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On a Differential Equation Arising in a Hele Shaw Flow Moving Boundary Problem

by

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I. INTRODUCTION

The present paper is mainly devoted to the following differential equation:

Given $f(\zeta)$, analytic and univalent in a neighbourhood of $|\zeta| \leqq 1$, find $f(\zeta,t)$, for $t \in \mathbb{R}$ small, analytic and univalent in a neighbourhood of $|\zeta| \leqq 1$, satisfying

(1) $$\mathrm{Re}[\dot{f}(\zeta,t)\cdot\overline{\zeta f'(\zeta,t)}] = 1 \qquad \text{for } |\zeta| = 1$$

and $f(\zeta,0) = f(\zeta)$ (for $|\zeta|<1$). In (1) $\dot{f}$ and f' denote derivates with respect to t and ζ respectively.

This differential equation arose in the paper [9] by S. Richardson as describing the solution of a two-dimensional moving boundary problem. The moving boundary in question then was the boundary of the domain $\Omega_t = f(\mathbb{D},t)$, where $\mathbb{D} = \{\zeta \in \mathbb{C} : |\zeta| < 1\}$ and t is time. Richardson did not prove existence or unicity for solutions of (1). However, this, essentially, was done in [12]. The existence of solutions was proved by using an iterative process, the proof of convergence of which was fairly complicated. Unicity was proved only with respect to solutions which depended analytically on t.

The aim of the present paper is primarily to give a more elementary proof of existence of solutions of (1) in the case that $f(\zeta)$ is a polynomial or a rational function. In that case (1) can be reduced to a finite system of ordinary differential equations (in t) and this system has a unique solution by standard theory. This solution is a polynomial or a rational function (as a function of ζ) of the same sort as $f(\zeta)$. (Theorem 4.)

We will also consider a generalization of the differential equation (1) in order to prove a result on the "moment map"

(2) $f \rightarrow (c_0, c_1, c_2, \ldots)$, where

(3) $c_n = \iint_{\Omega} z^n dxdy \; , \quad \Omega = f(\mathbb{D})$.

($\{c_n\}$ are the "complex moments" of the domain $\Omega = f(\mathbb{D})$). Namely, we prove (Theorem 6) that when (2) is viewed as a mapping from the set of univalent polynomials of degree $\leq r$, normalized by $f(0) = 0$ and $f'(0) > 0$, it is an immersion, i.e. its Fréchet derivative is one-to-one.

The first section in this paper is devoted to a (rather informal) sketch of how the differential equation (1) arises from the moving boundary problem which is the source for (1) in [9].

II. PHYSICAL BACKGROUND

a) Technical background

We start by sketching the physical and technical background of the moving boundary problem we are considering. This description essentially follows Richardson [9] complemented by facts from Lamb [8]. Aside from the background given here, there are other physical problems leading to the same or similar moving boundary problems, for example the dissolution of an anode under electrolysis (see [6], or [7], ex. 1 on p. 2), the propagation of a liquid front in a porous medium flow ([7], ex. 2 on p.2) and the melting of a solid in a one-phase Stefan problem with zero specific heat (see e.g. [4] and p. 4 and 26f of [7]).

Richardson considers the industrial process of production of thin lamina of plastics. Molten polymer is injected into a mould consisting of two large plates separated by a narrow gap. The injection takes place through a hole in one of the plates and the space between the plates is restricted by side walls to give the desired form to the lamina. When the melt has filled out the accessible space it is allowed to solidify. Air vents are placed at suitable points in order to let the air escape.

After the injection has begun, the melt will describe the form of an expanding circular disc until it reaches one of the side walls. Then the motion becomes more complex and there arises the mathematical problem of describing it.

This problem is very hard. A faithful mathematical model for it would have to take care of features of the fluid such as

1) compressibility due to high pressures, etc.
2) high viscosity which in turn will generate heat and therefore
3) non-constant temperature and thermal flow.
4) Moreover, a polymer melt is to be considered as a non-Newtonian fluid (i.e., roughly, the viscosity forces do not depend linearly on the velocity gradients).

To get a manageable mathematical problem Richardson makes the drastic simplification of replacing the polymer melt by a fluid with the following properties:

1) it is Newtonian,

2) incompressible and

3) thermal effects are negligible .

On the other hand, the fluid is still allowed to have high viscosity (in fact, the higher the better).

Still, however, the problem is too hard (to begin with at least) so Richardson makes one further simplification:

4) let there be no side walls in the mould and instead (to get a non-trivial problem) let the fluid initially occupy some given more or less arbitrary region.

A flow in the space between two parallel surfaces at a short distance from each other and with the fluid fulfilling 1)-3) above is called a Hele Shaw flow. Thus the problem is that of describing a Hele Shaw flow with free boundaries and with a source point.

b) Derivation of the Hele Shaw equation

Following Lamb [8] (p.581f) we now derive the equations governing a Hele Shaw flow starting from the Navier-Stokes equations. Let the two plates have equations

(1) $$\begin{cases} x_3 = h \quad \text{and} \\ x_3 = 0 \end{cases}$$

respectively in Euclidean 3-space with coordinates x_1, x_2, x_3 where $h > 0$ is the distance between the plates (h small). Let further

$\mathbf{v} = (v_1, v_2, v_3)$ be the velocity vector of the flow,

p the pressure of the fluid,

μ the viscosity coefficient,

$\bar{\mathbf{v}} = \frac{1}{h}\int_0^h \mathbf{v}\, dx_3$ the average over the gap of the velocity,

ρ the density of the fluid (constant by the imcompressibility assumption).

The Navier-Stokes equations for an incompressible, isothermal Newtonian fluid in the absence of outer forces (the influence of gravity is neglected) read

(2) $$\frac{\partial \mathbf{v}}{\partial t} + (\mathbf{v}\cdot\text{grad})\mathbf{v} = -\frac{1}{\rho}\,\text{grad}\, p + \frac{\mu}{\rho}\,\Delta\mathbf{v}$$

($\mathbf{v}$.grad is the operator $\sum_{i=1}^{3} v_i \frac{\partial}{\partial x_i}$ and Δ the Laplacian $\sum_{i=1}^{3} \frac{\partial^2}{\partial x_i^2}$).

Now we assume the injection of fluid is slow enough for the flow to be approximately stationary. This means that the term $\frac{\partial \mathbf{v}}{\partial t}$ can be neglected in (1):

(3) $$\frac{\partial \mathbf{v}}{\partial t} = 0 \,.$$

Moreover it is reasonable and customary to assume that the flow is entirely horizonatal, i.e. that

(4) $$v_3 = 0 \,.$$

With these assumptions (2) written up in component form becomes

(5) $$\left(v_1 \frac{\partial}{\partial x_1} + v_2 \frac{\partial}{\partial x_2}\right) v_1 = -\frac{1}{\rho}\frac{\partial p}{\partial x_1} + \frac{\mu}{\rho}\Delta v_1$$

(6) $$\left(v_1 \frac{\partial}{x_1} + v_2 \frac{\partial}{\partial x_2}\right) v_2 = -\frac{1}{\rho}\frac{\partial p}{\partial x_2} + \frac{\mu}{\rho}\Delta v_2$$

(7) $$0 = -\frac{1}{\rho}\frac{\partial p}{\partial x_3}$$

The boundary conditions are

(8) $$v_1 = v_2 = 0 \quad \text{whenever} \quad x_3 = 0 \quad \text{or} \quad x_3 = h \,.$$

If h is sufficiently small the first and second order derivatives of $\mathbf{v}$ in the x_1 and x_2 directions are negligibly small in comparison with the derivaties of the same order in the x_3 direction. Thus

(9) $$\left(\frac{\partial^2}{\partial x_1{}^2} + \frac{\partial^2}{\partial x_2{}^2}\right) v_j << \frac{\partial^2}{\partial x_3{}^2} v_j \qquad (j = 1,2) \,.$$

For similar reasons and assuming that μ is sufficiently large we also have

(10) $$\left(v_1 \frac{\partial}{\partial x_1} + v_2 \frac{\partial}{\partial x_2}\right) v_j << \frac{\mu}{\rho} \frac{\partial^2}{\partial x_3{}^2} v_j \qquad (j = 1,2) \,.$$

Inserting (9) and (10) into (5) - (7) there remains

(11) $$\frac{\partial p}{\partial x_1} = \mu \frac{\partial^2 v_1}{\partial^2 x_3}$$

(12) $$\frac{\partial p}{\partial x_2} = \mu \frac{\partial^2 v_1}{\partial^2 x_3}$$

(13) $$\frac{\partial p}{\partial x_3} = 0$$

Here (13) shows that p, and hence $\frac{\partial p}{\partial x_1}$ and $\frac{\partial p}{\partial x_2}$, do not depend on x_3. Therefore (11) and (12) imply that v_1 and v_2 considered as functions of x_3 are polynomials of degree at most two. The boundary conditions (8) yield that these polynomials are

(14) $$\begin{cases} v_1 = -\dfrac{6\bar{v}_1}{h^2}\, x_3(x_3-h) \\ v_2 = -\dfrac{6\bar{v}_2}{h^2}\, x_3(x_3-h)\ , \end{cases}$$

where $\bar{v}_j = \frac{1}{h}\int_0^h v_j\, dx_3$ is the averaged velocity over the gap.

Inserting (14) into (11), (12) gives

(15) $$\begin{cases} \dfrac{\partial p}{\partial x_1} = -\dfrac{12\mu}{h^2}\,\bar{v}_1 \\ \dfrac{\partial p}{\partial x_2} = -\dfrac{12\mu}{h^2}\,\bar{v}_2 \end{cases}$$

Since also $\frac{\partial p}{\partial x_3} = -\frac{12\mu}{h^2}\,\bar{v}_3$ (= 0) we have

(16) $$\bar{\mathbf{v}} = -\frac{h^2}{12\mu}\cdot \text{grad } p\ ,$$

where $\bar{\mathbf{v}} = \frac{1}{h}\int_0^h \mathbf{v}\, dx_3$. (16) is known as the Hele Shaw equation.

We notice the following consequences of (16).

1) The flow is irrotational, even a potential flow, despite high viscosity of the fluid. Moreover, the potential function for the velocity field is proportional to the pressure.

2) The pressure p is a harmonic function, as a consequence of div $\bar{\mathbf{v}} = 0$ (incompressibility of the flow).

c) Derivation of equations for the moving boundary

Let us now consider the Hele Shaw flow as a two-dimensional flow (in the (x_1,x_2)-plane). We then pass to complex variable notations as follows.

$\mathbb{C}$ = the (x_1,x_2)-plane

$z = x+iy = x_1+ix_2$ = the position variable in $\mathbb{C}$

$w = u+iv = \bar{v}_1-i\bar{v}_2$ = the complex conjugate of the averaged velocity vector (the bars on v_1,v_2 denote here averages; in the sequel bars will always denote complex conjugation).

$$2\frac{\partial}{\partial \bar{z}} = \frac{\partial}{\partial x} + i\frac{\partial}{\partial y} = \text{grad}$$

Let further

$\Omega = \Omega_t \subset \mathbb{C}$ denote the region in $\mathbb{C}$ occupied by fluid at time t and let

$0 \in \Omega$ be the point of injection.

Then the Hele Shaw equation (16) becomes

(17) $$w = -\frac{h^2}{6\mu}\cdot\frac{\partial p}{\partial z} \quad \text{in } \Omega .$$

We are now going to compute the function p, given Ω. Since p is harmonic in Ω, or, more correctly, in $\Omega\setminus\{0\}$, we only have to know the boundary values of p on $\partial\Omega$ and what kind of singularity p has at $z=0$ (the point of injection).

On $\partial\Omega$ we have

(18) p = the pressure of the fluid
= (the pressure outside the fluid) +
+ (the surface tension).

Now we may assume that

(19) the pressure outside = constant = the air pressure.

In fact, the pressure outside must at least be constant on each component of $\partial\Omega$. A more sophisticated model would allow the pressure to take different constant values on different components of $\partial\Omega$. One could, for example, think of the situation that in each bounded component of Ω^c there is a certain amount of air (namely the air which was there when the component in question was created) which is not able to escape and which therefore exerts a pressure on its component of $\partial\Omega$ that varies with the size of the component (e.g. according to Boyle's law). However, we shall assume (19) for simplicity.

As for the surface tension this is known ([8], §265, p455-56) to be (roughly) proportional to the mean curvature. Assuming that the curvature of $\partial\Omega$ is moderate it will be the radii of curvature in vertical sections through the flow which will have most influence on the surface tension and these radii will be of the order of magnitude h, hence will be essentially constant. Thus the surface tension will be approximately constant.

The above discussion results in

(20) $p = \text{constant}$ on $\partial\Omega$

(under the assumption that the curvature of $\partial\Omega$ is moderate).

To determine the singularity of p at the origin, let Q denote the rate by which area is created at $z=0$ ("area" in our two-dimensional model corresponds to "amount of fluid" in the three-dimensional model). This means that

(21) $$Q = \frac{d}{dt}|\Omega_t| .$$

Let U be a small neighbourhood of $z=0$ in Ω_t. Then, as is easily seen, the total flow of $\bar{w}$ per time unit through the infinitesimal element dz

of ∂U equals $\mathrm{Re}[w\cdot(-idz)]$ (cf. Fig. 1 below and observe that the scalar product of two complex numbers A and B, regarded as vectors, is $\mathrm{Re}[\bar{A}\cdot B]$), and it follows that

$$Q = \text{amount of area passing out through } \partial U \text{ per time unit}$$

$$= \int_{\partial U} \mathrm{Re}[w\cdot(-i\,dz)]$$

$$= \mathrm{Re}\,\frac{1}{i}\int_{\partial U} w\,dz = 2\pi\,\mathrm{Re}\,\operatorname*{Res}_{z=0} w\,dz$$

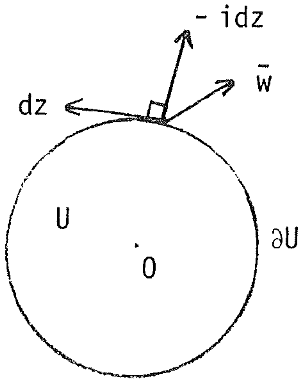

Fig. 1.

Moreover, since p is real and single-valued we have by (17)

$$\mathrm{Re}\int_{\partial U} w\,dz = \frac{1}{2}\int_{\partial U}(w\,dz + \bar{w}\,d\bar{z}) =$$

$$= -\frac{h^2}{12\mu}\int_{\partial U}\left(\frac{\partial p}{\partial z}\,dz + \frac{\partial p}{\partial \bar{z}}\,d\bar{z}\right) =$$

$$= -\frac{h^2}{12\mu}\int_{\partial U} dp = 0\ .$$

Thus $\mathrm{Im}\operatorname*{Res}_{z=0} w\,dz = 0$ and so

(22) $$\operatorname*{Res}_{z=0} w\,dz = \frac{Q}{2\pi}\ .$$

Assuming that w and p are no more singular at $z=0$ than necessary, it follows that

(23) $w = \frac{Q}{2\pi}\frac{1}{z}$ + regular analytic

and

(24) $p = -\frac{6\mu}{h^2} \cdot \frac{Q}{\pi} \cdot \log|z|$ + regular harmonic

at $z = 0$.

Now (20) and (24) (together with the fact that p is harmonic) give

(25) $p = \frac{6\mu Q}{\pi h^2} \cdot g_\Omega(z,0)$ + constant,

where $g_\Omega(z,\zeta)$ is the Green's function for Ω .

Also, by (17)

(26) $w = -\frac{Q}{\pi}\frac{\partial}{\partial z} g_\Omega(z,0)$

Equation (26) yields the rule according to which $\partial\Omega_t$ moves as t increases. In fact, a point z on $\partial\Omega_t$ moves with the velocity

(27) $\frac{\partial z}{\partial t} = \bar{w}(z)$ i.e.

(28) $\frac{\partial z}{\partial t} = -\frac{Q}{\pi}\frac{\partial}{\partial \bar{z}} g_\Omega(z,0)$

$= -\frac{Q}{2\pi}$ grad $g_\Omega(z,0)$.

Equation (28) can be said to constitute the moving boundary condition. It could be formulated in a more precise way, for example along the following lines:

$t \to \Omega_t$ is a solution of our moving boundary problem if the boundaries $\partial\Omega_t$ are sufficiently smooth (say C^2, so that grad $g_{\Omega_t}(z,0)$ have continuous extensions to $\partial\Omega_t$) and there exist parametrizations $s \to z(s,t)$ of them which depend smoothly on both s and t and for which

(29) $$\frac{\partial z(s,t)}{\partial t} = -\frac{Q}{2\pi}\,\mathrm{grad}\; g_{\Omega_t}(z(s,t),0)$$

holds.

Equation (28) is an equality between vectors (or complex numbers considered as vectors). The content of (28) is also expressed in the following scalar (or real) equation in which $\partial n_z/\partial t$ denotes the velocity of $\partial\Omega_t$ at the point z in the direction of the outward normal to $\partial\Omega_t$ and $\partial/\partial n_z$ denotes the outward normal derivative on $\partial\Omega_t$.

(30) $$\frac{\partial n_z}{\partial t} = -\frac{Q}{2\pi}\,\frac{\partial g_{\Omega_t}(z,0)}{\partial n_z} \quad .$$

There are a lot of other ways fo formulate the moving boundary condition (28). One such way is by the equation

(31) $$\frac{\partial}{\partial t}\, g_{\Omega_t}(z,0) = \frac{Q}{2\pi}\left(\frac{\partial g_{\Omega_t}(z,0)}{\partial n_z}\right)^2 ,$$

to hold on $\partial\Omega_t$. (31) is derived as follows. For pairs (z,t) with $z \in \partial\Omega_t$ we have

(32) $$g_{\Omega_t}(z,0) = 0 \;.$$

Thus

(33) $$d_{(z,t)}\, g_{\Omega_t}(z,0) = 0$$

with respect to variations of (z,t) subject to $z \in \partial\Omega_t$. ($d_{(z,t)} g_{\Omega_t}(z,0)$ denotes the total differential of the function $(z,t) \to g_{\Omega_t}(z,0)$.) On the other hand

$$(34)\qquad d_{(z,t)}g_{\Omega_t} = \frac{\partial g_{\Omega_t}}{\partial s}\,ds + \frac{\partial g_{\Omega_t}}{\partial n}\,dn + \frac{\partial g_{\Omega_t}}{\partial t}\,dt$$

where $\frac{\partial}{\partial s}$ denotes derivative with respect to arc-length s, and ds is the arc-length differential (this is not the same s as the one on p. 11-12). Since $\partial g_{\Omega_t}/\partial s = 0$ (33) and (34) yield

$$(35)\qquad \frac{\partial g_{\Omega_t}}{\partial n}\,dn + \frac{\partial g_{\Omega_t}}{\partial t}\,dt = 0 \;.$$

This gives

$$(36)\qquad \frac{\partial n_z}{\partial t} = -\,\frac{\partial g_{\Omega_t}/\partial t}{\partial g_{\Omega_t}/\partial n}$$

which by (30) is (31).

Since */

$$(37)\qquad \left(\frac{\partial g}{\partial n}\right)^2 = |\text{grad } g|^2 = \left|2\frac{\partial g}{\partial \bar{z}}\right|^2 = 4\frac{\partial g}{\partial z}\,\frac{\partial g}{\partial \bar{z}}$$

(31) also can be written

$$(38)\qquad \frac{\partial g}{\partial t} = \frac{2Q}{\pi}\,\frac{\partial g}{\partial z}\,\frac{\partial g}{\partial \bar{z}} \qquad \text{on } \partial\Omega_t \;.$$

(31) and (38) do not, however, seem to be particularly useful because they are equations which are to hold only on a set the location of which is not known a priori (namely $\partial\Omega_t$).

Still another way to forumulate the moving boundary condition is to write it as a partial differential equation to hold in the distribution sense

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*/ For convenience we now drop some subscripts. Thus $g = g_{\Omega_t}$, $\frac{\partial}{\partial n} = \frac{\partial}{\partial n_z}$ etc.

for the characteristic function χ_{Ω_t} of the set Ω_t.
With

$$u(x,y,t) = \begin{cases} g_{\Omega_t}(z,0) & \text{for } z = x+iy \in \Omega_t \\ 0 & \text{otherwise} \end{cases}$$

$$H(u) = \begin{cases} 1 & \text{when } u > 0 \\ 0 & u \leqq 0 \end{cases}$$

we have

$$\chi_{\Omega_t}(z) = H(u(x,y,t))$$

and the differential equation in question is

(39) $$\frac{\partial}{\partial t} H(u) = \frac{Q}{2\pi} \Delta u + Q \cdot \delta_0$$

Here Δ is the Laplace operator (Δu will be a distribution with support on $\partial\Omega_t$ and at the origin) and δ_0 is the Dirac measure at the origin. The derivation of (39) from (28) or (30) is left to the reader, or he/she is referred to [7] (p.22-25).

Now, and finally, we shall derive that equation for the moving boundary which will be the subject for investigation in the rest of this paper. This is an equation for the Riemann mapping function from the unit disc $\mathbb{D}$ onto Ω_t. Thus we shall assume from now on that Ω_t is simply connected.

Let $f_t(\zeta) = f(\zeta,t)$ be the (unique) conformal map

$$f_t : \mathbb{D} \to \Omega_t$$

subject to

$$f_t(0) = 0 \quad \text{and}$$

$$f_t'(0) > 0.$$

(recall that $0 \in \Omega_t$ for all t).

Then the equation we have in mind is

(40) $$\mathrm{Re}\,[\overline{\zeta f'(\zeta,t)}\cdot \dot f(\zeta,t)] = \frac{Q}{2\pi}\ ,$$

to hold for all $\zeta\in\partial\mathbb{D}$. Here, and in the sequel, f' denotes the derivative of (the analytic function) f with respect to ζ and $\dot f$ its derivative with respect to t.

In order for (1) to make sense $\partial\Omega_t$ must fulfil some regularity requirements, e.g. so that f' and $\dot f$ are continuously extendible to $\partial\mathbb{D}$. We shall, however, not bother to formulate any such conditions here. Also, the derivation of (40) will be quite informal but the reader might easily formulate for himself suitable conditions on f for which (40) and the derivation of it is valid. In the rest of this paper (sections III and IV) we are in any case going to work with conditions on f which are more than enough for (40) and the derivation of it to make sense, namely that f is analytically extendible to some neighbourhood of $\overline{\mathbb{D}}$ (for all t under consideration) and that it is continuously differentiable as a function of (ζ,t).

Now to the derivation of (40). We have, by well-known transformation properties of the Green´s function,

(41) $$g_{\Omega_t}(f(\zeta,t),0) = g_{\mathbb{D}}(\zeta,0) = -\log|\zeta| \qquad \text{for } \zeta\in\mathbb{D}$$

This gives

(42) $$\begin{aligned} 0 &= \frac{\partial}{\partial t}(-\log|\zeta|) = \\ &= \frac{\partial}{\partial t} g_{\Omega_t}(f(\zeta,t),0) = \frac{\partial g_{\Omega_t}}{\partial t}(f(\zeta,t),0) + \\ &+ \frac{\partial g_{\Omega_t}}{\partial z}(f(\zeta,t),0)\cdot\frac{\partial f(\zeta,t)}{\partial t} + \\ &+ \frac{\partial g_{\Omega_t}}{\partial \bar z}(f(\zeta,t),0)\cdot\overline{\frac{\partial f(\zeta,t)}{\partial t}} = \\ &= \frac{\partial g}{\partial t} + 2\,\mathrm{Re}\left[\frac{\partial g}{\partial z}\cdot\frac{\partial f}{\partial t}\right]\ . \end{aligned}$$

Further

(43) $$-\frac{1}{2\zeta} = \frac{\partial}{\partial\zeta}(-\log|\zeta|) =$$

$$= \frac{\partial g_{\Omega_t}}{\partial z}(f(\zeta,t),0)\cdot\frac{\partial f(\zeta,t)}{\partial\zeta} + \frac{\partial g_{\Omega_t}}{\partial\bar{z}}(f(\zeta,t),0)\cdot\frac{\overline{\partial f(\zeta,t)}}{\partial\zeta} =$$

$$= \frac{\partial g}{\partial z}\cdot\frac{\partial f}{\partial\zeta} \quad .$$

Hence

(44) $$\frac{\partial g}{\partial t} = -2\,\mathrm{Re}\left[\frac{\partial g}{\partial z}\cdot\frac{\partial f}{\partial t}\right]$$

and

(45) $$\frac{\partial g}{\partial z} = -\frac{1}{2\zeta\frac{\partial f}{\partial\zeta}} \quad .$$

Thus (38) gives, for $\zeta \in \partial\mathbb{D}$,

(46) $$\frac{Q}{2\pi} = \frac{\partial g/\partial t}{4\frac{\partial g}{\partial z}\cdot\frac{\partial g}{\partial\bar{z}}} = \frac{-2\,\mathrm{Re}\left[\frac{\partial g}{\partial z}\cdot\frac{\partial f}{\partial t}\right]}{4\frac{\partial g}{\partial z}\cdot\frac{\partial g}{\partial\bar{z}}} =$$

$$= -\frac{1}{2}\,\mathrm{Re}\left[\frac{\partial f/\partial t}{\partial g/\partial\bar{z}}\right] = \mathrm{Re}\left[\overline{\zeta\frac{\partial f}{\partial\zeta}}\cdot\frac{\partial f}{\partial t}\right]$$

which is (40).

<u>Remark</u>: Whether f satisfies (40) or not, the left member of (40) can be interpreted as

(47) $$\mathrm{Re}\left[\overline{\zeta\frac{\partial f}{\partial\zeta}}\cdot\frac{\partial f}{\partial t}\right] = \frac{Q}{2\pi}\cdot\frac{\text{the normal velocity of } \partial\Omega_t}{\text{the flow velocity at } \partial\Omega_t} \quad .$$

(47) follows from (46), (37), (36) and (26) in the following manner.

$$\mathrm{Re}\left[\zeta\overline{\frac{\partial f}{\partial \zeta}}\cdot\frac{\partial f}{\partial t}\right] = \frac{\partial g/\partial t}{4\frac{\partial g}{\partial z}\cdot\frac{\partial g}{\partial \bar{z}}} = \frac{-\partial g/\partial t\big/\partial g/\partial n}{-\partial g/\partial n} = \frac{\partial n/\partial t}{|2\,^{\partial g}/\partial \bar{z}|} = \frac{\partial n/\partial t}{^{2\pi}/_{Q}\cdot|\bar{w}|} \quad .$$

There are alternative ways to formulate (40). For $\zeta\in\partial\mathbb{D}$, ζ can be written $\zeta = e^{i\theta}$ with $\theta\in\mathbb{R}$. Then $\zeta\frac{\partial}{\partial\zeta} = -i\frac{\partial}{\partial\theta}$, and (40) becomes

(48) $$\mathrm{Im}\left[\overline{\frac{\partial f}{\partial\theta}}\cdot\frac{\partial f}{\partial t}\right] = -\frac{Q}{2\pi}\;, \qquad \theta\in\mathbb{R}\quad .$$

With $f = u + iv$ we have

(49) $$\mathrm{Im}\left[\overline{\frac{\partial f}{\partial\theta}}\cdot\frac{\partial f}{\partial t}\right] = \frac{\partial u}{\partial\theta}\cdot\frac{\partial v}{\partial t} - \frac{\partial v}{\partial\theta}\cdot\frac{\partial u}{\partial t} = \frac{\partial(u,v)}{\partial(\theta,t)}$$ = (the Jacobi-determinant

of the map $(\theta,t)\to(u,v)$)

Thus (48) takes the form

(50) $$\frac{\partial(u,v)}{\partial(\theta,t)} = -\frac{Q}{2\pi}\;, \quad \theta,t\in\mathbb{R}\;.$$

(50) can be regarded as a differential equation for the two real functions u and v defined on $\mathbb{R}\times\mathbb{R}$ and 2π-periodic in the θ variable. As such it expresses that the map $(\theta,t)\to(u,v)$ shall be area preserving up to a constant factor. The two functions u and v in (50) are, however, not independent of each other but (50) has to be supplemented by the condition that, as a function of $e^{i\theta}$, $u + iv$ has an analytic extension onto $\mathbb{D}$. That is, v shall be the Hilbert transform of u.

Remarkably enough it is possible to write down the "general solution" of (50). Namely, following [3] (Anhang zum ersten Kapitel, §4, s.49f), introduce new independent variables α and β and regard θ, t, u and v as functions of α and β :

(51) $$\begin{cases} \theta = \theta(\alpha,\beta) \\ t = t(\alpha,\beta) \end{cases} \qquad \begin{cases} u = u(\alpha,\beta) \\ v = v(\alpha,\beta) \end{cases}$$

Then

(52) $$\frac{\partial(u,v)}{\partial(\alpha,\beta)} = \frac{\partial(u,v)}{\partial(\theta,t)} \cdot \frac{\partial(\theta,t)}{\partial(\alpha,\beta)}$$

and (50) becomes

(53) $$\frac{\partial(u,v)}{\partial(\alpha,\beta)} = -\frac{Q}{2\pi} \frac{\partial(\theta,t)}{\partial(\alpha,\beta)} \quad .$$

Now the "general solution" of (53) is

(54) $$\begin{cases} \theta = \alpha + \dfrac{\partial\omega}{\partial\beta} & u = k \cdot (\beta + \dfrac{\partial\omega}{\partial\alpha}) \\ t = \beta - \dfrac{\partial\omega}{\partial\alpha} & v = k \cdot (\alpha - \dfrac{\partial\omega}{\partial\beta}) \end{cases} \quad ,$$

where $\omega = \omega(\alpha,\beta)$ is an "arbitrary" function satisfying

(55) $$1 - \left(\frac{\partial^2 \omega}{\partial\alpha\partial\beta}\right)^2 + \frac{\partial^2\omega}{\partial\alpha^2} \cdot \frac{\partial^2\omega}{\partial\beta^2} \neq 0 \ .$$

and where $k = \sqrt{Q/2\pi}$. (The expression in the left member of (55) is the same thing as $\frac{\partial(\theta,t)}{\partial(\alpha,\beta)}$.)

III. TREATMENT OF THE DIFFERENTIAL EQUATION

a) Notations and preliminaries

The rest of this paper is devoted to equation (II.40) and certain generalizations of it. The question which will interest us is that of existence and unicity of solutions (locally). To state the problem more accurately we introduce notations as follows.

$\mathbb{D}(a;r) = \{z \in \mathbb{C} : |z-a| < r\}$ (if $a \in \mathbb{C}$, $r > 0$),

$\mathbb{D} = \mathbb{D}(0;1)$,

$\mathbb{P} = \mathbb{C} \cup \{\infty\}$ = the Riemann sphere .

If $\Omega \subset \mathbb{C}$ is an open set

$H(\Omega)$ = {holomorphic functions on Ω},
$M(\Omega)$ = {meromorphic functions on Ω}.

For an arbitrary set $E \subset \mathbb{C}$ let

$H(E)$ = {functions, holomorphic in some open set containing E}.

In $H(E)$ two functions are identified if they agree on some neighbourhood of (i.e. open set containing) E.

$\mathcal{O} = \{f \in H(\overline{\mathbb{D}}) : f' \neq 0 \text{ on } \overline{\mathbb{D}}\}$,

$\mathcal{O}_0 = \{f \in \mathcal{O} : f(0) = 0\}$,

$\mathcal{O}_1 = \{f \in \mathcal{O} : f(0) = 0 \text{ and } f'(0) > 0\}$,

$H(\overline{\mathbb{D}})_1 = \{f \in H(\overline{\mathbb{D}}) : f(0) = 0 \text{ and } \operatorname{Im} f'(0) = 0\}$.

The following notations will not be needed just now but we gather them here in order to have all notations in one place.

If f is a function meromorphic in an open set $U \subset \mathbb{P}$ we set

$\operatorname{Div}_U f$ = the divisor of f in U
= the formal sum of the zeroes of f (occurring with plus signs) and poles (with negative signs), both counted with multiplicities.

$\mathrm{Pdiv}_U f$ = the pole divisor of f in U
= the formal sum of the poles of f in U counted with multiplicities (and with plus signs)

$\mathrm{Zdiv}_U f$ = the zero divisor of f in U
= the formal sum of the zeroes of f in U (counted with multiplicities)

Thus

$$\mathrm{Div}_U f = \mathrm{Zdiv}_U f - \mathrm{Pdiv}_U f$$

When $U = \mathbb{P}$ we just write Div in place of $\mathrm{Div}_{\mathbb{P}}$ etc.

The set (or abelian group) of divisors is partially ordered in a natural way, namely so that a divisor $\sum_{j=1}^{r} n_j \cdot (\zeta_j)$ (n_j integers, $\zeta_j \in \mathbb{P}$) is non-negative,

$$\sum_{j=1}^{r} n_j \cdot (\zeta_j) \geqq 0 \quad ,$$

if and only if $n_j \geqq 0$ for all j, assuming here that all the ζ_j are distinct. Then $D_1 \geqq D_2$ (D_1, D_2 divisors) means that $D_1 - D_2 \geqq 0$. With respect to this partial order the concepts max (= sup) and min (= inf) make sense and will be used.

$\mathcal{R}_n$ and $\mathcal{P}_r$ are defined on p. 28-29.

$*$ will denote the reflection map in $\partial\mathbb{D}$ and various associated maps, namely,

$\zeta^* = 1/\bar{\zeta}$ for points $\zeta \in \mathbb{P}$,

$\left(\sum_j n_j \cdot (\zeta_j)\right)^* = \sum_j n_j \cdot (\zeta_j^*)$ for divisors (n_j integers),

$E^* = \{\zeta^* \in \mathbb{P} : \zeta \in E\}$ for sets $E \subset \mathbb{P}$,

$F^*(\zeta) = \overline{F(\zeta^*)}$ for meromorphic functions F.

Now we can state the problem */:

Given $f_0 \in \mathcal{O}_1$, find an $\varepsilon > 0$ and a map

$(-\varepsilon,\varepsilon) \ni t \to f_t \in \mathcal{O}_1$

such that the function $f(\zeta,t) = f_t(\zeta)$ is continuously differentiable in a neighbourhood of $\overline{\mathbb{D}} \times (-\varepsilon,\varepsilon)$ and such that

(1) $$\mathrm{Re}[\dot{f}(\zeta,t) \cdot \overline{\zeta f'(\zeta,t)}] = 1$$

holds for $\zeta \in \partial\mathbb{D}$, $t \in (-\varepsilon,\varepsilon)$.

The requirement $f_t \in \mathcal{O}_1$ means that the mapping function f_t shall be normalized ($f_t(0) = 0$, $f_t'(0) > 0$), analytically extendible across $\partial\mathbb{D}$ and locally univalent on $\overline{\mathbb{D}}$ ($f_t' \neq 0$ on $\overline{\mathbb{D}}$). Since f_t originally appeared as a mapping function it is natural to require it to be univalent on $\mathbb{D}$ (or $\overline{\mathbb{D}}$). However, in the mathematical treatment of (1) it makes no difference whether f_t is univalent on $\overline{\mathbb{D}}$ or merely locally univalent and the latter condition being simpler to work with, we have preferred to use that one. Observe also that if f_0 actually is univalent on $\overline{\mathbb{D}}$ and $t \to f_t$ solves (1), then if $\varepsilon > 0$ is chosen small enough also all f_t are univalent on $\overline{\mathbb{D}}$.

It was shown in [12] that problem (1) always has a solution, and that the solution is unique **/ if it is required to be analytic in t (i.e. be expressible as a convergent power series also in t). The proof of this is complicated and consists of converting the problem into a system of integral equations which is solved by an iterative method. Here we are

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*/From now on we choose $Q = 2\pi$ in eq. II.40.

**/i.e. any two solutions coincide for small t .

going to give an elementary proof of the existence and unicity of solutions to (1) in the case that f_0 is a polynomial or a rational function. It turns out that the differential equation (1) then reduces to a finite dimensional system of ordinary differential equations (in t) which has a unique solution by standard theory. This solution f_t is in turn a polynomial or a rational function of the same kind as f_0.

b) Reformulation of the equation

Our first step will be to solve equation (1) for $\dot{f}$.

Proposition 1: Given $f \in \mathcal{O}_1$ the equation

(2) $$\mathrm{Re}[\overline{\zeta f'(\zeta)} \cdot g(\zeta)] = 1 \qquad \text{for } \zeta \in \partial\mathbb{D}$$

has a unique solution g in $H(\overline{\mathbb{D}})_1$. This solution is given by

(3) $$g = \mathcal{F}(f)$$

where $\mathcal{F}: \mathcal{O}_1 \to H(\overline{\mathbb{D}})_1$ is the operator defined by

(4) $$\mathcal{F}(f)(\zeta) = \zeta f'(\zeta) \cdot \frac{1}{2\pi i} \int_{\partial\mathbb{D}} |f'(z)|^{-2} \frac{z+\zeta}{z-\zeta} \frac{dz}{z} \; .$$

Proof: We transform the problem into a statement about two other analytic functions, F and G, related to f and g as follows.

(5), (6) $$\begin{cases} F(\zeta) = f'(\zeta) \\ G(\zeta) = \dfrac{g(\zeta)}{\zeta} \end{cases}$$

and, conversely,

(7), (8) $$\begin{cases} f(\zeta) = \displaystyle\int_0^{\zeta} F(z)dz \\ g(\zeta) = \zeta G(\zeta) \; . \end{cases}$$

Then the first statement in the proposition transforms into:

Given $F \in H(\bar{\mathbb{D}})$, non-vanishing on $\bar{\mathbb{D}}$ and with $F(0) > 0$, the equation

(9) $\quad \mathrm{Re}[\bar{F} \cdot G] = 1 \quad$ on $\partial\mathbb{D}$

has a unique solution G in $H(\bar{\mathbb{D}})$ satisfying $\mathrm{Im}G(0) = 0$.

On dividing by $|F|^2$ in (9) we get another equaivalent formulation:

Given $F \in H(\bar{\mathbb{D}})$, non-vanishing on $\bar{\mathbb{D}}$ and with $F(0) > 0$, the problem

(10) $\quad \mathrm{Re}[G/F] = |F|^{-2} \quad$ on $\partial\mathbb{D}$

(11) $\quad G/F \in H(\bar{\mathbb{D}})$, $\quad \mathrm{Im}\, G/F(0) = 0$

has a unique solution for G (or for G/F).

Now in this last formulation the statement is directly seen to be true. Namely, the solution for G/F of (10), (11) is

(12) $\quad G/F = P[|F|^{-2}]$

where P stands for "the Poisson integral of" . Explicitly

(13) $$\frac{G(\zeta)}{F(\zeta)} = \frac{1}{2\pi} \int_0^{2\pi} |F(e^{i\theta})|^{-2} \cdot \frac{e^{i\theta} + \zeta}{e^{i\theta} - \zeta}\, d\theta =$$

$$= \frac{1}{2\pi i} \int_{\partial\mathbb{D}} |F(z)|^{-2} \cdot \frac{z + \zeta}{z - \zeta} \frac{dz}{z}$$

Actually (13) only gives G/F as an analytic function in $\mathbb{D}$, but it is easy to see that G/F extends analytically across $\partial\mathbb{D}$ as required in (1). In fact, the function $|F(z)|^{-2}$ in the last integral of (13) can be replaced by $(F(z)F^*(z))^{-1}$, which is holomorphic in a neighbourhood of $\partial\mathbb{D}$, and then the path of integration for that integral can be moved slightly out from $\partial\mathbb{D}$ showing that the last member of (13) is an analytic function of ζ in some neighbourhood of $\bar{\mathbb{D}}$.

Thus the statement (10)-(11) is true and so also the statement (2) of the proposition is true. The expression (4) for the (hence well-defined) operator F follows from (5), (6) and (13). This proves the proposition.

Theorem 2: Let $f \in \mathcal{O}_1$.

(i) Let U be any open connected set containing $\overline{\mathbb{D}}$. Then if $f \in H(U)$ $F(f) \in H(U)$. Thus F is well-defined as an operator

$$F : \mathcal{O}_1 \cap H(U) \to H(U).$$

Moreover, with $H(U)$ provided with the topology of uniform convergence on compact sets, this operator is continuous.

(ii) If f is a polynomial of degree $\leq r$ then so is $F(f)$.

(iii) If f is a rational function with

(14) $$\operatorname{Pdiv} f \leqq \sum_{j=1}^{r} n_j \cdot (\zeta_j) + n_0 \cdot (\infty)$$

where $\zeta_j \in \mathbb{C} \smallsetminus \overline{\mathbb{D}}$, $n_j \geqq 0$ and $n_0 \geqq 1$ then $F(f)$ is a rational function with

(15) $$\operatorname{Pdiv} F(f) \leq \sum_{j=1}^{r} (n_j+1) \cdot (\zeta_j) + n_0 \cdot (\infty).$$

Proof: (ii) is a special case of (iii) so only (i) and (iii) need to be proven. Returning to the functions F and G used in the proof of Proposition 1, the relation

$$g = F(f) \qquad (\text{for } f \in \mathcal{O}_1,\ g \in H(\overline{\mathbb{D}})_1)$$

is equivalent to

$$\operatorname{Re}[\overline{F} \cdot G] = 1 \quad \text{on } \partial\mathbb{D}$$

for F and $G \in H(\overline{\mathbb{D}})$ satisfying $\operatorname{Im}F(0) = \operatorname{Im}G(0) = 0$, F non-vanishing on $\overline{\mathbb{D}}$ and related to f and g by (5) to (8). Apart from the continuity statement in (i), (i) and (iii) now follow from the following lemma.

Lemma 3: Suppose $F, G \in H(\overline{\mathbb{D}})$ satisfy

(16) $\mathrm{Re}[\overline{F} \cdot G] = 1$ on $\partial\mathbb{D}$

and that F has no zero on $\overline{\mathbb{D}}$. Then

(i) if $F \in H(U)$ then $G \in H(U)$ (for $U \supset \overline{\mathbb{D}}$ open and connected)

(ii) if F is a rational function then so is G, and

(17) $\mathrm{Pdiv}\, G \leqq \mathrm{Pdiv}\, F$.

How the theorem (except for the continuity statement) follows from the lemma:

(i) If $f \in H(U)$ then $F \in H(U)$ and, by (i) of the lemma, $G \in H(U)$. Hence $g \in H(U)$ by (8).

(iii) Suppose f is rational with

$$\mathrm{Pdiv}\, f \leqq \sum_{j=1}^{r} n_j \cdot (\zeta_j) + n_0 \cdot (\infty)$$

(where $n_0 \geqq 1$). Then $F = f'$ is rational with

$$\mathrm{Pdiv}\, F \leqq \sum_{j=1}^{r} (n_j+1) \cdot (\zeta_j) + (n_0-1) \cdot (\infty)$$

and it follows from (8) and from (ii) of the lemma that

$$\mathrm{Pdiv}\, g \leqq \mathrm{Pdiv}\, G + 1 \cdot (\infty) \leqq \sum_{j=1}^{r} (n_j+1) \cdot (\zeta_j) + n_0 \cdot (\infty)$$

as claimed.

It remains to prove the lemma and the continuity assertion in the theorem.

Proof of the lemma: Relation (16) can be written (see p. 19f for notations)

(18) $\mathrm{Re}[F^* \cdot G - 1] = 0 \quad \text{on } \partial\mathbb{D}$.

This shows that the function

(19) $H = F^* \cdot G - 1$

which is holomorphic in a neighbourhood of $\partial\mathbb{D}$ extends by reflection to be analytic in a domain which is symmetric with respect to $\partial\mathbb{D}$. In fact, (18) shows that

(20) $H = -H^*$ on $\partial\mathbb{D}$, and hence identically,

so that if H is a priori analytic in (say) V (20) defines an analytic extension of it to $V \cup V^*$.

To prove (i) of the lemma we observe that a priori the function H defined by (19) will be holomorphic in $U^* \cap \overline{\mathbb{D}}$ $(F \in H(U),\ G \in H(\overline{\mathbb{D}}))$. Thus it extends analytically to $(U^* \cap \overline{\mathbb{D}}) \cup (U \cap \overline{\mathbb{D}}^*)$ $(= U \cap U^*$, in view of $U \supset \overline{\mathbb{D}})$, in particular to $U \cap \overline{\mathbb{D}}^*$. Since F^* is holomorphic and has no zeroes in $U \cap \overline{\mathbb{D}}^*$ it follows from (19) that G is holomorphic there. Thus G is holomorphic in $U = (U \cap \overline{\mathbb{D}}^*) \cup \overline{\mathbb{D}}$, and (i) is proven.

To prove (ii) of the lemma observe first that for F a rational function, H defined by (19) is meromorphic in $\overline{\mathbb{D}}$, hence by reflection is meromorphic on all $\mathbb{P}$. This means that H is a rational function. Hence also G is rational (by (19)).

Now (17) follows from the following computation in which the first inequality depends on $Z\,\mathrm{div}_{\mathbb{D}^*} F^* = 0$, the second one on $P\,\mathrm{div}_{\mathbb{D}}\, G = 0$ and where also the symmetry (20) of $H = F \cdot G^* - 1$ is used.

$$\mathrm{P\,div}\,G = \mathrm{P\,div}_{\mathbb{D}^*}G \leqq \mathrm{P\,div}_{\mathbb{D}^*}(F^*\cdot G - 1) = [\mathrm{P\,div}_{\mathbb{D}}(F^*\cdot G - 1)]^* \leqq$$

$$\leqq (\mathrm{P\,div}_{\mathbb{D}}F^*)^* = \mathrm{P\,div}_{\mathbb{D}^*}F = \mathrm{P\,div}\,F\ .$$

This proves Lemma 3.

It remains to prove the continuity of $F : \mathcal{O}_1 \cap H(U) \to H(U)$ for $U \cup \bar{\mathbb{D}}$ open and connected. So suppose $f_n \to f$ uniformly on compact subsets of U $(f_n, f \in \mathcal{O}_1 \cap H(U))$ and we shall prove that $F(f_n) \to F(f)$ in the same topology. It is clear (by the maximum principle) that it is enough to prove that $F(f_n) \to F(f)$ uniformly on every compact subset of U which does not contain any zero of f' (in $U \setminus \bar{D}$).

Let K be such a compact subset. Then we can choose an open connected set V with nice boundary, such that $K \cup \bar{\mathbb{D}} \subset V \subset \bar{V} \subset U$, and such that also $\bar{V}$ avoids all zeroes of f'. Since the function $(f'(z)f'^*(z))^{-1}$ then is holomorphic in a neighbourhood of $\bar{V} \setminus \mathbb{D}$ and equals $|f'(z)|^{-2}$ on $\partial\mathbb{D}$ we have, for $\zeta \in \mathbb{D}$

$$(21)\qquad F(f)(\zeta) = \zeta f'(\zeta)\cdot\frac{1}{2\pi i}\int_{\partial\mathbb{D}}|f'(z)|^{-2}\cdot\frac{z+\zeta}{z-\zeta}\,\frac{dz}{z} =$$

$$= \zeta f'(\zeta)\cdot\frac{1}{2\pi i}\int_{\partial\mathbb{D}}\frac{1}{f'(z)f'^*(z)}\cdot\frac{z+\zeta}{z-\zeta}\,\frac{dz}{z} =$$

$$= \zeta f'(\zeta)\cdot\frac{1}{2\pi i}\int_{\partial V}\frac{1}{f'(z)f'^*(z)}\cdot\frac{z+\zeta}{z-\zeta}\,\frac{dz}{z}\ .$$

Both the first and the last members of this equation are functions (in ζ) holomorphic in V. Thus the equality between these is valid for all $\zeta \in V$.

Formula (21), with equality holding between the extreme members for all $\zeta \in V$, also is valid with f_n in place of f whenever n is large enough. For $f_n \to f$ on compacts implies that f_n' has no zeroes on $\overline{V}$ for n large (since f' has none), and so all that has been said about f above also applies to f_n (for large n).

Thus

$$F(f_n)(\zeta) = \zeta f_n'(\zeta)\frac{1}{2\pi i}\int_{\partial V}\frac{1}{f_n'(z)f_n'^*(z)}\cdot\frac{z+\zeta}{z-\zeta}\cdot\frac{dz}{z} \tag{22}$$

for $\zeta \in V$, n large.

Now $f_n \to f$ uniformly on compacts implies $f_n' \to f'$ uniformly on K and on ∂V, and $f_n'^* \to f'^*$ uniformly on ∂V. Therefore, since $|\frac{z+\zeta}{z-\zeta}|$ is bounded above for $z \in \partial V$, $\zeta \in K$ and $f_n'(z)f_n'^*(z)$ is bounded away from zero for $z \in \partial V$ and n large, (22) and (21) show that $F(f_n)(\zeta) \to F(f)(\zeta)$ uniformly for $\zeta \in K$ as $n \to \infty$.

This proves the continuity of F and finishes the proof of Theorem 2.

c) Existence and uniqueness of rational solutions to the equation

We now apply Theorem 2 to the differential equation $\dot{f} = F(f)$. Given integers $n_0, n_1, \ldots, n_r \geq 1$ let

$n = (n_0, n_1, \ldots, n_r) \in \mathbb{Z}^{r+1}$,

$|n| = n_0 + n_1 + \ldots + n_r$,

R_n = {rational functions f which have r distinct poles $\zeta_1, \ldots, \zeta_r$ (depending on f) in $\mathbb{C}$ of orders exactly $n_1, \ldots, n_r$ respectively, a pole of order at most n_0 at ∞ and no other poles}.

Thus $f \in R_n$ means that there exist $\zeta_j = \zeta_j(f) \in \mathbb{C}$ $(j=1,\ldots,r)$, $a_{jk} = a_{jk}(f) \in \mathbb{C}$ $(k=1,\ldots,n_j,\ j=1,\ldots,r)$ and $a_k = a_k(f) \in \mathbb{C}$ $(k=0,\ldots,n_0)$ with $\zeta_i \neq \zeta_j$ for $i \neq j$ and with $a_{jn_j} \neq 0$ $(j=1,\ldots,r)$ such that

(23) $$f(\zeta) = \sum_{j=1}^{r} \sum_{k=1}^{n_j} \frac{a_{jk}}{(\zeta-\zeta_j)^k} + \sum_{k=0}^{n_0} a_k \zeta^k \quad .$$

For $r=0$, $n=(n_0)$ $\mathcal{R}_n$ reduces to

$$\mathcal{R}_n = \mathcal{P}_{n_0} = \{\text{polynomials of degree} \leq n_0\} \quad .$$

With (ζ_j, a_{jk}, a_k) in (23) as local coordinates $\mathcal{R}_n$ is given the structure of a complex differentiable manifold of dimension $r + \sum_1^r n_j + n_0 + 1 = |n+1|$. We shall regard $\mathcal{R}_n$ as a real manifold, hence of dimension $2|n+1|$. Then $\mathcal{R}_n \cap \mathcal{O}_1$ and $\mathcal{R}_n \cap H(\overline{\mathbb{D}})_1$ are submanifolds of $\mathcal{R}_n$ of dimension $2|n+1| - 3$ (the conditions $f(0)=0$ and $\operatorname{Im} f'(0)=0$ reduce the dimension by 3).

Now we may consider the operator $F: \mathcal{O}_1 \to H(\overline{\mathbb{D}})_1$ as a vector field on $\mathcal{O}_1$ (the tangent space of $\mathcal{O}_1$ at any point $f \in \mathcal{O}_1$ may be identified with $H(\overline{\mathbb{D}})_1$ in a natural way) and the content of part (iii) of Theorem 2 then is that the restriction of this vector field to the submanifold $\mathcal{O}_1 \cap \mathcal{R}_n$ is tangent to $\mathcal{O}_1 \cap \mathcal{R}_n$. (We shall motivate this in a moment.) Thus $F|_{\mathcal{O}_1 \cap \mathcal{R}_n}$ may be considered as a vector field on (along) $\mathcal{O}_1 \cap \mathcal{R}_n$.

Moreover, this vector field is very smooth as can easily be seen from, say, (4). Now a smooth vector field on a finite dimensional manifold always admits a unique integral curve through any point on the manifold and so it follows that given $f_0 \in \mathcal{O}_1 \cap \mathcal{R}_n$ there is a unique smooth map $t \to f_t \in \mathcal{O}_1 \cap \mathcal{R}_n$ defined in some interval around $t=0$ and satisfying $\dot{f}_t = F(f_t)$. This is roughly the proof of Theorem 4 below, asserting existence and uniqueness of rational solutions of $\dot{f}_t = F(f_t)$, given f_0 rational.

In order to work out the details of the above discussion consider an arbitrary differentiable curve

(24) $t \to f_t$

in R_n. With

(25) $$f_t(\zeta) = \sum_{j=1}^{r} \sum_{k=1}^{n_j} \frac{a_{jk}(t)}{(\zeta-\zeta_j(t))^k} + \sum_{k=0}^{n_0} a_k(t)\zeta^k$$

the tangent vector of this curve at the point f_t becomes

(26) $$\dot{f}_t(\zeta) = \sum_{j=1}^{r} \sum_{k=1}^{n_j} \frac{\dot{a}_{jk}(t)}{(\zeta-\zeta_j(t))^k} + \sum_{j=1}^{r} \sum_{k=1}^{n_j} \frac{ka_{jk}(t)\dot{\zeta}_j(t)}{(\zeta-\zeta_j(t))^{k+1}} + \sum_{k=0}^{n_0} \dot{a}_k(t)\zeta^k =$$

$$= \sum_{j=1}^{r} \frac{n_j a_{jn_j}(t)\dot{\zeta}_j(t)}{(\zeta-\zeta_j(t))^{n_j+1}} + \sum_{j=1}^{r} \sum_{k=1}^{n_j} \frac{(k-1)a_{jk-1}(t)\dot{\zeta}_j(t) + \dot{a}_{jk}(t)}{(\zeta-\zeta_j(t))^k} +$$

$$+ \sum_{k=0}^{n_0} \dot{a}_k(t)\zeta^k \quad .$$

Now for any fixed $f \in R_n$ consider all curves with $f_0=f$. As (24) varies over all such curves the derivatives $\dot{\zeta}_j(0)$, $\dot{a}_{jk}(0)$, $\dot{a}_k(0)$ range over all $\mathbb{C}^r \times \mathbb{C}^{\sum_1^r n_j} \times \mathbb{C}^{n_0+1} = \mathbb{C}^{|n+1|}$ and it follows from the last member of (26) (observing that $n_j a_{jn_j}(t) \neq 0$ there) that the tangent vector $\dot{f}_0$ then ranges over all

(27) $$T_f(R_n) = \{\text{rational functions } g \text{ with } \operatorname{Pdiv} g \leq \sum_{j=1}^{r} (n_j+1)\cdot(\zeta_j) + n_0\cdot(\infty)\}$$

($\zeta_j=\zeta_j(0)$ are the poles of $f=f_0$). This means that the linear space $T_f(R_n)$ is the tangent space of R_n at $f \in R_n$ (whence the notation for it).

Considering $\mathcal{R}_n \cap \mathcal{O}_1$ instead of $\mathcal{R}_n$ it is easy to see that the tangent space of $\mathcal{R}_n \cap \mathcal{O}_1$ at $f \in \mathcal{R}_n \cap \mathcal{O}_1$ is

(28) $$T_f(\mathcal{R}_n \cap \mathcal{O}_1) = \{\text{rational functions } g \text{ with } \operatorname{Pdiv} g \leqq \sum_{j=1}^{r} (n_j+1)\cdot(\zeta_j) + n_0\cdot(\infty)$$

$$\text{and with } g(0) = 0,\ \operatorname{Im} g(0) = 0\,\}$$

$$= T_f(\mathcal{R}_n) \cap H(\overline{\mathbb{D}})_1$$

$(\zeta_1,\dots,\zeta_r,\infty$ are the poles of f).

In terms of the above notation, part (iii) of Theorem 2 (together with the fact that $\mathcal{F}(f) \in H(\overline{\mathbb{D}})_1$) say that

(29) $$\mathcal{F}(f) \in T_f(\mathcal{R}_n \cap \mathcal{O}_1) \qquad \text{for} \quad f \in \mathcal{R}_n \cap \mathcal{O}_1 \;.$$

Thus $\mathcal{F}(f)$ is tangent to $\mathcal{R}_n \cap \mathcal{O}_1$ for $f \in \mathcal{R}_n \cap \mathcal{O}_1$, i.e. $\mathcal{F}|_{\mathcal{R}_n \cap \mathcal{O}_1}$ is a smooth vector field on $\mathcal{R}_n \cap \mathcal{O}_1$, and so the problem

(30) $$\begin{cases} \dot{f}_t = \mathcal{F}(f_t) \\ f_0 \in \mathcal{R}_n \cap \mathcal{O}_1 \ \text{ given} \end{cases}$$

has a unique smooth solution

(31) $$t \to f_t \in \mathcal{R}_n \cap \mathcal{O}_1$$

defined in a neighbourhood of $t = 0$.

We have now proved

<u>Theorem 4</u>: Given any rational function f which is holomorphic and locally univalent on $\overline{\mathbb{D}}$ and satisfies $f(0) = 0$, $f'(0) > 0$, choose $n = (n_0, n_1, \dots, n_r)$

with $n_j \geq 1$ so that $f \in \mathcal{R}_n$ (i.e. $n_1,\dots,n_r$ shall be some enumeration of the exact orders of the finite poles of f and n_0 shall be greater or equal to the order of the pole of f at infinity). Then the problem

(45) $$\begin{cases} \dot{f}_t = F(f_t) \\ f_0 = f \end{cases}$$

or, equivalently, the problem (1) has a unique solution

(46) $$t \to f_t \in \mathcal{R}_n \cap \mathcal{O}_1$$

defined in a neighbourhood of $t = 0$.

d) <u>The moment property of solutions</u>

The next theorem shows the existence of an infinite number of simple constants at motion for a solution $t \to f_t$ of our differential equation, namely the analytic moments

(48) $$c_n = \iint_{\Omega_t} z^n dxdy = \iint_{\mathbb{D}} f(\zeta)^n |f'(\zeta)|^2 d\xi d\eta \qquad (\zeta = \xi + i\eta)$$

for n=1,2,... . Here $\Omega_t = f_t(\mathbb{D})$, which is regarded as a Riemann surface over $\mathbb{C}$ if f_t is not (globally) univalent on $\mathbb{D}$. The zeroth order moment

(49) $$c_0 = \iint_{\Omega_t} dxdy = |\Omega_t|$$

will increase linerly with t, This moment property of solutions of (III.1) was discovered by Richardson ([9]).

Since the c_n are linear in z^n we also obtain constants of motion by taking linear combinations of the z^n. Thus define, for arbitrary polynomials $P(z)$ and for $f \in \mathcal{O}_1$

(50) $$I_P(f) = \iint_{\Omega} P(z)dxdy = \iint_{\mathbb{D}} P(f(\zeta))|f'(\zeta)|^2 d\xi d\eta$$

where $\Omega = f(\mathbb{D})$ (as a Riemann surface). Then we have

Theorem 5: Suppose $(-\varepsilon,\varepsilon) \ni t \to f_t \in \mathcal{O}_1$ solves (1) (equivalently $\dot{f}_t = F(f_t)$). Then

(51) $$\frac{d}{dt} I_P(f_t) = 2\pi P(0)$$

for each polynomial $P(z)$.

Proof: Let $Q'(z) = P(z)$. Then we have, using the formalism of differential forms (see [5], Section I.3, p19ff e.g.)

$$\frac{d}{dt} 2iI_P(f_t) = \frac{d}{dt} 2i \iint_{\Omega_t} P(z)dxdy = \frac{d}{dt} \iint_{\Omega_t} Q'(z)d\bar{z}dz =$$

$$= \frac{d}{dt} \iint_{\Omega_t} d\bar{z}dQ(z) = \frac{d}{dt} \iint_{\mathbb{D}} d\overline{f_t(\zeta)}dQ(f_t(\zeta)) =$$

$$= \iint_{\mathbb{D}} d\overline{\dot{f}_t(\zeta)}dQ(f_t(\zeta)) + \iint_{\mathbb{D}} d\overline{f_t(\zeta)}d(\frac{d}{dt}Q(f_t(\zeta))) =$$

$$= \int_{\partial\mathbb{D}} \overline{\dot{f}_t(\zeta)}dQ(f_t(\zeta)) - \int_{\partial\mathbb{D}} \frac{d}{dt}(Q(f_t(\zeta)))d\overline{f_t(\zeta)} =$$

$$= \int_{\partial\mathbb{D}} Q'(f_t(\zeta))(\overline{\dot{f}_t(\zeta)}f_t'(\zeta)d\zeta - \dot{f}_t(\zeta)\overline{f_t'(\zeta)d\zeta}) =$$

$$= 2i \int_{\partial\mathbb{D}} P(f_t(\zeta))\,\mathrm{Im}\left[\overline{\dot{f}_t(\zeta)}f_t'(\zeta)d\zeta\right] =$$

$$= 2i \int_{\partial\mathbb{D}} P(f_t(\zeta)) \operatorname{Re}\left[\overline{\dot{f}_t(\zeta)\zeta f_t'(\zeta)}\right] \frac{d\zeta}{i\zeta} =$$

$$= 2i \int_0^{2\pi} P(f_t(e^{i\theta}))d\theta = 2i \cdot 2\pi P(f_t(0)) =$$

$$= 2i \cdot 2\pi P(0) .$$

In the last few lines above we used the fact that along $\partial\mathbb{D}$, $d\zeta = ie^{i\theta}d\theta = i\zeta d\theta \quad (\zeta = e^{i\theta})$.

This proves Theorem 5.

Remark: The proof of Theorem 5 gives hints for possible geometric interpretations of the differential equation

(52) $$\operatorname{Re}\left[\dot{f}(\zeta)\zeta\overline{f'(\zeta)}\right] = 1 \qquad \text{on } \partial\mathbb{D} .$$

Namely, we may extract from the proof the formula

(53) $$\frac{d}{dt}\int_{\Omega_t} P(z)dxdy = \int_{\partial\mathbb{D}} P(f_t(z)) \operatorname{Im}\left[\overline{\dot{f}_t(\zeta)}\; f_t'(\zeta)d\zeta\right]$$

(where $\Omega_t = f_t(\mathbb{D})$). On using the symbol δ to denote variations (or differentials) with respect to t (53) can be written

(54) $$\delta\int_{\Omega} P(z)dxdy = \int_{\partial\mathbb{D}} P(f(\zeta)) \operatorname{Im}\left[\overline{\delta f(\zeta)}\cdot df(\zeta)\right]$$

or, by pulling the right member back to $\partial\Omega$

(55) $$\delta\int_{\Omega} P(z)dxdy = \int_{\partial\Omega} P(z) \operatorname{Im}\left[\delta\bar{z}\cdot dz\right] .$$

Now, δz is the increment up to $\mathcal{O}((\delta t)^2)$ of $f(\zeta)$ for fixed $\zeta \in \partial\mathbb{D}$ as t increases by δt. Since

(56) $\operatorname{Im}[\delta\bar{z}\cdot dz]$ = the area of the parallelogram spanned by δz and dz

(see Fig. 2 below) (55) can be interpreted as saying that $\operatorname{Im}[\delta\bar{z}\cdot dz]$ is that mass distribution on $\partial\Omega$ the density of which is proportional to the local increment of the area of Ω caused by the motion of $\partial\Omega$. Expressed in a careless way

(57) $\operatorname{Im}[\delta\bar{z}\cdot dz] = |\delta\Omega|$ along $\partial\Omega$

($|\ldots|$ = area with sign here).

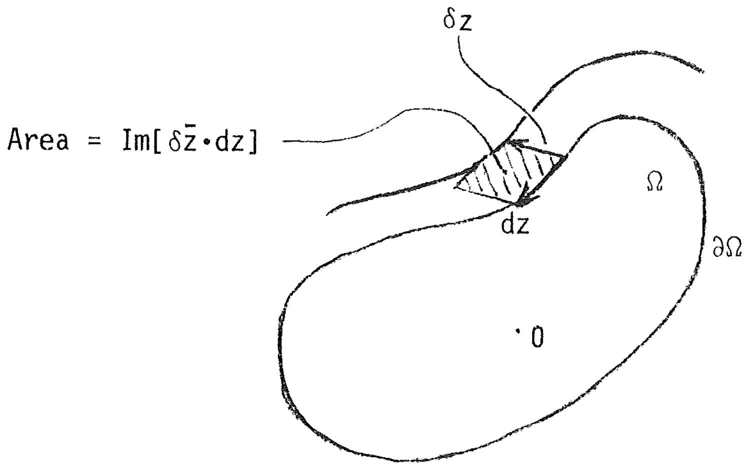

Fig. 2.

Equation (52) can be written

(58) $\mathrm{Im}[\delta\bar{z}\cdot dz] = \delta t\cdot d\theta$

(with $\zeta = e^{i\theta}$, $z = f(\zeta)$) and thus by (57)

(59) $|\delta\Omega| = \delta t\cdot d\theta$.

This is the interpretation of (52) we had in mind. Actually (59) is the same as (II.50).

IV. GENERALIZATIONS AND APPLICATIONS

a) The Generalized Differential Equation

Now we are going to consider a generalization of equation (III.1), obtain similar results for this equation as for (III.1), and apply these results to prove that a certain mapping is non-singular (Theorem 6). The generalized equation is

$$(1) \qquad \mathrm{Re}[\dot{f}(\zeta)\cdot\overline{\zeta f'(\zeta)}] = \begin{cases}\cos n\theta \\ \sin n\theta\end{cases}$$

where $\zeta = e^{i\theta} \in \partial\mathbb{D}$ and $n \in \mathbb{Z}$. For the case with $\cos n\theta$ in the right member and $n = 0$ (1) is the old equation (III.1). Each $n > 0$ in $\mathbb{Z}$ and each choice of $\cos n\theta$ and $\sin n\theta$ (1) gives a new equation. For negative $n \in \mathbb{Z}$ the same set of equations appears again (essentially).

Let us therefore make the following <u>convention</u> in order to be able to speak of the equations (1) conveniently:

Choose the right member of (1) to be
$\cos n\theta$ for $n = 0,1,2,\ldots$.
$\sin n\theta$ for $n = -1,-2,\ldots$.

Thus there is precisely one equation (1) for each $n \in \mathbb{Z}$ and, aside from the zero function and up to a sign, all possible choices of right members in (1) are covered,

(1) can be rewritten as

$$(2) \qquad \mathrm{Im}[\overline{\dot{f}(\zeta)}f'(\zeta)d\zeta] = \mathrm{Re}[\dot{f}(\zeta)\overline{\zeta f'(\zeta)}]d\theta$$

$$= \begin{cases}\cos n\theta\, d\theta & (n \geq 0) \\ \sin n\theta\, d\theta & (n < 0)\end{cases}$$

$$= \begin{cases} \frac{1}{2}(\zeta^n + \zeta^{-n}) \frac{d\zeta}{i\zeta} & (n \geqq 0) \\ \frac{1}{2i}(\zeta^n - \zeta^{-n}) \frac{d\zeta}{i\zeta} & (n < 0) \end{cases}$$

$$= \begin{cases} -\frac{1}{n} \cdot \mathrm{Im}\, d(\zeta^{-n}) & (n > 0) \\ \mathrm{Im}\, d \log \zeta & (n = 0) \\ -\frac{1}{n}\ \mathrm{Re}\, d(\zeta^{-n}) & (n < 0) \end{cases}$$

for $\zeta = e^{i\theta} \in \partial\mathbb{D}$.

Using (III.53) this shows that if $t \to f = f_t \in \mathcal{O}_1$ is a solution of (1), $n \geqq 0$ and $P(z)$ is a polynomial, then

(3) $$\frac{d}{dt} I_P(f) = \int_{\partial\mathbb{D}} P(f(\zeta)) \mathrm{Im}[\overline{\dot{f}(\zeta) f'(\zeta) d\zeta}] = \frac{1}{2i} \int_{\partial\mathbb{D}} P(f(\zeta))(\zeta^n + \zeta^{-n}) \frac{d\zeta}{\zeta} =$$

$$= \begin{cases} 2\pi\ P(f(0)) & (n = 0) \\ \pi \operatorname*{Res}_{\zeta=0} \frac{P(f(\zeta))}{\zeta^{n+1}} & (n > 0) \end{cases}$$

$$= \begin{cases} 2\pi\ P(0) & \\ \pi[A_1(f)P'(0) + \ldots + A_n(f)P^{(n)}(0)] & (n > 0) \end{cases}$$

where $A_j(f)$ $j=1,\ldots,n$ are complex numbers that depend on f and, in particular,

(4) $$A_n(f) = \frac{1}{n!} f'(0)^n \neq 0 .$$

Similarly, for $n<0$

(5) $$\frac{d}{dt} I_P(f) = i\pi \operatorname*{Res}_{\zeta=0} P(f(\zeta))\cdot\zeta^{n-1} = i\pi[A_1(f)P'(0)+\ldots+A_{-n}(f)P^{-n}(0)] \ .$$

Just as on p.22 (Proposition 1) equation (1) can be solved for $\dot{f}$ (uniquely with the requirement $\dot{f}\in H(\bar{\mathbb{D}})_1$) whenever $f\in\mathcal{O}_1$. Namely,

(6) $$\operatorname{Re}\left[\dot{f}(\zeta)\cdot\overline{\zeta f'(\zeta)}\right] = \begin{cases}\cos n\theta & n=0,1,2,\ldots \\ \sin n\theta & n=-1,-1,\ldots\end{cases}$$

for $\zeta=e^{i\theta}\in\partial\mathbb{D}$ is equivalent to

(7) $$\dot{f} = F_n(f)$$

where $F_n:\mathcal{O}_1\to H(\bar{\mathbb{D}})_1$ are the operators defined by

(8) $$F_n(f)(\zeta) = \zeta f'(\zeta)\cdot\frac{1}{2\pi i}\int_{\substack{\partial\mathbb{D}\\(z=e^{i\theta})}} |f'(z)|^{-2}\cdot\begin{Bmatrix}\cos n\theta\\ \sin n\theta\end{Bmatrix}\cdot\frac{z+\zeta}{z-\zeta}\,\frac{dz}{z} =$$

$$= \zeta f'(\zeta)\ P\left[|f'(e^{i\theta})|^{-2}\cdot\begin{Bmatrix}\cos n\theta\\ \sin n\theta\end{Bmatrix}\right](\zeta)$$

($\cos n\theta$ for $n\geq 0$, $\sin n\theta$ for $n<0$).
Thus $F_0 = F$.

Also Theorem 2, Lemma 3 and Theorem 4 have their generalizations to the equation (1) for arbitrary $n\in\mathbb{Z}$. Theorem 2 generalizes to

Theorem 2': Theorem 2 (on p.24) holds true with any F_n $(n\in\mathbb{Z})$ in place of F under the following assumptions:
For (ii) $r>|n|$ $(|n|=\pm n\geq 0)$, and
for (iii) $n_0>|n|$.

Proof: The proof of Theorem 2' is essentially the same as that of Theorem 2 with the role of Lemma 3 now played by the more general Lemma 3' below.

Lemma 3': Suppose $F, G \in H(\overline{\mathbb{D}})$ satisfy

(9) $$\mathrm{Re}[\overline{F} \cdot G] = H \quad \text{on } \partial\mathbb{D}$$

where H is a rational function which is real on $\partial\mathbb{D}$, and suppose that F has no zero on $\overline{\mathbb{D}}$. Then

(i) if $U \supset \overline{\mathbb{D}}$ is open and connected, $F \in H(U)$ implies $G \in H(U)$

(ii) if F is rational then also G is rational, and

(10) $$\mathrm{Pdiv}_{\mathbb{D}^*} G \leqq \max\{\mathrm{Pdiv}_{\mathbb{D}^*} F, \mathrm{Pdiv}_{\mathbb{D}^*} H\} .$$

How the theorem (except for the continuity statement in (i)) follows from the lemma:

We only treat the case $n \geqq 0$, the case $n < 0$ being similar. With

(11) $$F(\zeta) = f'(\zeta)$$

(12) $$G(\zeta) = \frac{\dot{f}(\zeta)}{\zeta} = \frac{F_n(f)(\zeta)}{\zeta}$$

(13) $$H(\zeta) = \frac{1}{2}(\zeta^n + \zeta^{-n})$$

(6) (or (7)) reads

(14) $$\mathrm{Re}[\overline{F} \cdot G] = H \quad \text{on } \partial\mathbb{D}$$

Now (i) of Theorem 2' is proved in exactly the same way as (i) of Theorem 2, with Lemma 3' in place of Lemma 3. (ii) of the theorem is a special case of (iii).

To prove (iii) let f be rational with

$$\mathrm{Pdiv}\, f \leqq \sum_{j=1}^{r} n_j \cdot (\zeta_j) + n_0 \cdot (\infty)$$

where $n_0 > |n|$. Then $F = f'$ is rational with

$$\mathrm{Pdiv}\, F \leq \sum_{j=1}^{r} (n_j+1)\cdot(\zeta_j) + (n_0-1)\cdot(\infty)$$

and since

$$\mathrm{Pdiv}_{\mathbb{D}^*} H = n\cdot(\infty) \leq (n_0-1)\cdot(\infty)$$

(10) shows that

$$\mathrm{Pdiv} F_n(f) \leq \mathrm{Pdiv} G + 1\cdot(\infty) \leq \sum_{j=1}^{r} (n_j+1)\cdot(\zeta_j) + (n_0-1)\cdot(\infty) + 1\cdot(\infty) =$$
$$= \sum_{j=1}^{r} (n_j+1)\cdot(\zeta_j) + n_0\cdot(\infty).$$

This proves (iii) of Theorem 2', and also finishes the proof of that theorem.

Proof of Lemma 3': The relation (9) shows that the function $F^*\cdot G - H$ is purely imaginary on $\partial\mathbb{D}$ and hence extends by reflection to be holomorphic in some region which is symmetric with respect to $\partial\mathbb{D}$. This gives (i) of the lemma exactly as in the proof of Lemma 3 (i). (Observe that H is holomorphic in $\mathbb{C}\setminus\{0\}$.)

Moreover, it is clear (by a reflection argument) that if F is rational then so is G. Now the rest of (ii) follows from the following series of inequalities.

$$\mathrm{Pdiv}\, G = \mathrm{Pdiv}_{\mathbb{D}^*} G \leq \mathrm{Pdiv}_{\mathbb{D}^*} F^*\cdot G \leq$$

$$\leq \max\{\mathrm{Pdiv}_{\mathbb{D}^*}(F^*\cdot G - H),\ \mathrm{Pdiv}_{\mathbb{D}^*} H\} =$$

$$= \max\{[\mathrm{Pdiv}_{\mathbb{D}}(F^*\cdot G - H)]^*,\ \mathrm{Pdiv}_{\mathbb{D}^*} H\} \leq$$

$$\leq \max\{[\mathrm{Pdiv}_{\mathbb{D}} F^*\cdot G]^*,\ [\mathrm{Pdiv}_{\mathbb{D}} H]^*,\ \mathrm{Pdiv}_{\mathbb{D}^*} H\} \leq$$

$$\leq \max\{[\mathrm{Pdiv}_{\mathbb{D}} F^*]^*,\ \mathrm{Pdiv}_{\mathbb{D}^*} H\} =$$

$$= \max\{\mathrm{Pdiv}_{\mathbb{D}^*} F,\ \mathrm{Pdiv}_{\mathbb{D}^*} H\}.$$

b) Non-singularity of the moment mapping.

Now we want to apply Theorem 2' to prove the non-singularity of a certain mapping. For $f \in \mathcal{O}_1$, $n \geq 0$ define

(15) $$c_n(f) = \iint_\Omega z^n dxdy = \frac{1}{2i} \iint_\Omega z^n d\bar{z}dz = \iint f^n \cdot |f'|^2 d\xi d\eta$$

where $\Omega = f(\mathbb{D})$ and where Ω is regarded as a Riemann surface over $\mathbb{C}$ in the first two integrals above if f is not globally univalent. The numbers $c_0, c_1, c_2, \ldots$ are called the complex moments*/ of Ω or of f. The map

(16) $$f \to (c_0, c_1, c_2, \ldots \quad)$$

has attracted some attention in recent years. For example, H.S. Shapiro raised the question ([2], Problem 1, p.193) whether the map (16), defined on the set of univalent functions mapping $\mathbb{D}$ onto Jordan domains, was one-to-one. Shapiro conjectured that the answer was "no", and this was confirmed in 1978 by M. Sakai ([10]), who constructed two different Jordan domains having the same set of moments c_n.

Here we shall prove a modest result in the other direction, namely that when restricted to the set of locally univalent polynomials of any given degree the map (16) is at least locally one-to-one (Theorem 6 below).

Recall that P_N denotes the set (or linear space) of polynomials of degree $\leq N$. It is easy to see (by a computation) that for $f \in \mathcal{O}_1 \cap P_N$

(17) $$c_n(f) = 0 \quad \text{for } n \geq N.$$

Conversely (but somewhat deeper), if (17) holds for some $f \in \mathcal{O}_1$ which is univalent on $\overline{\mathbb{D}}$ then $f \in P_N$. (See [1] where the result on p. 16 easily implies the assertion above.)

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*/ or "analytic moments".

By (17) only the moments $c_0(f),\dots,c_{N-1}(f)$ are of interest for $f \in \mathcal{O}_1 \cap \mathcal{P}_N$. Thus we consider the map

(18) $$J : f \to (c_0(f),\dots,c_{N-1}(f))$$

for $f \in \mathcal{O}_1 \cap \mathcal{P}_N$. Since $c_0(f) = |\Omega| \geq 0$, hence is real, we may consider J as a map

(19) $$J : \mathcal{O}_1 \cap \mathcal{P}_N \to V_N$$

where $V_N = \mathbb{R} \times \mathbb{C}^{N-1}$. Notice that

(20) $$\dim_{\mathbb{R}} V_N = \dim_{\mathbb{R}} \mathcal{O}_1 \cap \mathcal{P}_N = 2N - 1$$

(V_N is a linear space, $\mathcal{O}_1 \cap \mathcal{P}_N$ is an open subset of a linear space).

Clearly J enjoys all kinds of regularity properties one may wish for (e.g. it is real analytic). Now we have

Theorem 6: The Fréchet derivative of J is everywhere non-singular. Hence J is a local diffeomorphism.

Remark: C. Ullemar has proved special cases of Theorem 6. Namely, when $N = 3,4$ or 5 she proves that the restriction of J to polynomials $f \in \mathcal{O}_1 \cap \mathcal{P}_N$ with real coefficients is locally one-to-one ([11] p. 14-16). She also conjectures an expression for the Jacobian of J for arbitrary N (p.16 in [11]), and for $N = 3$ she proves that J is globally one-to-one on the set of those $f \in \mathcal{P}_N$ which are univalent on $\mathbb{D}$ and have real coefficients (p.17-23 in [11]).

Proof: Observe first that

(21) $$c_n(f) = I_P(f)$$

with $P(z) = z^n$, where I_P was defined on p.33. If $t \to f_t \in \mathcal{O}_1 \cap \mathcal{P}_N$ is any differentiable curve with $f_0 = f$ given, then

(22) $$\frac{d}{dt} I_P(f_t) = \int_{\partial \mathbb{D}} P(f_t(\zeta)) \operatorname{Im}\left[\overline{\dot{f}_t(\zeta)} f_t'(\zeta) d\zeta\right]$$

according to (III.53). Thus

(23) $$\frac{d}{dt} c_n(f_t) = \int_{\partial \mathbb{D}} f_t(\zeta)^n \cdot \operatorname{Im}\left[\overline{\dot{f}_t(\zeta)} f_t'(\zeta) d\zeta\right]$$

(n=0,1,..., N-1), and

(24) $$\frac{d}{dt} J(f_t) = \left(\int_{\partial \mathbb{D}} f_t(\zeta)^n \cdot \operatorname{Im}\left[\overline{\dot{f}_t(\zeta)} f_t'(\zeta) d\zeta\right]\right)_{n=0}^{N-1} =$$

$$= \left(\int_{\partial \mathbb{D}} \operatorname{Im}\left[\overline{\dot{f}}_t df_t\right], \ldots, \int_{\partial \mathbb{D}} f_t^{N-1} \operatorname{Im}\left[\overline{\dot{f}}_t df_t\right]\right) .$$

As $t \to f_t \in \mathcal{O}_1 \cap \mathcal{P}_N$ traces through all curves with $f_0 = f$ the tangent vector $\dot{f}_0$ at f traces through all of

(25) $$(\mathcal{P}_N)_1 = \{h \in \mathcal{P}_N : h(0) = 0 , \operatorname{Im} h'(0) = 0 \}.$$

In other words $(\mathcal{P}_N)_1$ is the tangent space of $\mathcal{O}_1 \cap \mathcal{P}_N$ at the point f.

Now (24) shows that the Frĕchet derivative of J at $f \in \mathcal{O}_1 \cap \mathcal{P}_N$ is the linear (over $\mathbb{R}$) map.

(26) $$dJ_f : (\mathcal{P}_N)_1 \to V_N$$

defined by

(27) $$dJ_f(h) = \left(\int_{\partial \mathbb{D}} f(\zeta)^n \cdot \operatorname{Im}\left[\overline{h(\zeta)} f'(\zeta) d\zeta\right]\right)_{n=0}^{N-1} =$$

$$= \left(\int_{\partial \mathbb{D}} \operatorname{Im}\left[\overline{h} df\right], \ldots, \int_{\partial \mathbb{D}} f^{N-1} \operatorname{Im}\left[\overline{h} df\right]\right) .$$

We have to show that this linear map is non-singular. Since the domain and range spaces have the same dimension ($= 2N-1$ over $\mathbb{R}$) is is enough to show that dJ_f is surjective. For that purpose we shall make use of the operators F_n defined on p. 39.

Namely, for $m=-N+1,\dots,0,1,\dots,N-1$ choose $h=h_m=F_m(f)$ in (27) (Observe that $F_m(f)\in(P_N)_1$ for $|m|<N$ by Theorem 2'.) Then, by the definition of F_m,

$$\mathrm{Im}\left[\overline{h_m(\zeta)}f'(\zeta)d\zeta\right] = \begin{cases}\cos m\theta\ d\theta & (m\geq 0)\\ \sin m\theta\ d\theta & (m<0)\end{cases}$$

$$= \begin{cases}\frac{1}{2}(\zeta^m+\zeta^{-m})\ \frac{d\zeta}{i\zeta} & (m\geq 0)\\ \\ \frac{1}{2i}(\zeta^m-\zeta^{-m})\ \frac{d\zeta}{i\zeta} & (m<0)\end{cases}$$

as in (2). This shows that the n:th component ($n=0,\dots,N-1$) of $dJ_f\ (h_m)$ for $m>0$ is

$$(dJ_f(h_m))_n = \int_{\partial\mathbf{D}} f(\zeta)^n\mathrm{Im}\left[\overline{h_m(\zeta)}f'(\zeta)d\zeta\right] = \frac{1}{2i}\int_{\partial\mathbf{D}} f(\zeta)^n(\zeta^m+\zeta^{-m})\ \frac{d\zeta}{\zeta} =$$

$$= \pi\ \operatorname*{Res}_{\zeta=0}\ \frac{f(\zeta)^n}{\zeta^m}\,\frac{d\zeta}{\zeta} =$$

$$= \begin{cases}* & \text{for } n<m\\ \\ \pi f'(0)^m & n=m\\ \\ 0 & n>m\end{cases}$$

Here, and in the sequel, $*$ stands for complex numbers whose values are unimportant for us. For $m=0$ we obtain

$$(dJ_f(h_m))_n = \begin{cases}2\pi & \text{for } n=0\\ 0 & \text{for } n>0\end{cases}$$

and for $m<0$

$$(dJ_f(h_m))_n = \frac{1}{2i} \int_{\partial\mathbb{D}} f(\zeta)^n (\zeta^m - \zeta^{-m}) \frac{d\zeta}{i\zeta} = -i\pi \operatorname*{Res}_{\zeta=0} \frac{f(\zeta)^n}{\zeta^{|m|}} \cdot \frac{d\zeta}{\zeta} =$$

$$= \begin{cases} * & \text{for } n < |m| \\ -i\pi f'(0)^{|m|} & n = |m| \\ 0 & n > |m| \end{cases}$$

In summary, the range of dJ_f contains the vectors

$(2\pi, 0, \ldots \qquad 0)$	$(m = 0)$
$(*, \ldots, *, \quad \pi f'(0)^m, 0, \ldots, 0)$	$(m = 1, \ldots, N-1)$
$(*, \ldots, *, -i\pi f'(0)^{\|m\|}, 0, \ldots, 0)$	$(m = -1, \ldots, -N+1)$

↑ component 0 ↑ $|m|$ ↑ $N-1$

Since $f'(0) \neq 0$ these vectors span $V_N = \mathbb{R} \times \mathbb{C}^{N-1}$ over $\mathbb{R}$. Thus dJ_f is surjective, hence non-singular and so Theorem 6 is proven.

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