value property for analytic (or harmonic) functions. For a short survey of earlier results on this, and for further references, see [AS 2, 1.3]. Corollary 4.3 is a generalization of Corollary 4.2 which is also proved (in essentially the same way) in [AS 2] (Theorem 7).

Corollary 4.5 is a generalization of Theorem 4 in [AS 2], which (essentially) agrees with part (ii) of Corollary 4.5. Aharonov - Shapiro however use quite another method of proof, and this method has also been used by C Ullemar to prove part (i) with the a priori assumption that Ω is symmetric with respect to the straight line through z_1 and z_2.

Theorem 4.1: If f is meromorphic of order $= 1$ on $\hat{W} = W \cup \Gamma \cup \tilde{W}$ then:

a) $\hat{W} = \mathbb{P}$

b) $W =$ a disc or a half-plane

c) $f =$ a Möbius transformation

Proof: Well-known.

Corollary 4.2: If

$$\int_\Omega f dxdy = a \cdot f(z_o) \quad \text{for all} \quad f \in L_a^2(\Omega) \ ,$$

then Ω is a disc and z_o its center.

As a corollary of the statement in comment 4 after Theorem 2.2, that the meromorphic differential df on $\hat{W}$ with a double pole at $\tilde{\zeta}_1 \in \tilde{W}$ and with $df|_W \in \Gamma_{ae}(W)$ cannot be univalent on W unless $\hat{W}$ has genus 0 (in which case f is single-valued on $\hat{W}$ and Theorem 4.1 applies), we have:

Corollary 4.3: If

$$\int_\Omega f dxdy = a \cdot f(z_o) \quad \text{for all} \quad f \in L_{as}^2(\Omega) \ ,$$

then Ω is a disc and z_o its center.

Theorem 4.4: If $\hat{W} = W \cup \Gamma \cup \tilde{W}$ has genus $p > 0$, there is no meromorphic function, f, on $\hat{W}$ with the properties:

a) f has order 2

b) f is univalent on W

c) $\bar{\Omega} \subsetneqq \mathbb{P}$, where $\Omega = f(W)$.

Corollary 4.5: There is no domain $\Omega \subset \mathbb{C}$ of connectivity > 1 which admits a quadrature formula of the type

(i) $\int_\Omega f dxdy = a_1 f(z_1) + a_2 f(z_2)$

or

(ii) $\int_\Omega f\,dxdy = a_1 f(z_0) + a_2 f'(z_0)$ for all $f \in L^2_a(\Omega)$

Proof (of theorem 4.4): Any meromorphic function, f, of order 2 on $\hat{W}$ gives rise to an automorphism $\sigma_f : \hat{W} \to \hat{W}$ of order 2 (i.e. $\sigma_f \circ \sigma_f$ = identity) defined by:

$$\sigma_f(\zeta_1) = \zeta_2, \text{ where } \{\zeta_1, \zeta_2\} = f^{-1}(z) \text{ and } z \text{ ranges over } \mathbb{P} .$$

Now, suppose f satisfies the hypothesis in the theorem. Then it is easily seen that:

1) σ_f maps W bijectively onto a region $U_1 \subset \tilde{W}$ ($f^{-1}(\Omega) = W \cup U_1$)

2) σ_f maps $U_2 = \tilde{W} \setminus \bar{U}_1$ bijectively onto itself ($f^{-1}(\mathbb{P} \setminus \bar{\Omega}) = U_2$).

3) $U_2 \neq \phi$ (by c)).

$$\left(\text{if } \Omega = f(W), \begin{cases} f^{-1}(\Omega) = W \cup U_1 \\ f^{-1}(\mathbb{P} \setminus \bar{\Omega}) = U_2 \end{cases}\right)$$

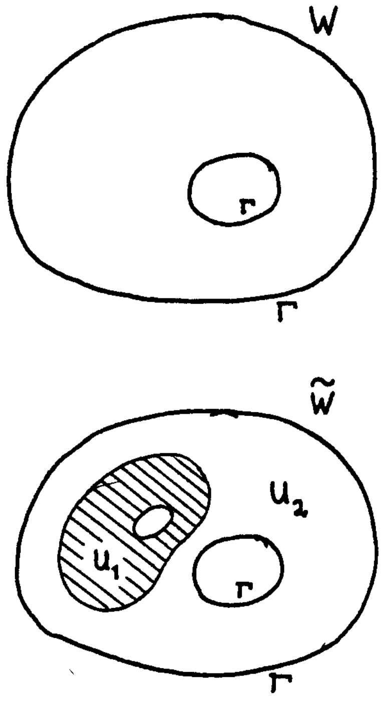


fig 4.1

This is however impossible:

the case p > 1: $\hat{W}$ is a hyperelliptic Riemann surface, and it is known ([G 1, p. 246]) that $\sigma = \sigma_f$ is unique*, i.e. does not depend on the choice of the function f (σ is the hyperelliptic automorphism of $\hat{W}$).

Hence $\sigma_f = \sigma_{f*}$ (recall $f*(\zeta) = \overline{f(\tilde{\zeta})}$),

i.e. $f(\zeta_1) = f(\zeta_2) \iff \overline{f(\tilde{\zeta}_1)} = \overline{f(\tilde{\zeta}_2)}$.

But take $\tilde{\zeta}_1 \in U_2$ (U_2 is not empty since $f(W \cup \Gamma) \subsetneq \mathbb{P}$).

---

* the footnote appears on p. 4.4.

Then

$$\tilde{\zeta}_2 = \sigma_f(\tilde{\zeta}_1) \in U_2$$

Hence

$$f(\tilde{\zeta}_1) = f(\tilde{\zeta}_2) \text{ and therefore } f(\zeta_1) = f(\zeta_2) .$$

Since $\zeta_1, \zeta_2 \in W$ and we could have chosen $\zeta_1 \in U_2$ so that $\tilde{\zeta}_2 \neq \tilde{\zeta}_1$ (σ_f has only finitely many fixed points), this contradicts the univalency of f.

Remarks:

① $\tilde{\zeta}_2 = \tilde{\zeta}_1$ ($\tilde{\zeta}_1$ fixed point of σ_f) means that $f'(\tilde{\zeta}_1) = 0$ and also $f'(\zeta_1) = 0$, so the univalency of f would have been contradicted also in that case.

② $\sigma_f = \sigma_{\tilde{f}}$ implies that $\sigma \circ \phi = \phi \circ \sigma$, which means that the hyperelliptic automorphism, σ, preserves the symmetry of $\hat{W}$.

---

* I have recently found a very short and nice proof of this in [A 2, p. 51]. I cannot resist the temptation to reproduce it:

Suppose f and g are two functions of order 2 on $\hat{W}$. We shall prove that $\sigma_f = \sigma_g$. This is equivalent to the statement that the meromorphic function

$$G(\zeta) = g(\zeta) - g(\sigma_f(\zeta))$$

is identically zero.

Suppose $G(\zeta) \not\equiv 0$.

Then G has at most 4 poles, since g has only 2 poles. On the other hand G has a zero at each fixed point of σ_f; and the fixed points of σ_f are exactly the branch points of f, the number of which is

$$2 \cdot (2 + p - 1) = 2p + 2 > 4$$

by a well-known formula ([W 1, p. 150]).

Hence G has more zeroes than poles which is impossible.
Hence $G(\zeta) \equiv 0$, what was to be proved.

<u>the case $p = 1$</u>: a different argument is needed since σ_f <u>does</u> depend on f in this case. Now $\hat{W}$ is a torus and can be represented by a period parallellogram, M (fig 4.2).

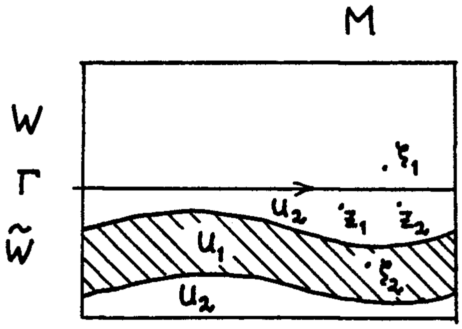


fig 4.2

If $z_1, z_2 \in \mathbb{C}$ are the poles of f, $\zeta_2 = \sigma_f(\zeta_1)$ then Abels theorem ([A 1, p. 263]) gives:

$$\zeta_1 + \zeta_2 \equiv z_1 + z_2 \pmod{M}$$

Hence σ_f is of the form:

$$\sigma_f(\zeta) \equiv -\zeta + a \pmod{M} \qquad (a = z_1 + z_2 = \text{constant})$$

Hence it is clearly impossible for σ_f to map W (exactly half of $\hat{W}$) onto U_1 (strictly included in the other half, $\tilde{W}$, of $\hat{W}$)

(for example σ_f preserves the area element in the period parallellogram, while area (U_1) < area (W)).

This conclues the proof of the theorem.

<u>Remarks</u>:

Corollaries 4.2, 4.3 and 4.5 can be put together to give the following:

No multiply connected domain Ω admits a quadrature identity of any of the following three types, for the test class $L_a^2(\Omega)$:

$$\int_\Omega f dxdy = a_1 f(z_1) + a_2 f(z_2) \ ,$$

$$\int_\Omega f dxdy = a_1 f(z_1) + a_2 f'(z_1) \ ,$$

$$\int_\Omega f dxdy = a_1 f(z_1) + a_2 \int_\gamma f dz \ .$$

Here z_1, $z_2 \in \Omega$ and γ is a closed curve in Ω.

On the other hand, according to [AS 2, 6.2], A. Levin has constructed a doubly connected domain Ω with

$$\int_\Omega f dxdy = a_1 f(z_1) + a_2 f'(z_1) \quad \text{for all} \quad f \in L^2_{as}(\Omega)$$

or, equivalently,

$$\int_\Omega f dxdy = a_1 f(z_1) + a_2 f'(z_1) + a_3 \int_\gamma f dz \quad \text{for all} \quad f \in L^2_a(\Omega)$$

Also, we have strong reasons to believe, but have as yet not been able to prove, that there exist doubly connected domains Ω with a quadrature identity of the kind

$$\int_\Omega f dxdy = a_1 f(z_1) + a_2 f(z_2) + a_3 f(z_3) \quad \text{for all} \quad f \in L^2_a(\Omega) \ .$$

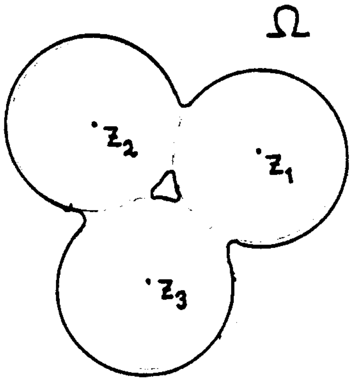


fig 4.3

Here is the intuitive idea: Ω should be a domain barely containing three tangent circular discs with centers z_1, z_2 and z_3, as indicated in fig. 4.3. To be slightly more precise, choose radii r_1, r_2, r_3 such that the interiors of the circles

$$C_j = \{z : |z - z_j| = r_j\} \ , \ j=1,2,3$$

intersect pair-wise, but have no common point of intersection.

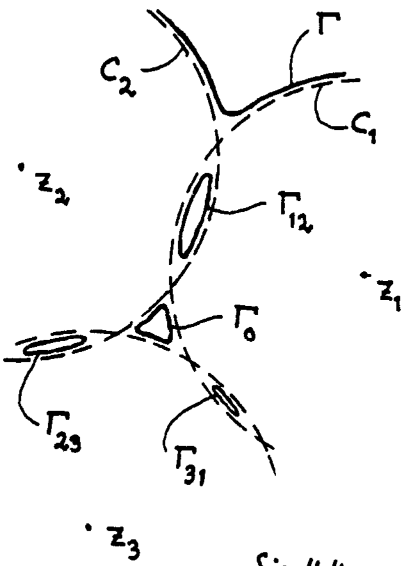


fig 4.4

Put

$$Q(x, y) = \prod_{j=1}^{3} (|z - z_j|^2 - r_j^2) \ .$$

Thus Q is a polynomial of degree 6 with zero-set $C_1 \cup C_2 \cup C_3$.

Let $R(x, y)$ denote a "perturbation polynomial" (with real coefficients) of degree ≤ 4. Then the zero-set of the polynomial

$$P(x, y) = Q(x, y) - R(x, y)$$

in general consists of 5 components Γ, Γ_0, Γ_{12}, Γ_{23}, Γ_{31} lying close to $C_1 \cup C_2 \cup C_3$ (if R is sufficiently small) as indicated by fig. 4.4.

Choosing R properly, however, the components Γ_{12}, Γ_{23} and Γ_{31} can be made to disappear (if, for example,

$$z_j = e^{\frac{2\pi i}{3}\cdot j} \quad , \quad j = 1, 2, 3 \ , \quad r_1 = r_2 = r_3 \ ,$$

the polynomial $R(x, y) = \alpha \cdot (x^2 + y^2)$ would do, for some $\alpha > 0$), so that the remaining part, $\Gamma \cup \Gamma_0$, looks like $\partial\Omega$ in fig. 4.3.

With such a choice of $R(x, y)$

$$\Omega = \{z \in \mathbb{C} : P(x, y) < 0\}$$

should be the quadrature domain in question.

Similar constructions with more than 3 circles should also give quadrature domains. Since each new circle gives one more interior component of $\partial\Omega$ and one more point in the quadrature formula, the following <u>conjecture</u> is natural:

For each $n > 0$, there exists quadrature domains Ω of connectivity n satisfying a quadrature identity of order $n + 1$:

$$\int_\Omega f dxdy = \sum_{j=1}^{n+1} a_j f(z_j) \quad \text{for all} \quad f \in L^2_a(\Omega) \ .$$

5. Quadrature domains bounded by algebraic curves

We shall study domains $\Omega \subset \mathbb{C}$ with quadrature formulas of the type:

$$\int_\Omega f dxdy = \sum_{k=1}^{m} \sum_{j=o}^{n_k-1} c_{kj} f^{(j)}(z_k) \quad , \quad f \in L^2_a(\Omega)$$

in somewhat closer detail.

Let us first summarize what we know about such domains:

① They are produced in the following way:

Take a plane bordered Riemann surface $W \cup \Gamma$ and a meromorphic function G on the double $\hat{W} = W \cup \Gamma \cup \tilde{W}$ such that G has all its poles on $\tilde{W}$ and is univalent on W.

Then $\Omega = G(W)$ is a quadrature domain of the above type.

If $\tilde{\zeta}_1, \ldots, \tilde{\zeta}_m \in \tilde{W}$ are the poles of G then

$$\begin{cases} z_k = G(\zeta_k) \\ n_k = \text{the order of the pole at } \tilde{\zeta}_k \end{cases}$$

(In this section we always assume that $c_{k,n_k-1} \neq 0$).

② With G as in 1 , G and G^* $(G^*(\zeta) = \overline{G(\tilde{\zeta})})$ generate the function-field on $\hat{W}$, and they satisfy an irreducible, self-conjugate polynomial equation

$$P(z, w) = \sum_{j,k=o}^{n} a_{jk} z^j w^k \qquad (a_{jk} = \bar{a}_{kj})$$

of degree $n = \Sigma_{k=1}^m n_k$ = order (G) in each of z and w, i.e.:

$$P(G(\zeta), G^*(\zeta)) \equiv 0 \quad , \quad \zeta \in \hat{W} \quad *$$

---

* The proofs of the statements ② and ③ are contained in the proof of Lemma 1.1, part iv).

③ $\partial\Omega$ = the algebraic curve of P, i.e.:

$$\partial\Omega = V_1 \ ,$$

where

$$V_1 \cup V_o = V = \{z \in \mathbb{P} : P(z, \bar{z}) = 0\}$$

and

$$V_o = \{\text{isolated points of } V\} \ .$$

④ Working in the domain Ω itself rather than on $\hat{W}$ we can put:

$$\begin{cases} z = G(\zeta) \ , \\ S(z) = G^*(\zeta) = G^*(G|_W^{-1}(z)) \ , \quad (\zeta \in W, \ z \in \Omega) \ . \end{cases}$$

Then:

$$\begin{cases} S(z) \text{ is meromorphic in } \Omega \text{ with poles of orders } n_k \text{ at } z_k, \ k = 1,..,m \ , \\ S(z) = \bar{z} \quad \text{on} \quad \partial\Omega \\ P(z, S(z)) \equiv 0 \ , \quad z \in \Omega \ . \end{cases}$$

S(z) is the Schwartz-function of $\partial\Omega$, i.e.:

$$S(z) = \overline{\zeta(z)}$$

where $\zeta(z)$ is the anticonformal reflection in $\partial\Omega$.

⑤ Conversely, if $\Omega \subset \mathbb{C}$ is a domain of finite area and bounded by a finite number of continua, and there exists a meromorphic function S(z) on Ω such that

$$S(z) = \bar{z} \quad \text{on} \quad \partial\Omega \quad (\text{i.e. } S(z) - \bar{z} \to 0 \quad \text{as} \quad z \to \partial\Omega) \ ,$$

then Ω is a quadrature domain, the formula, S(z), P(z, w) and $\partial\Omega$ being related as in ③ and ④ .

Now, suppose we have $\Omega \subset \mathbb{C}$ with the formula

$$\int_{\Omega} f dxdy = \sum_{k=1}^{m} \sum_{j=o}^{n_k-1} c_{kj} f^{(j)}(z_k) \ , \quad f \in L_a^2(\Omega) \ .$$

Since the polynomial P(z, w) obtained in ② is uniquely determined up to a real multiple $\neq 0$, it is natural to try to find the explicit relation between the quadrature data $z_1, \ldots, n_1, \ldots, \{c_{kj}\}$ and the coefficients $\{a_{kl}\}$ of P.

To begin with, the explicit relation between the coefficients $\{c_{kj}\}$ and the singular parts of S(z) is obtained from the quadrature formula (2.2) or, more easily, directly:

If:

$$S(z) = \sum_{k=1}^{m} \sum_{j=1}^{n_k} \frac{b_{kj}}{(z - z_k)^j} + \text{holomorphic function} \ ,$$

Stokes formula gives:

$$\int_{\Omega} f dxdy = \frac{1}{2i} \int_{\Omega} f d\bar{z} dz = \frac{1}{2i} \int_{\partial\Omega} f(z) \cdot \bar{z} dz = \frac{1}{2i} \int_{\partial\Omega} f(z) S(z) dz =$$

$$= \pi \cdot \sum_{z \in \Omega} \text{res } f(z) S(z) = \pi \cdot \sum_{k=1}^{m} \sum_{j=1}^{n_k} b_{kj} \cdot \frac{f^{(j-1)}(z_k)}{(j - 1)!} \ .$$

Hence:

$$c_{k,j} = \frac{\pi}{j!} \cdot b_{k,j+1} \ , \quad j = 0, 1, \ldots, n_k - 1, \quad k = 1, \ldots, m \ .$$

Next, let

$$P(z, w) = \sum_{j,k=o}^{n} a_{jk} z^j w^k = \sum_{k=o}^{n} P_k(z) w^k \ ,$$

where

$$a_{jk} = \overline{a_{kj}} \ , \quad P_k(z) = \sum_{j=o}^{n} a_{jk} z^j \ .$$

Here

$$n = \sum_{k=1}^{m} n_k = \text{the order of } S(z) \ .$$

Since $P(z,\ S(z)) \equiv 0$ we have:

$$0 \equiv \frac{P(z,\ S(z))}{S(z)^{n-1}} = \frac{P_o(z)}{S(z)^{n-1}} + \ldots + \frac{P_{n-2}(z)}{S(z)} + P_{n-1}(z) + P_n(z)S(z) =$$

$$= R(z) + P_{n-1}(z) + P_n(z)S(z)\ ,$$

where

$$R(z) = \frac{P_o(z)}{S(z)^{n-1}} + \ldots + \frac{P_{n-2}(z)}{S(z)}$$

Clearly $R(z) = O((z - z_k)^{n_k})$ as $z \to z_k$.

In particular $P_n(z)S(z) = - R(z) - P_{n-1}(z)$ is bounded as $z \to z_k$. This implies that $p_n(z)$ contains the factor $(z - z_k)^{n_k}$. Hence, since $P_n(z)$ has degree $n = \Sigma\ n_k$, $P_n(z)$ must be

$$P_n(z) = a_{nn}(z - z_1)^{n_1} \cdot \ldots \cdot (z - z_m)^{n_m}$$

As a consequence $a_{nn} \neq 0$, for otherwise $P(z\ ,\ w)$ would have degree less than n in w, which is impossible since $P(z,\ S(z)) \equiv 0$ and $S(z)$ has order = n. Another consequence is that

$$\frac{R(z)}{P_n(z)} \text{ is bounded as } z \to z_k\ .$$

Hence:

$$S(z) = - \frac{R(z)}{P_n(z)} - \frac{P_{n-1}(z)}{P_n(z)} = - \frac{P_{n-1}(z)}{P_n(z)} + O(1)\ , \quad \text{as} \quad z \to z_k\ ,$$

implying that the principal parts of $S(z)$ agree with those of

$$- \frac{P_{n-1}(z)}{P_n(z)}\ .$$

Explicitly:

$$\sum_{k=1}^{m} \sum_{j=1}^{n_k} \frac{b_{kj}}{(z - z_k)^j} = - \frac{P_{n-1}(z)}{P_n(z)} + C$$

where

$$C = \lim_{z\to\infty} \frac{P_{n-1}(z)}{P_n(z)} = \frac{a_{n,n-1}}{a_{n,n}} .$$

If the polynomial $P(z, w) = \Sigma\, a_{jk} z^j w^k$ $(a_{jk} = \overline{a_{kj}})$ is given, this gives the data in a possible quadrature formula for Ω, associated to P as in ③ from the last two columns in the coefficient matrix:

$$A = (a_{jk}) = \left[\begin{array}{ccc|cc} a_{00} & a_{01} & \cdots & a_{0,n-1} & a_{0,n} \\ a_{10} & a_{11} & & a_{1,n-1} & a_{1,n} \\ \vdots & & & \vdots & \vdots \\ a_{n-1,0} & & \cdots\cdots & a_{n-1,n-1} & a_{n-1,n} \\ a_{n,0} & & \cdots\cdots & \underset{\substack{\uparrow \\ \text{coefficients of } P_{n-1}(z)}}{a_{n,n-1}} & \underset{\substack{\uparrow \\ \text{coefficients of } P_n(z)}}{a_{nn}} \end{array}\right]$$

Suppose, conversely, that the quadrature formula, i.e. the principal part:

$$\sum_{k=1}^{m} \sum_{j=1}^{n_k} \frac{b_{kj}}{(z - z_k)^j}$$

of S(z) is given, Since the polynomial P(z, w) is determined only up to a real multiplicative factor, and we know that $a_{nn} \neq 0$, it is natural to normalize P(z, w) by requiring:

$$a_{nn} = 1 .$$

This will always be assumed in the sequel. Thus the last column in A is obtained from

$$P_n(z) = z^n + a_{n-1,n} z^{n-1} + \ldots + a_{o,n} \equiv (z - z_1)^{n_1} \cdot \ldots \cdot (z - z_m)^{n_m} .$$

Since P(z, w) shall be self-conjugate, this also gives:

$$a_{n,n-1} = \overline{a_{n-1,n}} \ .$$

The remaining coefficients in the (n - 1):st column of A are now uniquely obtained from

$$\sum_{k=1}^{m} \sum_{j=1}^{n_k} \frac{b_{kj}}{(z - z_k)^j} = - \frac{P_{n-1}(z)}{P_n(z)} + a_{n,n-1}$$

$$(a_{n,n-1} = C) \ .$$

It is interesting that there is one condition of the self-conjugacy of P(z, w) left over, namely that $a_{n-1,n-1}$ shall be real. To see what that condition means on the quadrature-formula side, note that

$$- \frac{P_{n-1}(z)}{P_n(z)} + a_{n,n-1} = \frac{- \sum_{o}^{n} a_{j,n-1} z^j + a_{n,n-1} \cdot \sum_{o}^{n} a_{j,n} z^j}{\sum_{o}^{n} a_{j,n} z^j} =$$

$$= \frac{(- a_{n,n-1} + a_{n,n-1} a_{nn}) z^n + (- a_{n-1,n-1} + a_{n,n-1} a_{n-1,n}) z^{n-1} + \ldots}{a_{nn} z^n + \ldots}$$

$$= \frac{(|a_{n,n-1}|^2 - a_{n-1,n-1}) z^{n-1} + \ldots}{z^n + \ldots} \ .$$

This gives:

$$|a_{n,n-1}|^2 - a_{n-1,n-1} = - \operatorname*{res}_{z=\infty} \left(- \frac{P_{n-1}(z)}{P_n(z)} + a_{n,n-1} \right) dz =$$

$$= - \operatorname*{res}_{z=\infty} \Sigma \Sigma \frac{b_{kj}}{(z - z_k)^j} dz = \Sigma \operatorname*{res}_{z\in\Omega} \Sigma \Sigma \frac{b_{kj}}{(z - z_k)^j} dz =$$

$$= \Sigma \operatorname*{res}_{z\in\Omega} S(z)dz = \frac{1}{2\pi i} \int_{\partial\Omega} S(z)dz = \frac{1}{2\pi i} \int_{\partial\Omega} \bar{z}dz = \frac{1}{2\pi i} \int_{\Omega} d\bar{z}dz$$

$$= \frac{1}{\pi} \int_{\Omega} dxdy = \frac{|\Omega|}{\pi} \qquad (|\Omega| = \text{area of } \Omega).$$

Thus

$$a_{n-1,n-1} = |a_{n,n-1}|^2 - \frac{|\Omega|}{\pi}$$

will automatically become real. Using the relation

$$c_{k,j} = \frac{\pi}{j!} b_{k,j+1}$$

and observing that

$$|\Omega| = \sum_{k=1}^{m} c_{k,o}$$

we can summarize:

Theorem 5.1:

The identity:

$$\frac{1}{\pi} \sum_{k=1}^{m} \sum_{j=o}^{n_k-1} \frac{j!c_{k,j}}{(z - z_k)^{j+1}} \equiv a_{n,n-1} - \frac{P_{n-1}(z)}{P_n(z)} ,$$

$$\begin{cases} P_{n-1}(z) = a_{n,n-1}z^n + a_{n-1,n-1}z^{n-1} + \ldots + a_{o,n-1} \\ P_n(z) = z^n + a_{n-1,n}z^{n-1} + \ldots + a_{o,n} \end{cases}$$

gives a one-to-one correspondence

between the last two columns (and rows) of coefficient matrices, $A = (a_{jk})$, of normalized ($a_{nn} = 1$) selfconjugate polynomials

$$P(z, w) = \sum_{j,k=o}^{n} a_{j,k}z^j w^k$$

and quadrature data $z_1, \ldots, z_m, n_1, \ldots, n_m$,

$$(c_{k,j})_{\substack{o\leq j\leq n_k-1 \\ 1\leq k\leq m}} , \quad \sum_1^m n_k = n$$

with

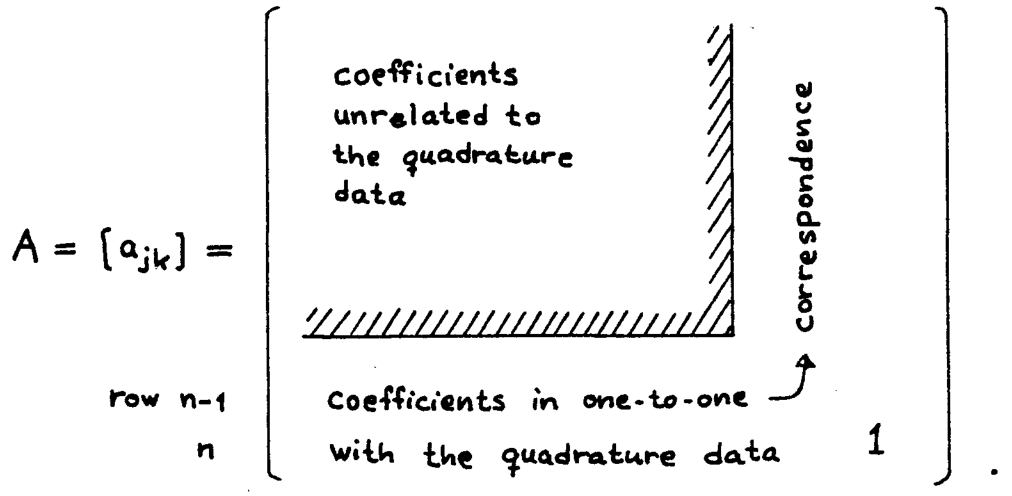
$$\sum_{k=1}^{m} c_{k,o} = \text{real} \ ,$$

<u>such that</u> whenever $\Omega \subset \mathbb{C}$ is a quadrature domain, the quadrature formula

$$\int_{\Omega} f dxdy = \sum_{k=1}^{m} \sum_{j=o}^{n_k-1} c_{k,j} f^{(j)}(z_k)$$

and the normalized, self-conjugate polynomial equation of the boundary: $\partial\Omega = \{z : P(z, \bar{z}) = 0\} \setminus$ (finite set) are related according to this correspondence.

The situation is most conveniently expressed by a picture of the coefficient matrix of P(z, w):



The coefficients unrelated to the quadrature data make up a $(n - 1)^2$-dimensional (over $\mathbb{R}$) vector space. Hence, given quadrature data, this vector space can be thought of as a parameter space for all algebraic curves γ which are candidates to be boundary curves for domains Ω admitting a quadrature formula with the given data. Of course, not every coefficient matrix of a self-conjugate polynomial P gives rise to a quadrature domain, even if the curve

$$\gamma = \{z : P(z, \bar{z}) = 0\}$$

happens to bound a domain $\Omega \subset \mathbb{C}$. In fact, if $\gamma = \partial\Omega$, Ω is a quadrature domain if and only if the Schwartz-function $S(z)$ of γ extends to a meromorphic function in Ω. But to decide whether this is the case from the coefficients of $P(z, w)$ seems not to be easy.

Our discussion however suggests the following: suppose Ω is a quadrature domain. Then the Schwartz-function $S(z)$ extends to a meromorphic function in Ω. A slight variation of the coefficients unrelated to the quadrature data in the coefficient matrix (a_{jk}) of the polynomial $P(z, w)$ associated to Ω gives a new polynomial $P'(z, w)$ and an algebraic curve $\gamma' = \{z : P'(z, \bar{z}) = 0\}$. One then expects that γ' does not differ very much from $\gamma = \partial\Omega$, hence bounds a domain Ω' and that the Schwartz function $S'(z)$ (not a derivative) of γ' extends meromorphically to Ω'. Hence Ω' would be a domain admitting the same quadrature formula as Ω.

Since one has $(n - 1)^2$ real parameters to ones disposal for such variations, this naive approach indicates that quadrature domains admitting a fixed quadrature formula of order n "in general" occur in $(n - 1)^2$-real-parameter families. That this naive approach is, at least not always, correct is shown by the example in section 6 (p. 6.3 - 6.8).

6. An example

We shall develop the details a little further for quadrature formulas with only one point, z_o, i.e.:

$$\int_\Omega f dxdy = c_o f(z_o) + c_1 f'(z_o) + \ldots + c_{n-1} f^{(n-1)}(z_o) \;,$$

$$f \in L_a^2(\Omega) \;, \qquad c_{n-1} \neq 0 \;.$$

For notational convenience, take $z_o = 0$.

Put

$$b_{j+1} = \frac{j!}{\pi} c_j \;, \; j = 0, 1, \ldots, n-1.$$

Then we know that equivalent for Ω to admit the above formula is that the Schwartz function, $S(z)$, of $\partial\Omega$ extends meromorphically to Ω with the singular part:

$$S(z) = \frac{b_1}{z} + \frac{b_2}{z^2} + \ldots + \frac{b_n}{z^n} + \text{holomorphic function} \;,$$

$$z \in \Omega \;, \qquad b_n \neq 0 \;.$$

The boundary, $\partial\Omega$, is an algebraic curve:

$$\partial\Omega = \{z \in \mathbb{C} : P(z, \bar{z}) = 0\} \setminus (\text{a finite set})$$

where

$$P(z, w) = \sum_{j,k=o}^{n} a_{j,k} z^j w^k$$

is self-conjugate ($a_{j,k} = \bar{a}_{k,j}$) and normalized ($a_{n,n} = 1$).

The last two columns (and rows) of the coefficient matrix $A = (a_{jk})$ of $P(z, w)$ are obtained from $b_1, \ldots, b_n$ by the identity (Theorem 5.1):

$$\frac{b_1}{z} + \frac{b_2}{z^2} + \dots + \frac{b_n}{z^n} \equiv$$

$$\equiv a_{n,n-1} - \frac{a_{n,n-1}z^n + a_{n-1,n-1}z^{n-1} + \dots + a_{o,n-1}}{z^n + a_{n-1,n}z^{n-1} + \dots + a_{o,n}}$$

giving the matrix:

$$A = (a_{jk}) = \begin{bmatrix} a_{oo} & \dots & a_{o,n-2} & -b_n & 0 \\ \cdot & & \cdot & \cdot & \cdot \\ \cdot & & \cdot & \cdot & \cdot \\ \cdot & & \cdot & \cdot & \cdot \\ a_{n-2,o} & & a_{n-2,n-2} & -b_2 & 0 \\ -\bar{b}_n & \dots & -\bar{b}_2 & -b_1 & 0 \\ 0 & & 0 & 0 & 1 \end{bmatrix} .$$

(Observe that $b_1 = \frac{1}{\pi} c_o = \frac{1}{\pi} |\Omega|$ is real.)

I. For $n = 1$ this becomes:

$$A = \begin{pmatrix} -b_1 & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} -\frac{c_o}{\pi} & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} -\frac{|\Omega|}{\pi} & 0 \\ 0 & 1 \end{pmatrix} ,$$

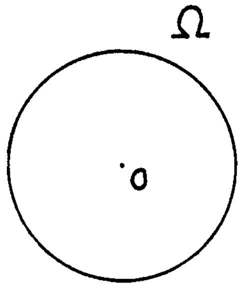


fig 6.1

so that

$$P(z, \bar{z}) = - \frac{|\Omega|}{\pi} + z \cdot \bar{z}$$

i.e. Ω is the disc around 0 with radius $= \sqrt{|\Omega|/\pi}$ as expected.

$$\left(\int_\Omega f dx dy = |\Omega| \cdot f(0) \right) .$$

II. For n = 2 we have:

$$A = \begin{bmatrix} a_{00} & -b_2 & 0 \\ -\bar{b}_2 & -b_1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

i.e. $P(z, \bar{z}) = a_{00} - \bar{b}_2 z - b_2\bar{z} - b_1 z\bar{z} + (z\bar{z})^2$.

From Corollary 4.5 we know that all quadrature domains of order 2 (n = 2) are simply connected. Hence they can all be produced as images of for example the unit disc, $\mathbb{D}$, under functions meromorphic on the double $\hat{\mathbb{D}}$. Realizing $\hat{\mathbb{D}}$ as the Riemannsphere $\mathbb{P}$ (the involution being $z \to 1/\bar{z}$) the mapping function:

$$G : \mathbb{D} \to \Omega$$

is thus going to be a rational function. The point in $\mathbb{D}$ to be mapped on $0 \in \Omega$ can be prescribed at will, say $0 \in \mathbb{D}$. By Theorem 3.1, G shall then have a double pole at the conjugate point of $0 \in \mathbb{D}$, i.e. at $z = \infty$, and be regular elsewhere. Hence:

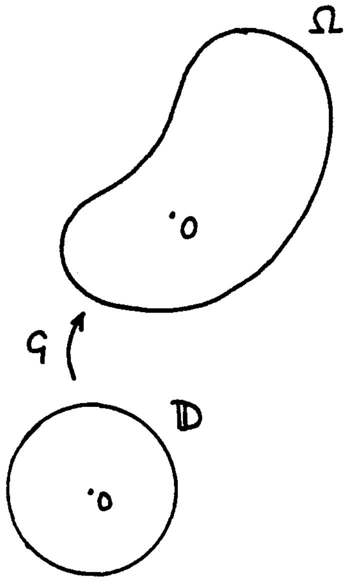


fig 6.2

$$G(t) = \beta_1 t + \beta_2 t^2 , \quad t \in \mathbb{P} ,$$

$$G^*(t) = \overline{G(1/\bar{t})} = \bar{\beta}_1 \cdot t^{-1} + \bar{\beta}_2 \cdot t^{-2} , \quad t \in \mathbb{P} .$$

Here one has the degree of freedom to choose β_1 real and positive. It is easily established that G is univalent in $\mathbb{D}$ iff

$$|\beta_2/\beta_1| \leq 1/2 .$$

Since the Schwartz function of $\partial\Omega$ is:

$$S(z) = G^*(G|_{\mathbb{D}}^{-1}(z)) = \frac{b_1}{z} + \frac{b_2}{z^2} + \text{holomorphic}, \qquad z \in \Omega ,$$

we must have:

$$G^*(t) = \frac{b_1}{G(t)} + \frac{b_2}{G(t)^2} + \text{holomorphic} , \qquad t \in \mathbb{D} .$$

A slight computation gives:

$$\begin{cases} b_1 = |\beta_1|^2 + 2|\beta_2|^2 & (6.1) \\ b_2 = \beta_1^2\bar{\beta}_2 & (6.2) \end{cases}$$

Moreover, the coefficient a_{oo} can be computed from β_1 and β_2 since G, G* shall satisfy:

$$P(G(t), G^*(t)) \equiv 0 .$$

It turns out to be:

$$a_{oo} = |\beta_2|^4 - |\beta_1\beta_2|^2 = |\beta_2|^2(|\beta_2|^2 - |\beta_1|^2) . \qquad (6.3)$$

Hence, from equations (6.1) - (6.3) one sees that a_{oo} is essentially determined by b_1 and b_2, i.e. for each choice of b_1, b_2 there are at most finitely many values for a_{oo}. In fact, computations carried out in [AS 2, pp. 25 - 27] show that the above equations together with the condition that G shall be univalent in $\mathbb{D}$ determine a_{oo} uniquely from b_1 and b_2.

According to the reasoning on page 5.8 - 5.9, however, a_{oo} should be a "free" parameter which one ought to be able to vary continuously to give a 1-real-parameter family of domains with the same quadrature formula. To see what goes wrong in this case, we compute the Schwartz function, S(z), for the algebraic curve:

$$\partial\Omega = \{z \in \mathbb{C} : P(z, \bar{z}) = 0\}$$

and study what happens to it when a_{oo} is varied around a value related to b_1 and b_2 according to equations (6.1) - (6.3).

$S(z)$ is obtained from $P(z, S(z)) \equiv 0$ i.e.:

$$z^2 S(z)^2 - (b_1 z + b_2) S(z) - (\bar{b}_2 z - a_{oo}) \equiv 0$$

giving:

$$S(z) = \frac{1}{2z^2} [b_1 z + b_2 \pm \sqrt{B(z)}] ,$$

where

$$B(z) = 4\bar{b}_2 \cdot z^3 + (b_1^2 - 4a_{oo}) z^2 + 2b_1 b_2 z + b_2^2 .$$

Let e_1, e_2, e_3 be the zeroes of $B(z)$, so that:

$$B(z) = 4\bar{b}_2 (z - e_1)(z - e_2)(z - e_3) .$$

In general e_1, e_2, e_3 are distinct, and when this is the case the Riemann surface, $\hat{W}$, canonically associated with the irreducible polynomial $P(z, w)$ has genus = 1. This follows for example from the genus-formula:

$$\text{genus} = 1 - (\text{number of sheets over } \mathbb{P}) + \frac{\text{total branching order}}{2} =$$

$$= 1 - 2 + \frac{4}{2} = 1 .$$

The branch points are (lie over) e_1, e_2, e_3 and ∞.

That $P(z, w)$ is self-conjugate implies that $\hat{W}$ is a symmetric Riemann surface, the involution being that induced by the mapping

$$(z, w) \rightarrow (\bar{w}, \bar{z})$$

on

$$\text{loc}(P) = \{(z, w) \in \mathbb{C}^2 : P(z, w) = 0\}$$

Moreover, when Ω is a quadrature domain, i.e. when the branches of $S(z)$ which take the values $S(z) = \bar{z}$ on $\partial\Omega$ fit together to a single-valued meromorphic function on Ω, the above Riemann surface $\hat{W}$ is conformally equivalent to the Schottky double $\hat{\Omega}$.

Indeed, the mapping

$$\hat{\Omega} \to \text{loc } (P)$$

defined by:

$$\begin{cases} z \to (z,\ S(z)) & \text{for } \ z \in \Omega \\ \tilde{z} \to (\overline{S(z)},\ \bar{z}) & \text{for } \ \tilde{z} \in \tilde{\Omega} \quad (\text{i.e. } z \in \Omega) \end{cases}$$

(here $S(z)$ stands for the branch on Ω which $= \bar{z}$ on $\partial\Omega$; since $(z,\ S(z)) = (z,\ \bar{z}) = (\overline{S(z)},\ \bar{z})$ for $z \in \partial\Omega$, the mapping extends continuously to $\hat{\Omega} = \Omega \cup \partial\Omega \cup \tilde{\Omega}$)

induces a conformal mapping $\hat{\Omega} \to \hat{W}$. This is a conformal equivalence because the order of the mapping is

$$= \frac{\text{number of poles of } S(z) \text{ in } \Omega}{\text{number of points of the kind } (z,\ \infty) \text{ in loc } (P)} \overset{1)}{\leq} 1 \text{ , hence } = 1 \text{ .}$$

From this discussion it follows that when Ω is a quadrature domain with e_1, e_2, e_3 distinct, $\hat{\Omega}$ must have genus 1, i.e. Ω must be doubly connected. Knowing that such domains do not exist (corollary 4.5) one concludes that the special values of a_{oo} actually giving quadrature domains, are values for which two of e_1, e_2, e_3 coincide. This is confirmed by a computation giving:

$$B(z) = (\bar{\beta}_1 z + \beta_1 \bar{\beta}_2)^2 \cdot (4\beta_2 z + \beta_1^2) = 4\beta_1^2 \beta_2 (z + \bar{\beta}_2)^2 \left(z + \frac{\beta_1^2}{4\beta_2}\right)$$

(assuming β_1 real)

for

$$\begin{cases} b_1 = |\beta_1|^2 + 2|\beta_2|^2 \\ b_2 = \beta_1^2 \bar{\beta}_2 \\ a_{oo} = |\beta_2|^2 \cdot (|\beta_2|^2 - |\beta_1|^2) \ , \quad |\beta_2/\beta_1| \leq 1/2 \end{cases}$$

Hence:

$$\begin{cases} e_1 = e_2 = -\bar{\beta}_2 \\ e_3 = -\dfrac{\beta_1^2}{4\beta_2} = -\bar{\beta}_2 \cdot \dfrac{1}{4} \left|\dfrac{\beta_1}{\beta_2}\right|^2 . \end{cases}$$

Thus the failure of the reasoning on p. 5.8 - 5.9 can be explained.

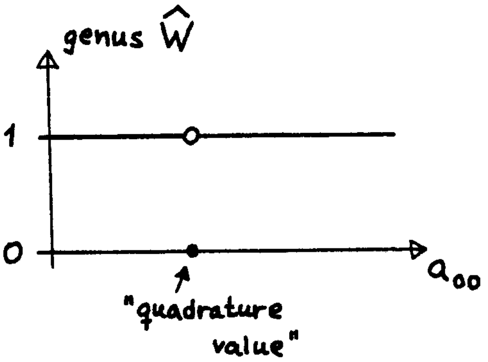


fig 6.3

The Riemann surface $\hat{W}$ of $P(z, w) = 0$ does not depend continuously on the coefficients of $P(z, w)$. More precisely, the genus of $\hat{W}$ is only a lower semicontinuous function of the coefficients of $P(z, w)$, and in our case the values of a_{00} giving quadrature domains are values for which the genus is exceptionally low (= 0).

Hence $\hat{W}$ "explodes" when a_{00} is varied about

$$a_{00} = |\beta_2|^2 \cdot (|\beta_2|^2 - |\beta_1|^2) .$$

This "explosion" can be studied through the set

$$\gamma_\varepsilon = \{z \in \mathbb{C} : P_\varepsilon(z, \bar{z}) = 0\}$$

where $P_\varepsilon(z, \bar{z})$ is the polynomial with the coefficients:

$$\begin{cases} b_1 = |\beta_1|^2 + 2|\beta_2|^2 \\ b_2 = \beta_1^2\bar{\beta}_2 \\ a_{00} = |\beta_2|^2 \cdot (|\beta_2|^2 - |\beta_1|^2) \\ + \varepsilon \quad (\varepsilon \text{ real}) \end{cases}$$

For $\varepsilon = 0$ this set consists not only of the curve $\partial\Omega$ (where Ω is the quadrature domain) but also of an isolated point at $z = e_1 = e_2$ (fig 6.4).

The set γ_ε :

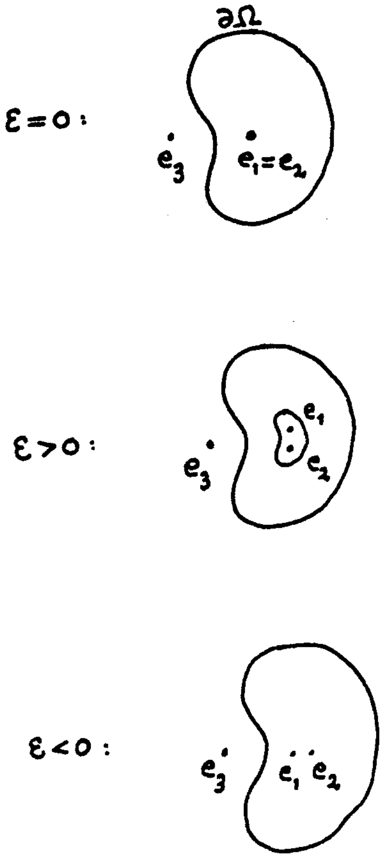


fig 6.4

When $\varepsilon > 0$, e_1 and e_2 have split up into two points and the isolated point of γ_0 has blown up to a whole curve, so that γ_ε ($\varepsilon > 0$) consists of two disjoint curves. This agrees with the fact that the Riemann surface $\hat{W}_\varepsilon$ of $P_\varepsilon(z, w) = 0$ is a symmetric torus. γ_ε is the "track" of the two symmetry-lines (the fixed-point set under the involution) on $\hat{W}_\varepsilon$.

For $\varepsilon < 0$, however, the isolated point of γ_0 has disappeared, so that γ_ε ($\varepsilon < 0$) only consists of one curve. By our earlier discussion, $\hat{W}_\varepsilon$ still is a symmetric torus, but this time with only one symmetry-line. Hence, although symmetric (i.e. admitting an anticonformal involution), $\hat{W}_\varepsilon$ cannot be the double of a plane domain. The symmetry must therefore be of the kind which arises from doubling a "non-orientable Riemann surface", in this case a Möbius strip.

Remark: A torus can be symmetric in 3 essentially different ways, namely the symmetries coming from doubling

1)	a ring-domain in $\mathbb{C}$	(giving 2 symmetry-lines),
2)	a Möbius strip	(1 symmetry-line),
3)	Klein's bottle	(no symmetry-points).

The simpliest example of a symmetric Riemann surface which is not the double of a plane domain is the Riemann sphere, $\mathbb{P}$, with the involution:

$$\phi : z \to - 1/\bar{z} .$$

ϕ has no fixed points and ($\mathbb{P}$, ϕ) can be considered as the double of the projective plane, which is a non-orientable Riemann surface.

For further discussion of these fascinating topics, see [K 1] and [AG 1].

III Next, let $n > 2$ on p. 6.1 - 6.2. It follows from the Existence Theorem 3.3. that there exist domains Ω of arbitrary conformal type with

$$(6.4)\quad \int_\Omega f dxdy = c_0 f(0) + c_1 f'(0) + \dots + c_{n-1} f^{(n-1)}(0)$$

$$\text{for all}\quad f \in L_a^2(\Omega)$$

if n is sufficiently large.

For example, to get quadrature domains of connectivity = 2 one could start with a symmetric torus represented as a period parallellogram, $\hat{W}$, in $\mathbb{C}$ with vertices $\pm i\omega$, $1 \pm i\omega$ $(\omega > 0)$. The involution is complex conjugation $(\tilde{t} = \bar{t})$, and:

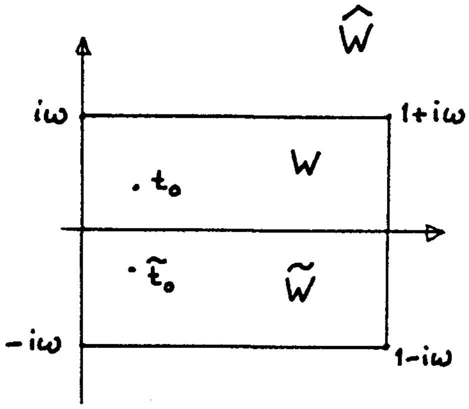


fig 6.5

$$\begin{cases} W = \hat{W} \cap \mathbb{C}^+ \\ \tilde{W} = \hat{W} \cap \mathbb{C}^- . \end{cases}$$

The meromorphic functions on $\hat{W}$ are represented as doubly periodic functions (elliptic functions) on $\mathbb{C}$. If $t_0 \in W$, the most general such function with a single pole of order n at $t = \tilde{t}_0 = \bar{t}_0 \in \tilde{W}$ is:

$$G(t) = \beta_0 + \sum_{j=o}^{n-2} \beta_{j+2} p^{(j)}(t - \bar{t}_0) ,$$

where $\beta_j \in \mathbb{C}$ and p(t) is the Weierstrass p-function ([A 1 p. 264 f]) with double poles at $m_1 + m_2 \cdot 2i\omega$, $m_1, m_2 \in \mathbb{Z}$.

According to the approximation argument on pp. 3.8 - 3.10 the coefficients $\beta_2, \dots, \beta_n$ can be chosen so that G becomes univalent on W if n is sufficiently large. Hence, choosing β_0 so that $G(t_0) = 0$, such a G will map W onto a domain Ω with a quadrature formula

$$\int_\Omega f dxdy = c_0 f(0) + c_1 f'(0) + \dots + c_{n-1} f^{(n-1)}(0) , \quad f \in L_a^2(\Omega) .$$

The conjugate function to G(t) is:

$$G^*(t) = \bar{\beta}_0 + \sum_{j=o}^{n-2} \bar{\beta}_{j+2} \cdot p^{(j)}(t - t_0) ,$$

and since G(t) is univalent on W, the singular part of $G^*(t)$ (at $t = t_o$) can be expanded in G(t):

$$G^*(t) = \frac{b_n}{G(t)^n} + \dots + \frac{b_1}{G(t)} + \text{regular terms} , \quad t \in W .$$

As on p. 6.4 this means that the Schwartz function of $\partial\Omega$ is:

$$S(z) = \frac{b_n}{z^n} + \dots + \frac{b_1}{z} + \text{regular terms} , \quad z \in \Omega ,$$

so that the coefficients $b_1, \dots, b_n$ are those occuring in the coefficient matrix of P(z, w) on p. 6.2, and

$$b_{j+1} = \frac{j!}{\pi} c_j .$$

Now, it is interesting that varying the parameters in G(t) one finds that <u>there must be a whole 1-real-parameter family of doubly connected domains</u> Ω_τ, $-\varepsilon < \tau < \varepsilon$, $\Omega_o = \Omega$, <u>all admitting the same quadrature formula</u>

$$\int_{\Omega_\tau} f dxdy = c_o f(0) + \dots + c_{n-1} f^{(n-1)}(0) , \quad f \in L^2_a(\Omega_\tau) .$$

To see this, observe first that the definition of G(t) depends on the following parameters:

$$\begin{cases} \omega \in \mathbb{R}^+ & \text{(the p-function depends on } \omega) \\ t_o \in W \subset \mathbb{C} & \\ (\beta_2, \dots, \beta_n) \in \mathbb{C}^{n-1} & \end{cases}$$

These parameters can be varied freely in the indicated domains, always giving a unique function $G = G_{(\omega, t_o, \beta_2, \dots, \beta_n)}$:

$$G(t) = \beta_o + \sum_{j=o}^{n-2} \beta_{j+2} p^{(j)}(t - \bar{t}_o) ,$$

where β_o is chosen so that $G(t_o) = 0$.

Moreover, the initial G could have been chosen to be univalent in a whole neighbourhood of the closure of W in $\hat{W}$, and it is clear that all sufficiently small variations of $(\omega, t_o, \beta_2, \ldots, \beta_n)$ $\in \mathbb{R} \times \mathbb{C} \times \mathbb{C}^{n-1}$ will then give functions $G = G_{(\omega, t_o, \beta_2, \ldots, \beta_n)}$ which are univalent on $W = W_\omega$ (observe that the domain W depends on ω). Hence the formula:

$$G^*(\cdot t) = \frac{b_n}{(G(t)^n} + \ldots + \frac{b_1}{G(t)} + \text{regular} , \quad t \in W$$

gives a mapping:

$$(\omega, t_o, \beta_2, \ldots, \beta_n) \to (b_1, \ldots, b_n)$$

from an open subset of $\mathbb{R} \times \mathbb{C} \times \mathbb{C}^{n-1}$ to $\mathbb{C}^n$.

Explicitly $b_1, \ldots, b_n$ are given by:

$$b_j = \frac{1}{2\pi i} \int_{\partial\Omega} G(t)^{j-1} G^*(t) dt , \quad j = 1, \ldots, n ,$$

from which it is seen that $(b_1, \ldots, b_n)$ depends analytically on $(\omega, t_o, \beta_2, \ldots, \beta_n)$.

Moreover, a direct computation, or the fact that

$$b_1 = \frac{1}{\pi} c_o = \frac{1}{\pi} \cdot |\Omega| \quad (\Omega = G(W)) ,$$

shows that b_1 is always real. Hence $(\omega, t_o, \beta_2, \ldots, \beta_n) \to (b_1, \ldots, b_n)$ is actually a map:

$$\text{open subset} \subset \mathbb{R} \times \mathbb{C} \times \mathbb{C}^{n-1} \to \mathbb{R} \times \mathbb{C}^{n-1} .$$

Hence counting dimensions, we see that there must be submanifolds of real dimension = 2 in $\mathbb{R} \times \mathbb{C} \times \mathbb{C}^{n-1}$ on which this map is constant. (See p. 7.5 for a more careful motivation of this.) This means that there is a 2-real-parameter subfamily of the functions $G = G_{(\omega, t_o, \beta_2, \ldots, \beta_n)}$ which map $W = W_\omega$ on domains $\Omega = \Omega_{(\omega, t, \beta_2, \ldots, \beta_n)}$ all of which admit exactly the same quadrature formula:

$$\int_{\Omega} f dxdy = c_o f(0) + c_1 f'(0) + \dots + c_{n-1} f^{(n-1)}(0) \quad , \quad f \in L_a^2(\Omega) \ .$$

However, different G can produce the same Ω. In fact, if $\omega = \omega'$ there is a 1-real-parameter-group of conformal equivalences:

$$\varphi : W_{\omega'} \to W_{\omega}$$

and it is clear that $G = G_{(\omega,t_o,\beta_2,\dots,\beta_n)}$ and $G' = G \circ \varphi$ ($G' = G_{(\omega',t_o',\beta_2',\dots,\beta_n')}$) give the same Ω for each such φ.

Conversely, if

$$\Omega = G(W_{\omega}) = G'(W_{\omega'}) \ ,$$

then necessarily $\omega = \omega'$ (modulo the modular group), and $\varphi = G^{-1} \circ G'$ is a conformal equivalence:

$$\varphi : W_{\omega'} \to W_{\omega} \ .$$

Since φ = identity (if and) only if $G = G'$, these considerations show that the automorphism-group of W absorbs only one real parameter in the 2-real-parameter family of functions G. Hence there remains a 1-real-parameter family of different domains Ω all having the same quadrature formula, as asserted.

7. Questions of uniqueness

At the end of section 5 (p. 5.9) we obtained an upper bound for the multitude of domains Ω admitting a quadrature formula

$$\int_\Omega f dxdy = L(f) \quad , \qquad f \in L_a^2(\Omega)$$

for a fixed functional L. Namely, if

$$n = \Sigma n_k = \text{order of } L \ ,$$

we associated to L a $(n - 1)^2$-real-parameter family of self-conjugate polynomials $P(z, w)$, such that the algebraic curves $P(z, \bar{z}) = 0$ represented all possible boundary curves for domains having L as quadrature functional.

In this section we shall study uniqueness questions from another point of view. Let us however begin with two elementary remarks:

i) Given $\Omega \subset \mathbb{C}$, there can be at most one functional:

$$L(f) = \sum_{k=1}^{m} \sum_{j=o}^{n_k-1} c_{kj} f^{(j)}(z_k)$$

such that:

$$\int_\Omega f dxdy = L(f)$$

holds for all $f \in L_a^2(\Omega)$, or even for all polynomials f, since otherwise one would get a non-trivial identity of the kind

$$\Sigma \Sigma c_{kj} f^{(j)}(z_k) = \Sigma \Sigma c'_{kj} f^{(j)}(z'_k)$$

holding for all polynomials.

ii) Given the functional $L(f) = \Sigma \Sigma c_{kj} f^{(j)}(z_k)$, there need not exist any domain Ω with

$$\int_\Omega f dxdy = L(f) \quad \text{for all} \quad f \in L_a^2(\Omega) \ .$$

Putting $f \equiv 1$ gives the necessary condition that

$$\sum_{k=1}^{m} c_{k,o} = |\Omega| \quad \text{is real and} > 0 ,$$

but even if this consistency condition is fulfilled there need not exist any Ω. For example there is no domain Ω with

$$\int_{\Omega} f dxdy = f(0) + cf'(0) , \qquad f \in L_a^2(\Omega)$$

if $|c| > 1/2$, as is easily seen from the example on p.6.3 - 6.4 in section 6 (the mapping function G on p. 6.3 will not be univalent, since $b_1 = 1 \Rightarrow |\beta_1| \leq 1$ (eq. (6.1)), hence $|c| = |b_2| = |\beta_1|^2|\beta_2| \leq$ $\leq \frac{|\beta_2|}{|\beta_1|} \leq 1/2$ by eq. (6.2) if G is univalent).

Now, let $W \subset \mathbb{C}$ be a standard domain representing a certain conformal type and let $\hat{W} = W \cup \Gamma \cup \tilde{W}$ be its Schottky double. We shall consider quadrature identities

$$(7.1) \quad \int_{\Omega} f dxdy = L(f)$$

holding for the class $L_{as}^2(\Omega)$, and for domains Ω conformally equivalent to W.

We know (Theorem 3.1) that the existence of such a quadrature identity for such Ω is equivalent to the existence of a meromorphic differential dG on $\hat{W}$ such that dG has all its poles on $\tilde{W}$, and such that $G(t) = \int^t dG$ is single-valued and univalent on W, mapping W onto Ω. That dG has its poles on $\tilde{W}$ means that dG*
has its poles $t_1, \ldots, t_m$ on W, and if:

$$(7.2) \quad L(f) = \sum_{k=1}^{m} \sum_{j=o}^{n_k-1} c_{kj} f^{(j)}(z_k) ,$$

the order of the pole at $t_k \in W$ is $n_k + 1$, and $G(t_k) = z_k$.

Hence dG* is of the form:

$$dG^*(t) = \sum_{k=1}^{m} \sum_{j=1}^{n_k} \beta_{kj} \, d \frac{1}{(t - t_k)^j} + \text{holomorphic differential} ,$$

$t \in W$.

Moreover, G single-valued on W is equivalent to:

$$\int_{\alpha_k} dG = \int_{\alpha_k} dG^* = 0 , \quad k = 1, \ldots, p .$$

Here p = genus of $\hat{W}$, and $\alpha_1, \ldots, \alpha_p$ are the cycles homologuous to the boundary curves of W in a canonical homology basis of $\hat{W}$.

By the Runge Theorem on p. 3.9 we know that given $t_1, \ldots, t_m \in W$ and $n_1, \ldots, n_m$ with $n = n_1 + \ldots + n_m$ large enough, there always exists a differential dG with the above properties. We can even require dG to be exact on $\hat{W}$ and G to be univalent on a neighbourhood of $W \cup \Gamma$ in $\hat{W}$. Observe that the last property is a simple geometric property of the domain $\Omega = G(W)$ as illustrated in figure 7.1.

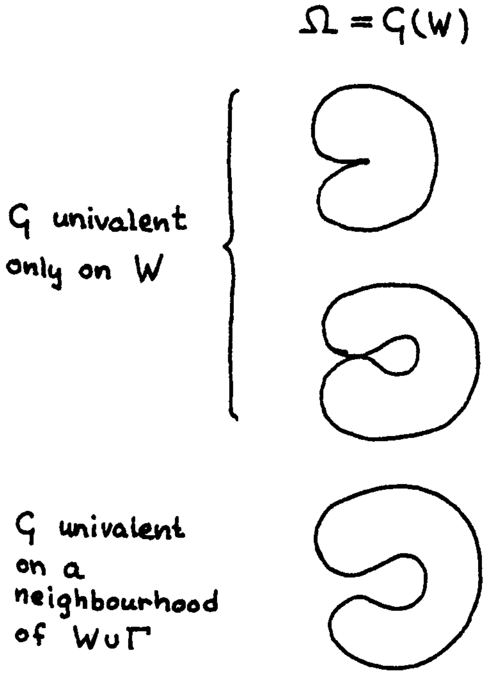


fig 7.1

Starting with such a dG, we shall now let the parameters $t_1, \ldots, t_m \in W$, $(\beta_{kj}) \in \mathbb{C}^{n_1+\ldots+n_m}$ vary.

For each choice of the data $t_1, \ldots, t_m$, (β_{kj}) (with $t_1, \ldots, t_m$ distinct) there is a unique meromorphic differential dG on $\hat{W}$ such that:

① $\int_{\alpha_k} dG = 0 , \quad k = 1, \ldots, p$

(G single-valued on W)

② dG* has the singular parts $\sum_{k=1}^{m} \sum_{j=1}^{n_k} \beta_{kj} \, d \frac{1}{(t - t_k)^j}$.

Moreover, if the data $t_1, \ldots, t_m, (\beta_{kj})$ are sufficiently close to the data $t_1^{(0)}, \ldots, t_m^{(0)}, (\beta_{kj}^{(0)})$ of the initial $dG = dG^{(0)}$, it is obvious that G will remain univalent on W. Hence G(t) can be used as global parameter on W, so that the singularities of dG* can be expressed:

$$dG^*(t) = \sum_{k=1}^{m} \sum_{j=1}^{n_k} b_{kj}\, d \frac{1}{(G(t) - z_k)^j} + \text{holomorphic differential} ,$$

$t \in W$.

Here $z_k = G(t_k)$ $k = 1, \ldots, p$, where the integration constant in $G(t) = \int^t dG$ is chosen for example so that the point

$$z_1 = G(t_1)$$

is kept fixed $(z_1 = z_1^{(0)})$ under the variations of $t_1, \ldots, t_m, (\beta_{kj})$.

Hence we have a mapping

$$\tau : U \subset W^m \times \mathbb{C}^n \rightarrow \mathbb{C}^m \times \mathbb{C}^n$$

$$((t_1, \ldots, t_m), (\beta_{kj})) \rightarrow ((z_1, \ldots, z_m), (b_{kj}))$$

from an open neighbourhood, U, of

$$\left((t_1^{(0)}, \ldots, t_m^{(0)}), (\beta_{kj}^{(0)})\right) \text{ in } W^m \times \mathbb{C}^n \text{ into } \mathbb{C}^m \times \mathbb{C}^n .$$

It is clear that $((z_k), (b_{kj}))$ depends analytically on $((t_k),(\beta_{kj}))$. Explicitly one gets the formulas:

$$\begin{cases} z_k = z_1^{(0)} + \int_{t_1}^{t_k} dG & k = 1, \ldots, m \\ b_{kj} = - \frac{1}{j} \frac{1}{2\pi i} \oint_{t_k} (G(t) - z_k)^j \, dG^*(t) . \end{cases}$$

A simple computation shows that the coefficients $\{b_{k,j}\}$ and the coefficients $\{c_{k,j}\}$ of the quadrature functional L for $\Omega = G(W)$ (eq (7.1), (7.2)) are related by

$$c_{k,j} = \frac{\pi}{j!} b_{k,j+1} , \quad j = 0, \ldots, n_k - 1 , \quad k = 1, \ldots, m .$$

Hence, apart from the factors $\frac{\pi}{j!}$, τ is the mapping which to each $q = ((t_k), (\beta_{kj})) \in U \subset W^m \times \mathbb{C}^n$ associates the data of the quadrature formula for $\Omega_q = G_q(W)$, where dG_q is the differential determined by $q = ((t_k), (\beta_{kj}))$.

Since $z_1 = z_1^{(0)}$ is constant and

$$\sum_{k=1}^{m} b_{k,1} = \frac{1}{\pi} \sum_{k=1}^{m} c_{k,o} = \frac{|\Omega|}{\pi}$$

is always real (and > 0), τ actually maps $U \subset \mathbb{C}^m \times \mathbb{C}^n$ into an affine subspace of $\mathbb{C}^m \times C^n$ with real codimension = 3. This implies that U must contain submanifolds of real dimension ≥ 3 on which τ is constant. For if

$$d\tau_q : \mathbb{R}^{2m+2n} \to \mathbb{R}^{2m+2n}$$

is the (real) differential (Jacobian) of τ at the point $q \in U$, then necessarily rank $(d\tau_q) \leq 2m + 2n - 3$ for all $q \in U$, and its maximum:

$$r = \max_{q \in U} \text{rank } (d\tau_q)$$

is attained on an open subset $V \subset U$. By the implicit function theorem

$$M_c = \{q \in V : \tau(q) = \text{constant} = c\}$$

are then manifolds of dimension = $2m + 2n - r \geq 3$. If $q, q' \in M_c$, it follows that the associated quadrature domains

$$\Omega_q = G_q(W)$$

and

$$\Omega_{q'} = G_{q'}(W)$$

have the same quadrature functional, L_c.

Observe however that the argument we have used does not guarantee that the initial data $q^{(0)} = ((t_k^{(0)}, (\beta_{kj}^{(0)})) \in U$ belong to M_c for some c, since it could happen that $q^{(0)} \notin V$, i.e. that

$$\text{rank } (d\tau_{q^{(0)}}) < r .$$

We have to account for the possibility that different $q \in U$, i.e. different G_q, define the same domain $\Omega = G_q(W)$. So suppose that

$$\Omega = G_q(W) = G_{q'}(W) .$$

Then $\varphi = G_q^{-1} \circ G_{q'}$ is an automorphism $W \to W$, and φ = identity if and only if $q = q'$.

Conversely, if $\varphi : W \to W$ is an automorphism then clearly $G(W) = G_q(W)$ where $G = G_q \circ \varphi$, and $G = G_{q'}$ for a unique point $q' \in U$ if φ is sufficiently close to the identity.

Thus the automorphy-group aut (W) defines orbits on U such that two points in U define the same domain Ω if and only if they lie on the same orbit. Since we are only considering "small" sets U, the only interesting cases are when aut (W) contains continuous families of automorphisms, i.e. when $p = 0$ and $p = 1$. In these cases the orbits in U are submanifolds of real dimension 3 resp. 1.

It is easily seen, for example by choosing suitable representatives for the orbits, that identifying points lying on the same orbit gives a quotient manifold U/aut (W) of dimension

$$\dim_{\mathbb{R}} U/\text{aut } (W) = \begin{cases} 2(m+n) - 3 & \text{for } p = 0 , \\ 2(m+n) - 1 , & p = 1 , \\ 2(m+n) , & p > 1 . \end{cases}$$

The manifold U/aut (W) can be thought of as a parameter-manifold for a family of conformally equivalent domains $\Omega \subset \mathbb{C}$.

The mapping $\tau : U \to \mathbb{C}^m \times \mathbb{C}^n$ induces an analytic mapping

$$\tilde{\tau} : U/\text{aut } (W) \to \mathbb{C}^m \times \mathbb{C}^n$$

This is the mapping which, on the parameter-side associates to each Ω its quadrature functional, L.

Clearly each orbit of aut (W) which meets a $M_c \subset V$ for some c lies entirely in this M_c. Thus we also have the quotient manifolds M_c/aut (W) of dimension:

$$\dim_{\mathbb{R}} M_c/\text{aut } (W) = \begin{cases} 2(m+n) - r - 3 \geq 0 & \text{if } \quad p = 0 \ , \\ 2(m+n) - r - 1 \geq 1 & \qquad p = 1 \ , \\ 2(m+n) - r \quad \geq 3 \ , & \qquad p > 1 \ , \end{cases}$$

on which $\tilde{\tau}$ is constant = c.

Thus, to summarize in a more concrete language:

<u>Theorem 7.1</u>: If $W \subset \mathbb{C}$ is any given domain of finite connectivity = p + 1, and if $n = n_1 + \ldots + n_m$ is large enough, there exist functionals

$$L(f) = \sum_{k=1}^{m} \sum_{j=o}^{n_k-1} c_{kj} f^{(j)}(z_k)$$

with the property that the quadrature formula:

$$\int_{\Omega} f dxdy = L(f) \ , \qquad f \in L^2_{as}(\Omega)$$

holds for a family of domains Ω, conformally equivalent to W and depending on q real parameters, where:

$$q = \begin{cases} 0 & \text{if} \quad p = 0 \ , \\ 1 & \qquad p = 1 \ , \\ 3 & \qquad p > 1 \ . \end{cases}$$

Observe that the number q in the Theorem does not depend on the order n of the functional, L.

It is interesting, that insisting that the formula shall hold for the class $L_a^2(\Omega)$ poses another 2p real restrictions on the mapping function:

$$G : W \to \Omega ,$$

namely that it shall be single-valued on the double $\hat{W}$, i.e. that:

$$\int_{\beta_k} dG = 0 \qquad k = 1, \ldots, p .$$

Since $q - 2p \leq 0$ this suggests that there never exist continuous families of conformally equivalent domains having the same quadrature formula for the class L_a^2. There even remains the possibility that two conformally equivalent domains never can have the same quadrature formula for L_a^2.

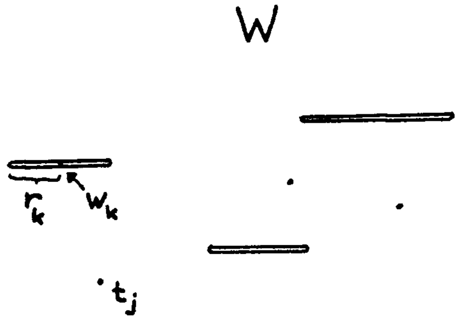


fig 7.2

Until now the domain $W = W^{(0)}$ has been kept fixed under the variations. The next step naturally is to let W also vary. Such variations are most easily described if the W:s are taken to be for example horizontal slit domains. So let

$$W = \mathbb{P} \setminus (p + 1 \text{ horizontal slits with endpoints } w_k \pm r_k , \quad k = 0, 1, \ldots p) ,$$

where

$$w_o, \ldots, w_p \in \mathbb{C} , \quad r_o, \ldots, r_p > 0 .$$

(fig 7.2). Given

$$t_1, \ldots, t_m \in W \text{ (distinct) and } (\beta_{kj}) \in \mathbb{C}^{n_1+\ldots+n_m}$$

we have the meromorphic differential dG on $\hat{W}$ defined as before (p. 7.3). The definition of dG now depends on the following parameters:

$$\begin{cases} (w_o, \ldots, w_p) \in \mathbb{C}^{p+1} \\ (r_o, \ldots, r_p) \in \mathbb{R}^{p+1} \\ (t_1, \ldots, t_m) \in W^m \subset \mathbb{P}^m \\ (\beta_{kj}) \in \mathbb{C}^{n_1+\ldots+n_m} = \mathbb{C}^n . \end{cases}$$

Starting at a point

$$q^{(0)} = ((w_k^{(0)}), (r_k^{(0)}), (t_k^{(0)}), (\beta_{kj}^{(0)})) \in \mathbb{C}^{p+1} \times \mathbb{R}^{p+1} \times \mathbb{P}^m \times \mathbb{C}^n$$

with the property that the corresponding integral

$$G = G^{(0)}$$

is univalent on a neighbourhood of

$$W^{(0)} \cup \Gamma^{(0)} \text{ in } \hat{W}^{(0)} ,$$

it seems very reasonable that there is an open neighbourhood U of $q^{(0)}$ such that for every $q \in U$, the corresponding G_q is univalent on W_q.

Suppose this is true. Then the mapping τ on page 7.4 becomes a well-defined and analytic mapping

$$\tau : U \subset \mathbb{C}^{p+1} \times \mathbb{R}^{p+1} \times \mathbb{P}^m \times \mathbb{C}^n \to \mathbb{C}^m \times \mathbb{C}^n$$

$$q = ((w_k), (r_k), (t_k), (\beta_{kj})) \to ((z_k), (b_{kj}))$$

with (z_k), (b_{kj}) defined as before.

We still have that the image of τ is contained in an affine subspace of real codimension = 3 in $\mathbb{C}^m \times \mathbb{C}^n$ ($z_1 = z_1^{(0)}$ = const.; $\Sigma\, b_{k,1} =$ = real).

Hence there must be submanifolds M_c in U of real dimension

$$2(p + 1) + (p + 1) + 2m + 2n - (2m + 2n - 3) = 3p + 6$$

on which τ is constant = c.

So if M_c is such a submanifold and $q, q' \in M_c$, then

$$\Omega_q = G_q(W_q)$$

and

$$\Omega_{q'} = G_{q'}(W_{q'})$$

are quadrature domains with the same quadrature functional $L_{\tau(q)} = L_{\tau(q')} = L_c$.

Also, $\Omega_q = \Omega_{q'}$ if and only if there is a conformal map:

$$\varphi : W_{q'} \to W_q \qquad (\varphi = G_q^{-1} \circ G_{q'})$$

It is known that such maps depend on 6 real parameters. More precisely, if W is a horizontal slit region then, given

$$\begin{cases} t_o \in W \\ a, b \in \mathbb{C} \quad (a \neq 0) \end{cases},$$

there is a unique univalent function φ on W with:

$$\varphi(t) = \frac{a}{t - t_o} + b + O(t - t_o) \quad \text{as} \quad t \to t_o ,$$

such that $\varphi(W)$ is also a horizontal slit region*. It follows that the set of all such mappings φ define orbit manifolds in M_c (and in U) of real dimension = 6 (if N is such a manifold and $q \in N$, then $q' \in N$ if and only if there is a conformal map $\varphi : W_{q'} \to W_q$ i.e. iff $\Omega_q = \Omega_{q'}$).

Hence, identifying points lying on the same orbit we get a quotient manifold Q_c for each M_c. Choosing a suitable representative for each

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* In fact, φ is just the function g_1 on p. 2.12. $d\varphi = dg_1$ is uniquely determined by a and t_o, and b is the arbitrary integration constant.

orbit, Q_c can be realized as a submanifold of M_c (fig. 7.3).

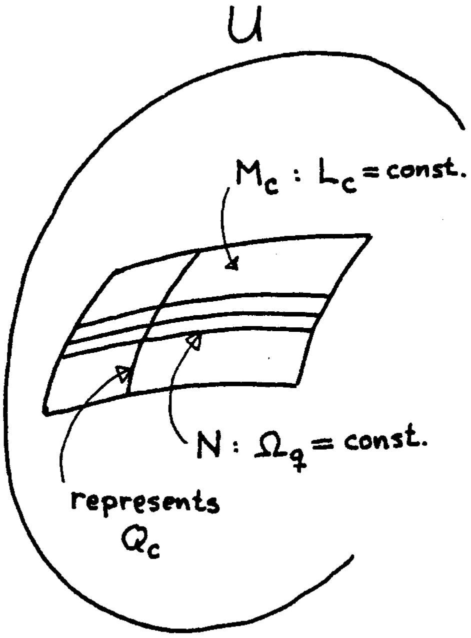

fig 7.3

Q_c has dimension = $(3p + 6) - 6 = 3p$, and the points on Q_c are in one-to-one correspondence to a family of quadrature domains, Ω_q $(q \in Q_c)$, all having the same quadrature functional L_c for the class L^2_{as}. Thus:

Theorem 7.2: For every $p \geq 0$ there exist functionals

$$L(f) = \sum_{k=1}^{m} \sum_{j=o}^{n_k - 1} c_{kj} f^{(j)}(z_k)$$

such that the quadrature formula

$$\int_{\Omega} f dxdy = L(f) \quad , \qquad f \in L^2_{as}(\Omega)$$

holds for a 3p-real-parameter family of domains Ω of connectivity $p + 1$.

Now we wish to adapt Theorem 7.2 for the test class L^2_a instead of L^2_{as}.

The mapping τ to be considered then is:

$$\tau : U \subset \mathbb{C}^{p+1} \times \mathbb{R}^{p+1} \times \mathbb{P}^m \times \mathbb{C}^n \to \mathbb{C}^m \times \mathbb{C}^n \times \mathbb{C}^p$$

$$q = ((w_k), (r_k), (t_k), (\beta_{kj})) \to ((z_k), (b_{kj}), (a_k)) \quad ,$$

where (z_k), (b_{kj}) are defined as earlier and

$$a_k = \frac{1}{2i} \int_{\beta_k} dG^*_q \quad , \qquad k = 1, \ldots, p \ .$$

If G_q is univalent the formula:

$$(7.3) \qquad \int_{\Omega_q} f dxdy = \sum_{k=1}^{m} \sum_{j=o}^{n_k-1} c_{k,j} f^{(j)}(z_k) + \sum_{k=1}^{p} a_k \int_{\alpha_k} f dz$$

then holds for $f \in L_a^2(\Omega_q)$, where $\Omega_q = G_q(W_q)$ and $c_{k,j} = \frac{\pi}{j!} b_{k,j+1}$.

Exactly the same reasoning as before yields that for every $p \geq 0$ there exist families of $(p + 1)$-connected domains, depending on $3p - 2p = p$ real parameters, admitting the above quadrature formula for a fixed right member.

We also know (Theorem 3.3 for instance) that there exist domains of arbitrary connectivity satisfying (7.3) with $a_k = 0$ for all $k = 1, \ldots, p$. Therefore it seems natural to conclude that such domains are members of p-real-parameter families of quadrature domains satisfying the same quadrature identity.

There is however a minor (probably) complication here. Namely, as in the proofs of Theorems 7.1 and 7.2, the construction of these families of domains consists of an application of the Implicit Function Theorem on the open subset of $U \subset \mathbb{C}^{p+1} \times \mathbb{R}^{p+1} \times \mathbb{P}^m \times \mathbb{C}^n$ on which rank $(d\tau)$ attains its maximum; and it may happen that rank $(d\tau_q)$ is strictly less than this maximum for all points $q \in U$ corresponding to quadrature functionals with all $a_k = 0$.

To illustrate this point, let us consider a (highly unrealistic) example:

Let $m = n = 1$, let N be some number $2, 3, \ldots, p$, suppose τ is given by:

$$\tau : \mathbb{C}^{p+1} \times \mathbb{R}^{p+1} \times \mathbb{C}^1 \times \mathbb{C}^1 \rightarrow \mathbb{C}^1 \times \mathbb{C}^1 \times \mathbb{C}^p$$

$$((w_k), (r_k), (t_1), (\beta_{11})) \rightarrow ((z_1), (b_{11}), (a_k)) ,$$

where:

$$\begin{cases} z_1 = 0 \\ b_{11} = r_o \\ a_1 = r_1 - 1 + i \cdot \sum_{k=2}^{N} (r_k - 1)^2 \\ a_2 = t_1 \\ a_3 = \beta_{11} \\ a_k = w_k , \quad k = 4, \ldots, p \end{cases}$$

and suppose that the 6-dimensional orbit manifolds are

$$\left\{((w_k), (r_k), (t_1), (\beta_{11})) : \begin{array}{l}\text{everything except } (w_1, w_2, w_3) \\ \text{is constant}\end{array}\right\}$$

This τ has the right properties in the sense that:

i) it is analytic (real analytic with respect to $(r_o, \ldots, r_p)$),

ii) z_1 is constant,

iii) $\sum_{k=1}^{m} b_{k,1} = b_{11}$ is real.

One readily sees that

$$\text{rank } (d\tau) = 2p + 1$$

at pre-images of points with Im $a_1 \neq 0$, while:

$$\text{rank } (d\tau) = 2p$$

when Im $a_1 = 0$, and in particular when $a_1 = \ldots = a_p = 0$.

Manifolds on which τ is constant have

$$\text{dimension} = \begin{cases} 8 + (p - 2) = p + 6 & \text{when Im } a_1 \neq 0 \\ 8 + (p - N) = p + 8 - N & \text{when Im } a_1 = 0 \end{cases}$$

since exactly

$$(w_o, w_1, w_2, w_3, r_3, \dots, r_p) \in \mathbb{C}^4 \times \mathbb{R}^{p-2}$$

can be varied freely in the former case, while only

$$(w_o, w_1, w_2, w_3, r_{N+1}, \dots, r_p) \in \mathbb{C}^4 \times \mathbb{R}^{p-N}$$

can be varied in the latter.

Thus, if the orbit manifolds are accounted for by freezing (w_1, w_2, w_3) at $(0, 0, 0)$, we see that this τ would correspond to a situation where the number of real parameters in the family of domains Ω with:

$$\int_\Omega f dx dy = L(f) \quad , \qquad f \in L^2{}_a(\Omega)$$

for a fixed

$$L(f) = \pi b_{11} \cdot f(z_1) + \sum_{k=1}^{p} a_k \int_{\alpha_k} f dz$$

would be

$$= \begin{cases} p & \text{when} \quad \operatorname{Im} a_1 \neq 0 \\ p + 2 - N & \text{when} \quad \operatorname{Im} a_1 = 0 \; . \end{cases}$$

Here $p + 2 - N = p$, $p - 1, \dots, 2$, depending on N.

This example shows that a somewhat deeper analysis is needed to prove the following

<u>Suggestion 7.3</u>: For every $p \geq 0$ there exist functionals

$$L(f) = \sum_{k=1}^{m} \sum_{j=o}^{n_k-1} c_{kj} f^{(j)}(z_k)$$

such that the quadrature formula

$$\int_{\Omega} f dx dy = L(f) \ , \qquad f \in L_a^2(\Omega)$$

holds for a p-real-parameter family of domains Ω of connectivity $p + 1$.

Observe that we have actually already proved this suggestion in the case $p = 1$, namely on pp. 6.10 - 6.12 (and for $p = 0$ of course; Theorem 3.3).

Suggestion 7.3 should also be compared with the upper bound for the number of free parameters obtained on p. 5.8 by considering the polynomials of the algebraic boundary curves $\partial\Omega$. Thus, if L is as in the suggestion and

$$n = \sum_{k=1}^{m} n_k = \text{the order of } L$$

then, according to p. 5.8

$$p \leq (n - 1)^2 \ .$$ 1)

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