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APPLICATIONS OF VARIATIONAL INEQUALITIES TO A MOVING BOUNDARY PROBLEM FOR HELE SHAW FLOWS

by

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I. INTRODUCTION

a. General introduction

The aim of the present paper is to prove a global existence and uniqueness theorem for a kind of weak solution to a moving boundary problem (MBP) arising in two-dimensional Hele Shaw flows. This problem was introduced by S. Richardson in [17] . Our main results are summarized in Theorem 13 (p.69). We also give some applications to quadrature domains (p.78f).

Our MBP occurs as a specific example in a class of MBP:s which can be described as follows:

> Let there to each region D in some suitable class of regions in $\mathbb{R}^n$ be associated (by some rule to be specified) a function g_D defined, harmonic and positive in a neighbourhood in D of the boundary ∂D, taking the boundary value zero on ∂D. Then, given an initial domain $D = D_0$ the problem it to find a family of domains $t \to D_t$ for $t \geq 0$ such that ∂D_t moves with the velocity $-(\text{grad } g_{D_t})|_{\partial D_t}$ (it is assumed here that grad g_{D_t} has a continuous extension to ∂D_t).
>
> More generally the role of the boundary ∂D above can be played by some distinguished subset S of it, so that only S_t moves while $\partial D_t \setminus S_t$ remains fixed.

This defines the class of MBP:s which we have in mind. In order to specify a single problem in it one has to specify three things: the class of regions D under consideration, the rule which associates g_D with D and the subset $S \subset \partial D$ for each D.

Let us at once notice that a family $t \to D_t$ solving a MBP of the above kind necessarily is an increasing family ($D_\tau \subset D_t$ for $\tau < t$) since the vector field $-\text{grad } g_{D_t}$ points perpendicularly out from D_t at ∂D_t (or S_t).

We shall give three examples in the class of MBP:s just described, the third of which is the one which shall be treated in this paper.

Ex. 1: Let K be a fixed compact set in $\mathbb{R}^2$ or $\mathbb{R}^3$ and let, whenever D is in the class of bounded domains containing K, g_D be that harmonic function in $D \setminus K$ which takes the boundary values

$$g_D = \begin{cases} 1 & \text{on } K \\ 0 & \text{on } \partial D \end{cases} ,$$

and let $S = \partial D$.

The MBP so arising occurs in electrochemistry, in which case K is a cathode, $D \setminus K$ is an electrolyte and $\mathbb{R}^n \setminus D$ is an anode, the surface of which gradually dissolves under electrolysis so that D grows according to the rule described above. Here $-g_D$ is the electric potential and $\operatorname{grad} g_D$ is the electric current (up to a positive factor). See [7] for details.

Ex. 2: Let L be a subspace of $\mathbb{R}^2$ (or $\mathbb{R}^3$) of codimension one, let H be one of the half-spaces in which it separates $\mathbb{R}^2$ (or $\mathbb{R}^3$) (think of L as being horizontal and H as being the lower half-space) and consider the class of subdomains D of H such that $\partial D = L \cup S$ for some sufficiently smooth curve (or surface respectively) S in H. Let h be a fixed positive function on L sufficiently small at infinity (typically $h(x) = ae^{-bx^2}$, $a,b > 0$ if $L = \mathbb{R}$) and define g_D to be that harmonic function in D which takes the boundary values

$$g_D = \begin{cases} h & \text{on } L \\ 0 & \text{on } S \end{cases} .$$

This model is suggested in [16] to describe the growth of a propagating water zone (the region D) penetrating a rock (H) when the surface (L) of the latter is impinged on by a heavy water jet. This example is an instance of porous medium flow governed by Darcy's law. g_D is proportional to the hydrostatic pressure of the water and $-\operatorname{grad} g_D$ is proportional to the flow velocity.

Ex. 3: Let D range over all bounded domains in $\mathbb{R}^2$ containing the origin, and let, for such D, g_D be the Green's function for D with respect to $0 \in D$, i.e.

$$g_D(z) = -\log|z| + \text{harmonic} \qquad \text{in } D\,,$$
$$g_D(z) = 0 \qquad \text{on } \partial D\,.$$

Here we have identified $\mathbb{R}^2$ with $\mathbb{C}$ and used $z = x+iy$ as variable. $S = \partial D$ in this example.

This is the MBP to which the present paper is devoted. It occurs in two-dimensional Hele Shaw flows with free boundaries and with a source at the origin.

More precisely, this means the following: Let two large surfaces be lying parallel to each other and to the $\mathbb{R}^2$-plane and at a small distance from each other. The space between the surfaces is to be partly occupied by a fluid (corresponding to the region D) which might well have high viscosity but which is supposed to be Newtonian and incompressible. At the point corresponding to the origin in $\mathbb{R}^2$ fluid is injected at a constant and moderate rate. Then the region of fluid D will grow according to the mathematical model described.

The function g_D is here proportional to the hydrostatic pressure of the fluid (the pressure turns out to be constant in the direction perpendicular to the $\mathbb{R}^2$-plane), and according to the so-called Hele Shaw equation $-\operatorname{grad} g_D$ is proportional to the fluid velocity, or more correctly to the average of it across the gap. This gives the moving boundary condition (that ∂D moves with the velocity $-\operatorname{grad} g_D$, up to a constant factor).

The assumption that the fluid is incompressible implies that g_D is a harmonic function ($\operatorname{div}(-\operatorname{grad} g_D) = 0$) and the injection of fluid gives a logarithmic singularity at the origin for g_D. Further, g_D takes a constant value on ∂D (a constant which can be chosen to be zero) because the pressure outside the fluid is constant (the air pressure) and because the pressure drop across ∂D due to the surface tension will be approximately constant along ∂D (this presupposes that the curvature of ∂D is not too high). This motivates that g_D equals the Green's function for D with respect to the origin.

For more details and technical background to this example, see [17] .

Aside from this Hele Shaw interpretation the choice for g_D in Ex. 3 (possibly with several logarithmic singularities) has been used to describe "migration of oil in hydrostatic environment". See [15], in particular Chapter XV, §7.

Another interpretation of the moving boundary conditions in the examples above is that they describe a one-phase Stefan problem with zero specific heat (i.e. $\mathbb{R}^n \setminus D$ is a melting solid held at its melting temperature, taken to be zero, D is the growing liquid phase with temperature g_D and the liquid is supposed to have zero specific heat). See e.g. [3] and p. 26 ff in the present paper.

The MBP to be treated in this paper (Ex. 3 above) was introduced by S. Richardson in [17]. In that paper the moving boundary condition is formulated as a differential equation for the Riemann mapping function from the unit disc onto the domain D_t (identifying $\mathbb{R}^2$ with the complex plane $\mathbb{C}$ and assuming that D_t is simply connected). This differential equation is of a quite uncommon type and no existence or uniqueness proof for solutions to it is given in [17]. (A local existence and a partial uniqueness proof for solutions to the same differential equation seems, however, to have been given in [21].)

Richardson also discovers in [17] a sequence of simple constants of motion (the complex moments, see p. 9) for his differential equation and uses them to obtain a functional equation for the Riemann mapping function.

In the present paper we re-formulate Richardson's MBP in a way which leads us into the theory of variational inequalities. By using this theory we are able to give a global existence and uniqueness proof for a kind of weak solution to the MBP.

The theory of variational inequalities has already proved to be useful in handling MBP:s of various types. Most relevant for us are the works by G. Duvaut and C.M. Elliott. In [5] Duvaut gives a variational inequality formulation for classical two-phase Stefan problems*/ and indicates existence and uniqueness proofs for their solutions. In [6] Elliott outlines variational inequality formulations for problems of our type (our Ex. 1 is treated in [6] §4). Elliott also announces a paper together with V. Janovsky on our Hele Shaw problem (Ex. 3).

Despite the works of Elliott and others there seems so far to exist no complete, detailed and relatively simple treatment of the Hele Shaw MBP (Ex. 3) from a purely mathematical point of view**/. The present paper is intended to partly fill this gap. Probably our paper and the announced paper by Elliott - Janovsky will complement each other rather than completely overlap since Elliott seems to work along somewhat different lines than we and since the work of Elliott seems to be more directed towards applications and numerical questions while the present work is completely theoretical.

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*/Here "classical" stands for the requirement that both phases shall have strictly positive specific heat. When our problem is interpreted as a Stefan problem the specific heat is zero (cf. p. 4 and 26 ff).

**/Recently there has appeared the work [19] of M. Sakai which contains a detailed theoretical study of the Hele Shaw problem. See p. 13 ff.

b. A survey of the contents

This paper consists of four sections of which the present introduction is Section I.

In Section II we formulate a number of properties, (A) to (F), of one-parameter families $t \to D_t$ of domains in $\mathbb{R}^2$ which in various ways are to express our moving boundary condition.

The first of these, (A), expresses that $t \to D_t$ solves our MBP in a classical sense, i.e. that ∂D_t moves with the velocity $-\operatorname{grad} g_{D_t}$. This formulation is not very flexible because it requires the boundary ∂D_t to be rather smooth and also involves a parametrization of it. Thus, this formulation cannot cover occasions at which D_t changes connectivity.

(B) is a weakened and somewhat more flexible way of formulating the moving boundary condition. It does not require (directly at least) a parametrization of ∂D_t, although it involves a contour integral along ∂D_t.

We then arrive at our concept of weak solution, (C). In (C) we have freed ourselves from all regularity requirements for the boundary. Further, (C) makes no explicit reference to the Green's function.

Let us indicate more closely what (C) is. It can rather easily be seen that the moving boundary condition can be expressed in distribution language by the equation

(1) $$\frac{\partial}{\partial t}\chi_{D_t} = \Delta g_{D_t} + 2\pi\delta_0$$

Here χ_{D_t} is the characteristic function of D_t, g_{D_t} is the Green's function for D_t with respect to the origin, extended to all $\mathbb{R}^2$ by defining it to be zero outside D_t, and δ_0 is the Dirac measure at the origin. These functions are considered as distributions on $\mathbb{R}^2$. Actually, (1) is nothing else than the condition (B) formulated within distribution language. Also (1) can be given a direct physical interpretation. See p. 26 ff.

By integrating (1) with respect to t and putting*/

(2) $$u_t = \int_0^t g_{D_\tau}\, d\tau$$

one obtains

(3) $$\chi_{D_t} - \chi_{D_0} = \Delta u_t + 2\pi t \cdot \delta_0$$

Thus, (1) is equivalent to (2) and (3) together. Since $g_{D_t} \geq 0$ and a solution $t \to D_t$ of (2), (3) must be increasing u_t must also satisfy

(4) $u_t \geq 0$ (for $t \geq 0$) and

(5) $u_t = 0$ outside D_t

Now (C) is essentially (3) combined with (4) and (5), i.e. we say that $t \to D_t$ (for $t \geq 0$) is a weak solution if for each $t \geq 0$ there exists a distribution u_t on $\mathbb{R}^2$ which satisfies (3) - (5). The conditions (4) and (5) replace the coupling (2) of u_t to the Green's function, and it will be seen that these conditions are strong enough to guarantee uniqueness for solutions of (3) - (5) (given D_0).

Observe the remarkable feature of the transformation leading from (1) to (3) - (5) of having liberated the problem from any essential dependence on t; the concept of weak solution is a concept which has an independent meaning for each fixed t, and a weak solution can always be found for any particular value of t without bothering about the solution for any other value of t.

*/The subscript t in u_t is used to indicate that u_t is a function on $\mathbb{R}^2$ which depends on t (i.e. t is a parameter). We never use subscripts to denote partial derivatives in this paper.

In Sections II and III we want to work in a Sobolev space $(H_0^1(\Omega))$ on a bounded domain $\Omega \subset \mathbb{R}^2$ (a sufficiently large disc). Therefore, condition (C) is not formulated exactly as above but we have been forced to make some minor modifications of it. Thus, the Dirac measure in (3) is replaced by an approximation to it $(\frac{1}{|\mathbb{D}_r|}\chi_{\mathbb{D}_r}$, $r>0$ small), we have to require that the D_t:s are relatively compact in Ω for all t under consideration and this t-set has to be a bounded set ($[0,T]$, $0 < T < \infty$). It is also required that $u_t \in H_0^1(\Omega)$ for all t.

Next to condition (C) come conditions (D1) and (D2) which express the moving boundary condition as a series of so-called linear complementarity problems (LCP:s), one for each t. In these problems the explicit occurrence of the domains D_t has disappeared. (D1) is obtained from (3) - (5) by replacing the occurrence of D_t in (3) by the inequality resulting from $\chi_{D_t} \leq 1$. Thus (D1) is the problem of finding u_t, defined on Ω with boundary value zero on $\partial\Omega$ such that

$$(6)\qquad \begin{cases} \Delta u_t + \Delta\psi_t \leq 0 \\ u_t \geq 0 \\ u_t \cdot (\Delta u_t + \Delta\psi_t) = 0 \end{cases}$$

Here ψ_t is the solution - essentially */ - of

$$(7)\qquad \begin{cases} \Delta\psi_t = \chi_{D_0} - 1 + 2\pi t \cdot \delta_0 & \text{in } \Omega \\ \psi_t = 0 & \text{on } \partial\Omega \end{cases}$$

(D2) is the same as (D1) but expressed in terms of the function $v_t = u_t + \psi_t$ instead:

*/ δ_0 in (7) has to be smoothed out. Then $\psi_t \in H_0^1(\Omega)$ and it is required that $u_t \in H_0^1(\Omega)$.

$$(8)\quad \begin{cases} \Delta v_t \leq 0 \\ v_t \geq \psi_t \\ (v_t - \psi_t) \cdot \Delta v_t = 0 \end{cases}$$

Each of the LCP:s (D1) and (D2) is equivalent to a series of variational inequalities, (E1) and (E2) respectively. (E1) is

$$(9)\quad \begin{cases} \text{Find (for each } t\text{) } u_t \in H_0^1(\Omega) \text{ such that } \Delta u_t + \Delta\psi_t \leq 0 \text{ and} \\ \int_\Omega \nabla(u-u_t) \cdot \nabla u_t \geq 0 \quad \text{for all} \quad u \in H_0^1(\Omega) \text{ with } \Delta u + \Delta\psi_t \leq 0 \end{cases},$$

and (E2) is

$$(10)\quad \begin{cases} \text{Find } v_t \in H_0^1(\Omega) \text{ such that } v_t \geq \psi_t \text{ and } \int_\Omega \nabla(v-v_t) \cdot \nabla v_t \geq 0 \text{ for all} \\ v \in H_0^1(\Omega) \text{ with } v \geq \psi_t \,. \end{cases}$$

One very nice property of solutions $t \to D_t$ to our MBP is the existence of an infinite number of simple constants of motion for it, namely the complex moments

$$(11)\quad c_n(D_t) = \int_{D_t} z^n \, dx\, dy \qquad\qquad (z = x + iy)$$

for $n = 1,2,3,\ldots$. The zeroth order moment $c_0(D_t) = |D_t|$ = area of D_t increases linearly with t. For the case that the D_t are simply connected the set (11) of constants of motion may also be expected to be complete, i.e. given D_0 simply connected, a map $t \to D_t$ (satisfying some suitable regularity requirement) should be completely determined by the property that the $c_n(D_t)$ remain constant.

Our final formulation (F) of the moving boundary condition is a strengthened form of this moment property. Namely, we say that $t \to D_t$ satisfies (F) if for each $t \geq 0$

(12) $$\int_{D_t} \varphi \geq 2\pi t \cdot \varphi(0) + \int_{D_0} \varphi$$

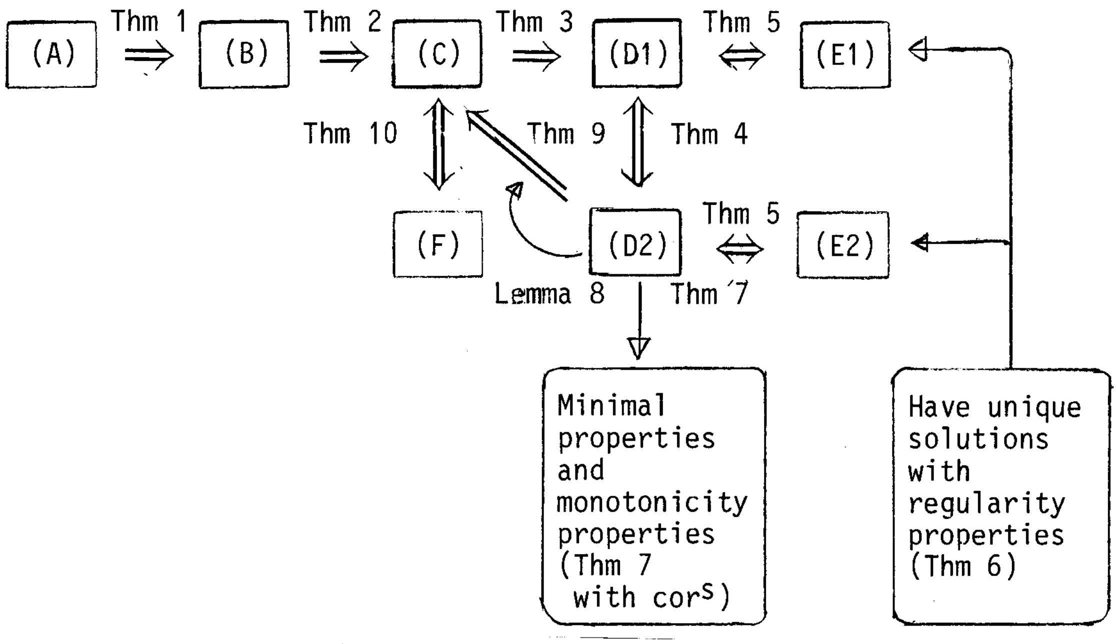
for every subharmonic function φ in D_t. To see why (12) strengthens (11) observe that a natural generalization of (11) is that

(13) $$\int_{D_t} \varphi = 2\pi t \cdot \varphi(0) + \int_{D_0} \varphi$$

for every harmonic function φ in D_t ($\varphi = z^n$ yields (11)).

The moment inequality (12) turns out to be sufficiently strong to be equivalent to (C).

Section III consists essentially of a number of theorems giving the precise relationships between the conditions (A) to (F). These relationships (and the flow of Section III in general) are as shown in the following diagramme.



Here an arrow $\Longrightarrow$, say (C) $\overset{\text{Thm 3}}{\Longrightarrow}$ (D1), means that a solution $t \to D_t$ in the sense of (C) in a simple way (made precise in Theorem 3) provides a solution $t \to u_t$ of (D1).

As the diagramme shows the conditions (C) to (F) are all equivalent while (A) implies (B) and (B) implies (C) to (F). The main goal of Section III is to show the existence and uniqueness of solutions to (C) (weak solutions). This result is stated as Corollary 9.1. The steps (A) $\Rightarrow$ (B) $\Rightarrow$ (C), showing that a classical solution is also a weak solution, together with the fact that a weak solution is unique, serve to motivate the concept of weak solution and also proves uniqueness for classical solutions.

The hardest step in the diagramme is to carry over a solution of (D1) or (D2) to a solution of (C)(Theorem 9). Since (D1) and (D2) concern functions only (u_t and v_t respectively) while (C) concerns sets (D_t) this step involves showing that a certain function (namely $-\Delta v_t$) is a characteristic function of a set (namely the set which is to be the complement of D_t). This is the content of Lemma 8. Its proof requires regularity properties of the solutions of (D1) and (D2) (Theorem 6) and also monotonicity properties possessed by them (Lemma 7 with corollaries).

Section III concludes by showing the equivalence between (C) and (F) (Theorem 10).

In Section III we work within a Sobolev space ($H_0^1(\Omega)$ on a fixed bounded domain $\Omega \subset \mathbb{R}^2$ (a disc)). This has forced us to smooth out the singularity of the Green's function (in order that it shall belong to that Sobolev space). Section IV begins by showing that the solutions of (C) do not depend in any essential way on the choice of Ω (as long as it is sufficiently large) and are not affected by the smoothing of the Green's function (Lemma 11).

We are then able to summarize much of the work in Section III be formulating a kind of Main Theorem - Theorem 13. It states that given an initial domain D_0 (an open bounded set containing the origin) there is a unique weak solution $t \to D_t$ defined for all $t \in [0,\infty)$ of our MBP. It also states that this solution satisfies the moment inequality and that it has those monotonicity properties which it is supposed to have, namely that D_t increases with t and that inclusions between initial domains are preserved for all $t \geq 0$.

The nature of our MBP is such that one expects D_t to become nicer and nicer with increasing t . For example, for all $t>0$, ∂D_t is expected to consist of analytic curves (even if ∂D_0 does not) and asymptotically as $t \to \infty$ the shape of D_t should approach that of a circular disc. We are not able to prove any complete result of that sort but we at least make some steps in that direction. Thus in Theorem 15 we prove that ∂D_t consists of analytic curves for $t > 0$ under the assumptions that D_t is finitely connected and that it contains D_0 compactly (i.e. that $\overline{D}_0 \subset D_t$). The last assumption can be replaced by the assumption that t is sufficiently large or the assumption that ∂D_0 is analytic (Remark 1 after the theorem).

This paper concludes with two applications to so-called quadrature domains (QD:s). The first result states that the property of being a QD is preserved by solutions of our MBP (Theorem 16), and the second result asserts the existence of QD:s of certain kinds (Corollary 16.1).

c. Some further bibliographical notes

Treatments of MBP:s of kinds similar to ours by the methods of variational inequalities are found in [5] and [6]. See also §3 of the survey article [11] and Chapter VIII of the recent book [12] (MBP:s in one space dimension).

In the context of Stefan problems there is an established notion of "weak solution", discussed e.g. in [8] and [3]. This concept of weak solution is not the same as that used in the present paper.

In [3] Crowley proves unicity for weak solutions (in the sense of [8] and [3]) to MBP:s of our type (in particular that of Ex. 1, p. 2) .

In the present paper two kinds of LCP:s ((D1) and (D2)) and two kinds of variational inequalities ((E1) and (E2)) are treated side by side. Our concept of weak solution (C) is most naturally connected with the problems (D1) and (E1). The main reason for also involving (D2) and (E2) is that these problems are of more conventional type than (D1) and (E1). For example the variational inequalities arrived at in [5], [6] and [11] are of the kind (E2), and most literature on existence, unicity and regularity of solutions of variational inequalities (e.g. [12], [13] and [2]) concern those of the type (E2).

Recently there has appeared the paper (preprint) [19] of M. Sakai in which (among other things) our Hele Shaw MBP is treated. The main subject of [19] is the construction and investigation of quadrature domains of rather general kinds, and the results on the MBP are obtained as applications of this. These results are similar to (or slightly stronger than) those of the present paper, while the methods used are quite different.

Let us indicate rather briefly what Sakai has done on the Hele Shaw problem.

1) Sakai formulates a concept of solution of the Hele Shaw flows.*/ This concept is similar to (but weaker than) our concept of classical solution in the sense that the core of it is an equation equivalent (essentially) to our equation (iii) of (A) (p. 22). Sakai´s concept of solution is, however, a global one (i.e. a solution is always to be defined for all $t \geq 0$ and it is formulated in a way that allows the domain to develop certain kinds of singularities and to change connectivity. As a consequence of this Sakai´s concept of solution is very complicated. Sakai expects that a solution in this sense always exists but he is not able to prove it. (Uniqueness of solutions is, however, proved; cf. 4 below.)

*/Whenever the term "solution" (of Hele Shaw flows) appears on the following pages, it is understood to mean solution in this strict sense of Sakai. The exact definition of this concept of solution is, however, too complicated to be reproduced here.

2) Sakai formulates a concept of weak solution which is as follows (with our notations):

Given an arbitrary bounded domain D_0, $\{D_t\}_{t\geq 0}$ is a weak solution if for each $t \geq 0$ D_t is a domain containing D_0 such that

(i) $\int_{D_0} \varphi \, d\sigma + t \cdot \varphi(0) \leq \int_{D_t} \varphi \, d\sigma$

for all function φ which are subharmonic and integrable on D_t,

(ii) D_t has finite area,

(iii) D_t is minimal with the properties (i) and (ii).

This concept of weak solution is similar to our moment inequality (F). In fact, the essential difference is that in (i) Sakai has a somewhat larger test class than we have. A further difference is that Sakai only requires finite area of the domains ((ii)) while we require boundedness. However, Sakai proves (Theorem 6.4 in [19]) that a D_t satisfying (i) to (iii) necessarily is bounded, so in effect there is no difference at that point. As regards (iii) it can be looked upon as a kind of normalisation. Sakai proves (essentially Theorem 3.7) that there is a minimum (not only minimal) domain satisfying (i) and (ii) (i.e. there is a domain D_t satisfying (i) and (ii) such that if G also satisfies (i) and (ii) then $D_t \subset G$).

It follows from what has been said above that a weak solution in the sense of Sakai satisfies our moment inequality (F) and so is a weak solution also in our sense (i.e. satisfies (C) on each finite interval [0,T] and for suitable R and r). On the other hand, a weak solution in our sense is unique up to null-sets (Corollary 9.1). Therefore the two concepts of weak solutions are identical modulo null-sets. Incidentally this gives a kind of answer to a question posed by Sakai (p.113 in [19]) concerning the relation between his concept of weak solution and that one obtained by variational inequality techniques.

3) Sakai shows existence and uniqueness for weak solutions. The existence is proved by simply constructing the domains "by hand" (more or less). This construction is rather lengthy and is done in such a way that the domain obtained can be proved to be the minimum of all domains having the properties (i) and (ii) under 2) above. Uniqueness is then trivial due to the requirement (iii). (Actually Sakai proves existence of quadrature domains for rather general classes of positive measures and for various classes of test functions (subharmonic, harmonic and analytic) in this way, and the weak solutions of Hele Shaw flows are just applications of this.)

4) Sakai proves that a solution is also a weak solution (Proposition 13.1 in [19]). Since Sakai's concept of solution is weaker than our concept of classical solution and his concept of weak solution is stronger (a priori) than that of ours, this result is better than our Theorems 1 and 2.

The weak solution being unique, the above result also shows uniqueness for solutions. Existence of solutions is, however, never proved although Sakai has a partial result in that direction asserting that a weak solution known a priori to fulfil certain of the requirements for a solution actually is a solution (Proposition 13.4).

5) Sakai shows that a solution essentially is bounded by analytic curves. More precisely, if $\{D_t\}_{t\geq 0}$ is a solution then, for $t>0$, every non-degenerate component of ∂D_t is analytic except at certain exceptional points (certain "corners" and ends of slits). The union over all $t>0$ of these exceptional points is a set which is at most countably infinite (Corollary 13.3.)

6) Finally, Sakai proves that for large t a solution of Hele Shaw flows is a simply connected domain bounded by a simple analytic curve and that it asymptotically as $t\to\infty$ converges to a disc in the sense that the Riemann mapping function from the unit disc (scaled in the proper way) converges uniformly on the closure of that disc to the identity map (Proposition 13.5).

II. FORMULATIONS OF THE MOVING BOUNDARY CONDITION

a. Notations and preliminaries

$\mathbb{R}^2 \cong \mathbb{C}$: we will identify $\mathbb{R}^2$ with the complex plane $\mathbb{C}$ whenever convenient. Often the only consequence of this will be the usage of letters such as z and ζ for variables in $\mathbb{R}^2$.

$\mathbb{D}(a;r)$ = the open disc in $\mathbb{R}^2$ with center a and radius r.

$\mathbb{D}_r = \mathbb{D}(0;r)$

$\mathbb{D} = \mathbb{D}(0;1)$

$(a,b) = \{x \in \mathbb{R} : a < x < b\}$

$[a,b] = \{x \in \mathbb{R} : a \leq x \leq b\}$

Ω will generally denote our "universe", a fixed boundary region in $\mathbb{R}^2$ containing the origin. (Actually Ω will always be a disc centered at the origin.)

∂D = the boundary of D in $\mathbb{R}^2$, if $D \subset \mathbb{R}^2$

$\partial_\Omega D = \partial D \cap \Omega$ = the boundary of D in Ω, if $D \subset \Omega$

$\overline{D} = D \cup \partial D$ = the closure of D in $\mathbb{R}^2$ $(D \subset \mathbb{R}^2)$

$D \subset\subset \Omega$ means : $\overline{D} \subset \Omega$ (if Ω is open and bounded, $D \subset \Omega$)

$d\sigma = dx\,dy$ = element of area measure in $\mathbb{R}^2$ (will often be omitted in integrals)

$|D|$ = area of D if $D \subset \mathbb{R}^2$
(also $|z| = \sqrt{x^2+y^2}$ if $z = x+iy \in \mathbb{C}$)

χ_D = the characteristic function of D
if $D \subset \mathbb{R}^2$ $\left(\chi_D(z) = \begin{cases} 1 & \text{for } z \in D \\ 0 & \text{for } z \notin D \end{cases}\right)$.

$\delta = \delta_0$ = the Dirac measure with respect to the origin in $\mathbb{R}^n$ (the unit point mass at $0 \in \mathbb{R}^n$)

$\nabla u = \text{grad } u = (\frac{\partial u}{\partial x}, \frac{\partial u}{\partial y})$

$g_D(z,\zeta)$ is the Green's function for the domain D
($= -\log|z-\xi|$ + harmonic in D,
$= 0$ on ∂D as a function of z)

$g_D(z) = g_D(z,0)$ (if $0 \in D$)

$C^m(\Omega)$: the (linear) space of functions $\Omega \to \mathbb{R}$ all of whose derivates of order $\leq m$ exist and are continuous ($m = 0,1,2,\ldots,\infty$).

$C^\alpha(\Omega)$: the space of functions $u : \Omega \to \mathbb{R}$ which are locally Hölder continuous with exponent α $(0 < \alpha < 1)$, i.e. which satisfy

$$\sup_{\substack{z,\zeta \in K \\ z \neq \zeta}} \frac{|u(z) - u(\zeta)|}{|z-\zeta|^\alpha} < +\infty$$

for each compact $K \subset \Omega$.

$C^{m,\alpha}(\Omega)$: those functions in $C^m(\Omega)$ whose derivates of order m belong to $C^\alpha(\Omega)$ $(m = 0,1,\ldots, \ 0 < \alpha < 1)$.

$\mathcal{S}$: the class of simply connected regions*/ $D \subset \mathbb{R}^2$ with $0 \in D$ and such that ∂D is a Jordan curve of class C^2
(i.e. admits a twice continuously differentiable parametrization).

$\mathcal{R}_{R,r}$: the class of open sets $D \subset \mathbb{R}^2$ with $\mathbb{D}_r \subset\subset D \subset\subset \mathbb{R}_R$ $(R,r > 0)$.

$\mathcal{R}$: the class of open bounded sets $D \subset \mathbb{R}^2$ with $0 \in D$.
Thus $\mathcal{R} = \bigcup_{R,r>0} \mathcal{R}_{R,r}$.

---

*/region = domain = open subset of $\mathbb{R}^2$

$\mathcal{D}(\Omega) = C_c^\infty(\Omega)$: the (linear) space of test functions on Ω (infinitely differentiable functions $\Omega \to \mathbb{R}$ with compact support).

$\mathcal{D}'(\Omega)$: the space of real-valued distributions on Ω .

$H^{m,p}(\Omega)$: the Banach space of real-valued distributions on Ω, all of whose distribution derivatives of order $\leq m$ belong to $L^p(\Omega)$ $(m \geq 0,\ 1 \leq p \leq \infty)$, normed by

$$\|u\|_{m,p} = \sum_{\substack{n,k \geq 0 \\ n+k \leq m}} \left\| \frac{\partial^{n+k} u}{\partial x^n \partial y^k} \right\|_{L^p(\Omega)}$$

$H^{-m,p}(\Omega)$: the space of distributions of the form

$$\sum_{\substack{n,k \geq 0 \\ n+k \leq m}} \frac{\partial^{n+k} u_{nk}}{\partial x^n \partial y^k} \quad \text{with } u_{nk} \in L^p(\Omega) \quad (m \geq 0,\ 1 \leq p \leq \infty).$$

$H^m(\Omega) = H^{m,2}(\Omega)$.

$H_0^1(\Omega) = H_0^{1,2}(\Omega)$: the closure of $C_c^\infty(\Omega)$ in $H^{1,2}(\Omega)$.

We will primarily work in the Sobolev space $H_0^1(\Omega)$ and $H^{-1}(\Omega)$. $H_0^1(\Omega)$ will always be considered as a real Hilbert space, equipped with the inner product

$$(u,v) = \int_\Omega \nabla u \cdot \nabla v = \int_\Omega \frac{\partial u}{\partial x}\frac{\partial v}{\partial x} + \frac{\partial u}{\partial y}\frac{\partial v}{\partial y} \quad .$$

This bilinear form is an inner product because Ω is bounded ([20], Prop.23.4). The norm it induces, $\|u\| = \sqrt{(u,u)}$, is equivalent to the norm $\|u\|_{1,2}$ defined above.

For $u \in \mathfrak{D}'(\Omega)$, $\varphi \in \mathfrak{D}(\Omega)$ we set

(1) $\langle u,\varphi\rangle = \langle \varphi,u\rangle = u(\varphi)$ (value of u on φ).

The following facts are well-known (and easy to verify):

(i) $v \in L^2(\Omega)$ if and only if the map $\mathfrak{D}(\Omega) \in \varphi \to \langle \varphi,v\rangle$ is continuous in the $L^2(\Omega)$ topology, and in that case the (unique) continuous extension of it to $L^2(\Omega)$ is given by

(2) $$\langle u,v\rangle = \int_\Omega u \cdot v \qquad (u,v \in L^2(\Omega)) .$$

(ii) $v \in H^{-1}(\Omega)$ if and only if $\mathfrak{D}(\Omega) \ni \varphi \to \langle \varphi,v\rangle$ is continuous in the $H_0^1(\Omega)$ topology. The (unique) continuous extension of this map to $H_0^1(\Omega)$ (when $v \in H^{-1}(\Omega)$) defines a bilinear pairing between $H_0^1(\Omega)$ and $H^{-1}(\Omega)$ (also denoted $\langle .,.\rangle$), by means of which $H^{-1}(\Omega)$ can be identified with the dual space of $H_0^1(\Omega)$ ([20], Prop. 23.1) .

The Laplacian operator Δ is an isomorphism of $H_0^1(\Omega)$ onto $H^{-1}(\Omega)$ with the property that

(3) $$(u,v) = - \langle u,\Delta v\rangle \qquad (u,v \in H_0^1(\Omega))$$

([20], Thm 23.1). Thus $-\Delta$ identifies two representations of the dual of $H_0^1(\Omega)$, namely $H_0^1(\Omega)$ (in being a Hilbert space) and $H^{-1}(\Omega)$.

If $u \in H_0^1(\Omega)$ and $v \in H^{-1}(\Omega)$ the product $u \cdot v$ has in general no meaning. Despite that it will (in similarity with (2)) sometimes be convenient to use the symbolic integral $\int u \cdot v$ as an alternative notation for $\langle u,v\rangle$ (the pairing defined under (ii) above). Thus,

(4) $$\int u \cdot v = \langle u,v\rangle \qquad \text{for } u \in H_0^1(\Omega),\ v \in H^{-1}(\Omega) .$$

If u and v in (4) happen to be such that the integral $\int_\Omega u \cdot v$ exists in the sense of measure theory, then one wants the symbolic integral in (4) to agree with $\int_\Omega u \cdot v$. This is actually the case, at least in the following two situations:

(i) $v \in L^2(\Omega)$ (then $u \cdot v \in L^1(\Omega)$ since $u \in H_0^1(\Omega) \subset L^2(\Omega)$) ;

(ii) u is a continuous function (in $H_0^1(\Omega)$) and v is a Radon measure (in $H^{-1}(\Omega)$) .

These two cases will cover our needs.

Using the above kind of notation formula (3) takes the form

(5) $$\int \nabla u \cdot \nabla v = - \int u \cdot \Delta v \qquad (u,v \in H_0^1(\Omega)) .$$

Primarily the elements of $H_0^1(\Omega)$, $H^{-1}(\Omega), \ldots$ will be considered as distributions. Thus statements such as

(6) $$u \geq 0 \quad \text{in} \quad D ,$$

if $D \subset \Omega$ is open, always has its meaning and shall be interpreted in the sense of distributions. On the other hand the sentence

(7) $$u > 0 \quad \text{in} \quad D$$

has no meaning in general and such sentences will be used only if the distribution u is known to have a representation in the form of a continuous function (necessarily unique) and shall then have its usual meaning in terms of that continuous function. Similarly, inequalities such as (6) and (7) when D is non-open will be used only when we have additional information about u and their meaning will depend on the context.

If $u \in H_0^1(\Omega)$, $v \in H^{-1}(\Omega)$ the following are true.

(8) $u \geq 0$ (in Ω) if and only if $\langle u,\varphi \rangle \geq 0$ for all $\varphi \in H^{-1}(\Omega)$ with $\varphi \geq 0$

and

(9) $v \geq 0$ if and only if $\langle \varphi,v \rangle \geq 0$ for all $\varphi \in H_0^1(\Omega)$ with $\varphi \geq 0$.

Some abbreviations sometimes used:

MBP moving boundary problem

LCP linear complementarity problem

QD quadrature domain

Formulas are numbered subsequently within each section (I to IV). When a formula is referred to from another section its number is preceded by the number of the section to which it belongs (e.g. (III.12) = formula 12 of section III).

b. A classical formulation of the moving boundary condition

We shall give an example of a precise formulation of the condition that the boundary ∂D of D moves with the velocity $-\nabla g_D$ in order to be able to prove later that a classical solution is also a weak solution. For simplicity we formulate this condition only for simply connected regions, but it is obvious how to extend it to the multiply connected case.

Let $\mathcal{S}$ denote the class of all simply connected regions $D \subset \mathbb{R}^2 \cong \mathbb{C}$ such that $0 \in D$ and such that ∂D is a Jordan curve of class C^2. For $D \in \mathcal{S}$ let g_D denote the Green's function for D with respect to the origin ($g_D(z) = -\log|z| +$harmonic in D, $g_D = 0$ on ∂D). The assumptions on D imply (aside from the existence of g_D) that g_D and ∇g_D have continuous extensions to $\bar{D} = D \cup \partial D$.

Let $(a,b) \subset \mathbb{R}$ be an open interval containing $0 \in \mathbb{R}$. Then we say that

(A) a map $(a,b) \ni t \to D_t \in \mathcal{S}$

satisfies condition (A) or is a classical solution of our moving boundary problem if there exists a map $\zeta : \mathbb{R}/\mathbb{Z} \times (a,b) \to \mathbb{R}^2$ of class C^2 (twice continuously differentiable) such that

(i) $\quad \zeta(s,t) \in \partial D_t \quad$ for all s,t ,

(ii) $\quad \zeta(\cdot,t) : \mathbb{R}/\mathbb{Z} \to \partial D_t$ is a diffeomorphism (of class C^2) for each $t \in (a,b)$,

(10) (iii) $\quad \dfrac{\partial\zeta(s,t)}{\partial t} = -\nabla g_{D_t}(\zeta(s,t))$ for all s,t .

Comment: (i) and (ii) say that for each fixed t , $\zeta(\cdot,t)$ parametrizes ∂D_t. The parameter variable s (in which ζ has period 1) numbers the particles on ∂D_t, and (iii) says that each such point moves with the velocity $-\nabla g_{D_t}(\zeta(s,t))$. Here ∇g_{D_t} is the continuous extension of the gradient of g_{D_t} to $\overline{D}_t$.

c. A semi-weak formulation of the moving boundary condition

Next we give a formulation of the moving boundary condition which serves as a link between the classical formulation and the final weak formulation. Let $\mathcal{S}$ be as on p. 21 and let $(a,b) \subset \mathbb{R}$ be an open interval containing the origin. Then we say that

(B) the map $(a,b) \ni t \to D_t \in \mathcal{S}$

satisfies condition (B) or is a semiweak solution of our moving boundary problem if, for each $\varphi \in C_c^\infty(\mathbb{R}^2)$, $\int_{D_t} \varphi \, dx\, dy$ is a continuously differentiable function of t with

(11) $$\frac{d}{dt}\int_{D_t} \varphi \, dx\, dy = -\int_{\partial D_t} \varphi \cdot \frac{\partial g_{D_t}}{\partial n}\, ds \; .$$

Comment: $\frac{\partial}{\partial n}$ denotes the derivate in the direction of the outward normal of ∂D_t , ds denotes the arc-length measure on ∂D_t .

To see that (B) expresses the same moving boundary condition as (A) does, just integrate (11) with respect to time from t to $t+\delta t$ ($\delta t > 0$ small). This gives

(12) $$\int_{D_{t+\delta t}} \varphi - \int_{D_t} \varphi = - \int_{\partial D_t} \varphi \cdot \frac{\partial g_{D_t}}{\partial n} ds \cdot \delta t + \mathcal{O}(\delta t^2)$$

expressing that the width of the strip $D_{t+\delta t} \setminus D_t$, that is the distance from ∂D_t to $\partial D_{t+\delta t}$, is $-\frac{\partial g_{D_t}}{\partial n} \cdot \delta t + \mathcal{O}(\delta t^2)$. This is precisely what the moving boundary condition (A) wants it to be.

d. <u>The weak formulation of the moving boundary condition</u>

This is the formulation we are going to work with and for which we shall prove existence and unicity of solutions. We give the formulation first and discuss it afterwards. Let $r,R,T > 0$, let $\Omega = \mathbb{D}_R$ and let $\mathfrak{R}_{R,r}$ denote the class of all open sets $D \subset \mathbb{R}^2$ with $\mathbb{D}_r \subset\subset D \subset\subset \mathbb{D}_R$. Then we say that

(C) the map $[0,T] \ni t \to D_t \in \mathfrak{R}_{R,r}$

satisfies (C) or is a <u>weak solution</u> (with parameters r,R,T) of our moving boundary problem if for all $t \in [0,T]$ the function $u = u_t \in H_0^1(\Omega)$ defined by

(13) $$\chi_{D_t} - \chi_{D_0} = \Delta u_t + 2\pi t \cdot \frac{1}{|\mathbb{D}_r|} \chi_{\mathbb{D}_r}$$

satisfies

(14) $u_t \geq 0$ and

(15) $\int u_t \cdot (1-\chi_{D_t}) = 0.$

Discussion: First, observe that equation (13) really defines u_t since $\chi_{D_t} - \chi_{D_0} - 2\pi t \cdot \frac{1}{|\mathbb{D}_r|} \chi_{\mathbb{D}_r} \in H^{-1}(\Omega)$ and Δ is an isomorphism $H_0^1(\Omega) \to H^{-1}(\Omega)$.

The left member of (15) can either be interpreted as the duality pairing between H_0^1 and H^{-1} ($u_t \in H_0^1$, $1 - \chi_{D_t} \in H^{-1}$) or as an ordinary Lebesgue integral ($1 - \chi_{D_t} \in L^2$; cf. p.19 f). Since $1 - \chi_{D_t} \geq 0$ (so that the integrand in (15) is non-negative) (14) and (15) together express that $u_t \geq 0$ in Ω and that $u_t = 0$ (a.e) outside D_t.

Let us motivate the concept of weak solution by sketching how it arises from condition (8) and also be giving a physical interpretation of it.

Observe first that equation (11) in (B) can be expressed in distribution language as

(16) $$\frac{\partial}{\partial t} \chi_{D_t} = - \frac{\partial g_{D_t}}{\partial n} \cdot (\text{arc length measure of } \partial D_t) .$$

Here the second factor in the right member is the distribution

(17) $$\varphi \to \int_{\partial D_t} \varphi \, ds \quad \text{on } \mathbb{R}^2.$$

Combine this with the fact that

(18) $$\Delta g_{D_t} = - 2\pi \delta_0 - \frac{\partial g_{D_t}}{\partial n} \cdot (\text{arc length measure of } \partial D_t) ,$$

where it is understood that the Green's function g_{D_t} is extended to all $\mathbb{R}^2$ by putting it to zero outside D_t, and where hence $\frac{\partial g_{D_t}}{\partial n}$ also equals the jump of the outward normal derivative of g_{D_t} across ∂D_t when passing from the outside of D_t into D_t (besides being just the normal derivative from the inside of D_t itself). Then we get

(19) $$\frac{\partial}{\partial t}\chi_{D_t} = \Delta g_{D_t} + 2\pi\,\delta_0$$

Now (13) is essentially (19) integrated with respect to time. That is with

(20) $$u_t = \int_0^t g_{D_\tau}\, d\tau$$

integration of (19) gives

(21) $$\chi_{D_t} - \chi_{D_0} = \Delta u_t + 2\pi t \cdot \delta_0$$

which essentially is (13). (14) and (15) are clearly also satisfied for (20) since $g_{D_\tau} \geq 0$, and $g_{D_\tau} = 0$ outside D_t for $\tau \in [0,t]$ (D_τ increases with τ).

The difference between (21) and (13) is that the Dirac measure δ_0 is smoothed out to $\frac{1}{|\mathbb{D}_r|}\chi_{\mathbb{D}_r}$ in (13). This replacement is necessitated by the fact that $\delta_0 \notin H^{-1}(\Omega)$ (equivalently $g_{D_t} \notin H_0^1(\Omega)$). Thus, the function u_t in (C) is not intended to be exactly (20) but rather

(22) $$u_t = \int_0^t \tilde{g}_{D_\tau}\, d\tau$$

where

(23) $$\tilde{g}_{D_\tau}(z) = \frac{1}{|\mathbb{D}_r|} \int_{\mathbb{D}_r} g_{D_\tau}(z,\zeta)\, d\sigma_\zeta \ .$$

(Here $g_D(z,\zeta)$ is the Green's function with respect to an arbitrary point $\zeta \in D$.) This modification of g_D does not affect the moving boundary condition since it is easily seen that $\tilde{g}_D = g_D$ in a neighbourhood of ∂D. In fact, one has

$$(24) \qquad \tilde{g}_D(z) = g_D(z) + \log|z| - \frac{1}{|\mathbb{D}_r|}\int_{\mathbb{D}_r} \log|z-\zeta|\,d\sigma_\zeta =$$

$$= \begin{cases} g_D(z) + \log|z| - \frac{1}{|\mathbb{D}_r|}\int_{\mathbb{D}_r} \log|z-\zeta|\,d\sigma_\zeta & \text{for } z \in \mathbb{D}_r \\ g_D(z) & z \in D \setminus \mathbb{D}_r \\ 0 & z \in \Omega \setminus D\,. \end{cases}$$

Next let us give a direct physical meaning to (13), or rather to (19) in the context of heat conduction problems and Stefan problems. Thus, consider the ordinary heat equation

$$(25) \qquad c(T)\frac{\partial T}{\partial t} = \nabla(k(T)\nabla T) + q \quad ,$$

describing the heat flow in some matter. Here $T = T(x,y,t)$ is the temperature, $c = c(T) > 0$ the specific heat, $k = k(T) > 0$ the thermal conductivity and $q = q(x,y,t)$ is a source term (the heat production per unit volume). The density of the matter is assumed to be constant and is taken as unity. It is also assumed that c and k do not depend explicitly on x,y,t.

If $c(T)$ and $k(T)$ are smooth, strictly positive functions (25) describes an ordinary heat conduction problem.

If $c(T)$ is of the form

$$(26) \qquad c(T) = L \cdot \delta(T-T_0) + \text{a smooth, strictly positive function}$$

(25) can be seen to describe a two-phase Stefan problem of a melting solid. Then T_0 is the melting temperature, $L > 0$ the latent heat, the inequalities $T < T_0$ and $T > T_0$ define the solid and liquid phases respectively and the equation $T(x,y,t) = T_0$ defines the moving boundary separating these two phases.

Now I claim that our equation (19) is of the type (25) with

$$(27) \qquad c(T) = \delta(T) \quad .$$

Thus our moving boundary problem will be a Stefan problem of generalized type (of "generalized type" because classically one requires the specific heat to be everywhere strictly positive). Our problem will also in practice be a one-phase problem since our boundary conditions are such that $T \equiv$ constant $(= 0)$ in one phase (the solid phase; the description of the solid phase as that defined by $T < T_0$ cannot be taken literally in this case).

To prove the claim we introduce the enthalpy (heat function) H and the temperature function ϕ defined by

(28) $$H(T) = \int^{T} c(T')\,dT' \quad \text{and}$$

(29) $$\phi(T) = \int^{T} k(T')\,dT' \quad \text{respectively.}$$

(The lower limits of the integrals are immaterial.) In terms of these (25) becomes

(30) $$\frac{\partial H}{\partial t} = \Delta\phi + q\,.$$

Now the choices

(31) $$c(T) = \delta(T)$$

(32) $$k(T) \equiv 1$$

yield

(33) $$\phi(T) = T$$

(34) $$H(T) = \text{the Heaviside function of } T$$

$$= \begin{cases} 0 & \text{for } T < 0 \\ 1 & \phantom{\text{for }} T > 0 \end{cases}$$

Thus by observing that χ_{D_t} is essentially the Heaviside function of g_{D_t} (there arises a tiny problem of how to define the Heaviside function at the jump point, but this problem is in any case unimportant since the functions g_{D_t}, χ_{D_t} in (19) and ϕ, H in (30) can be changed by additive constants without anything happening) we see that (30) goes over into (19) with

(35) $$\phi = T = g_{D_t} \quad ,$$

(36) $$H = \chi_{D_t} \quad \text{and}$$

(37) $$q = 2\pi\delta_0 \quad .$$

This proves the claim.

It is clear from the formulation of (C) that (given D_0) a weak solution $t \to D_t$ can be unique at most up to sets of two-dimensional Lebesgue measure zero; for (13) - (15) is not affected if D_t is replaced by another set $D_t' \in \mathfrak{R}_{R,r}$ such that $\chi_{D_t'} = \chi_{D_t}$ in the sense of distributions, i.e. almost everywhere. Thus phrases such as "unique up to null-sets" will appear in some theorem later on.

If we had wanted to we could have remedied this situation by choosing representatives in each equivalence class of D:s (calling D and D' $\in \mathfrak{R}_{R,r}$ equivalent if $\chi_D = \chi_{D'}$ a.e.). There is at least one natural choice for such representatives: in each equivalence class there is a unique maximal element, the union of all its members, or, what amounts to the same, let $D \in \mathfrak{R}_{R,r}$ be represented by

(38) $$D' = \mathbb{R}^2 \setminus \operatorname{supp}(1 - \chi_D) \quad ,$$

where supp means support in the sense of distributions. In other words, the representative D' of D consists of those points z which have a neighboorhood U with the property $|U \cap D| = |U|$.

One reason why we have chosen not to formulate (C) in terms of these canonical representative domains is that we later shall work with other choices of D_t, namely

(39) $$D_t = D_0 \cup \{z \in \Omega : u_t(z) > 0\}$$

and it is not quite clear whether these two choices always are the same.*/

A nice feature of (C) is that the time variable appears only as a parameter in it: (C) just consists of a number of uncoupled problems, one for each $t \in [0,T]$. Each of these problems can be viewed as a free boundary problem for ∂D_t . In fact, if D_t and u_t satisfy (C) then u_t must be continuously differentiable since (13) implies that $\Delta u \in L^\infty$ (cf. p. 45 f), and it follows that u_t and D_t solve the following free boundary problem (at least if it is assumed that $\partial D_t = \partial(\Omega \setminus D_t)$).

Find a region D_t (in $\mathfrak{R}_{R,r}$) and a continuously differentiable function u_t on $\overline{D}_t$ such that

(40) $$\Delta u_t = 1 - \chi_{D_0} - 2\pi t \cdot \frac{1}{|\mathbb{D}_r|} \chi_{D_r} \quad \text{in } D_t$$

(41) $$u_t = 0 \quad \text{and}$$

(42) $$\text{grad } u_t = 0 \quad \text{on } \partial D_t .$$

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*/ Sakai, in [19], works with another representative, namely that domain defined by (i) - (iii) on p. 14 (which he shows to be unique).

e. Two linear complementarity problems

These two problems constitute an intermediate step between the weak formulation of the moving boundary problem and variational inequalities. The expression "linear complementarity problem" has been used by Elliott ([6]) and has been adopted here.

Let $r,R,T>0$, put $\Omega = \mathbb{D}_R$ and let $D_0 \subset \mathbb{R}^2$ be a given domain with $\mathbb{D}_r \subset\subset D_0 \subset\subset \mathbb{D}_R$ (i.e. $D_0 \in \mathfrak{R}_{R,r}$). For each $t \in [0,T]$ introduce the function $\psi_t \in H_0^1(\Omega)$ defined by

(43) $$\Delta\psi_t = \chi_{D_0} - 1 + 2\pi t \cdot \frac{1}{|\mathbb{D}_r|} \chi_{\mathbb{D}_r} .$$

(Since the right member above belongs to $H^{-1}(\Omega)$ (43) defines a unique function ψ_t in $H_0^1(\Omega)$.)

Expressed with the aid of ψ_t condition (C) becomes

(44)–(46) $$\begin{cases} \Delta u_t + \Delta\psi_t = \chi_{D_t} - 1 & (44) \\ u_t \geq 0 & (45) \\ \int u_t \cdot (\Delta u_t + \Delta\psi_t) = 0 . & (46) \end{cases}$$

Now, the first of our linear complementarity problems is just (44) - (46), but with (44) weakened to the inequality $\Delta u_t + \Delta\psi_t \leq 0$, and the second one is the same thing expressed in the function $v_t = u_t + \psi_t$. Thus we say that

(D1) the map $[0,T] \ni t \to u_t \in H_0^1(\Omega)$ satisfies (D1) if for all $t \in [0,T]$

(47)–(49) $$\begin{cases} u_t \geq 0 & (47) \\ \Delta u_t + \Delta\psi_t \leq 0 & (48) \\ \int u_t \cdot (\Delta u_t + \Delta\psi_t) = 0 , & (49) \end{cases}$$

and we say that

(D2) the map $[0,T] \ni t \to v_t \in H_0^1(\Omega)$ satisfies (D2) if for all $t \in [0,T]$

$$\begin{cases} v_t \geq \psi_t & (50) \\ \Delta v_t \leq 0 & (51) \\ \int (v_t - \psi_t) \cdot \Delta v_t = 0 & (52) \end{cases}$$

Comments: A difference as compared with the earlier versions (A), (B), (C) is that the regions D_t have disappeared from the formulation, except for the initial domain D_0 which is now given beforehand and is implicitly contained in the function ψ_t .

It is clear that the conditions (D1) and (D2) are meaningful for arbitrary given functions $\psi_t \in H_0^1(\Omega)$ (not necessarily of the form given by (43)) and we will sometimes take the liberty to refer to (D1) and (D2) for $\psi_t \in H_0^1(\Omega)$ not of the form (43). We will also refer to (D1) and (D2) for isolated values of t without involving all the correspondences $[0,T] \ni t \to u_t, v_t$. That is, we will use formulations such as "let $u_t \in H_0^1(\Omega)$ satisfy (D1) for $t = t_0$" (given t_0) . Similar remarks hold for the problems (C) and (E1) and (E2).

For fixed $t \in [0,T]$ (D1) and (D2) can be considered as free boundary conditions. Consider (D2) for example. Introducing the coincidence set

$$I_t = \{z \in \Omega : v_t(z) = \psi_t(z)\} \tag{53}$$

one can see that (D2) implies that outside I_t, v_t satisfies the overdetermined problem

$$\Delta v_t = 0 \quad \text{in} \quad \Omega \setminus I_t \tag{54}$$

$$\begin{cases} v_t = \psi_t \quad \text{and} & (55) \\ \operatorname{grad} v_t = \operatorname{grad} \psi_t \quad \text{on} \quad \partial I_t \, . & (56) \end{cases}$$

Thus we have a free boundary condition for ∂I_t . This free boundary problem is known as "the obstacle problem" (the function ψ_t describes the obstacle). See [11] or [12] for example .

f. Two variational inequalities

We shall formulate two variational inequalities which are to be equivalent, respectively, to the linear complementarity problems (D1) and (D2).

Let r,R,T, Ω , D_0 and ψ_t be as on p.30 . Then we say that

(E1) the map $[0,T] \ni t \to u_t \in H_0^1(\Omega)$ satisfies (E1) if, for all $t \in [0,T]$,

(57) $$\begin{cases} \Delta u_t + \Delta\psi_t \leq 0 \\ \text{and} \\ \int_\Omega \nabla(u - u_t) \cdot \nabla u_t \geq 0 \quad \text{for all} \\ u \in H_0^1(\Omega) \quad \text{with} \quad \Delta u + \Delta\psi_t \leq 0 \ . \end{cases}$$ (58)

We say that

(E2) the map $[0,T] \ni t \to v_t \in H_0^1(\Omega)$ satisfies (E2) if, for all $t \in [0,T]$,

(59) $$\begin{cases} v_t \geq \psi_t \\ \text{and} \\ \int_\Omega \nabla(v - v_t) \cdot \nabla v_t \geq 0 \quad \text{for all} \\ v \in H_0^1(\Omega) \quad \text{with} \quad v \geq \psi_t \ . \end{cases}$$ (60)

g. The moment inequality

Let $\mathfrak{R}$ denote the class of all bounded open sets $D \subset \mathbb{R}^2$ with $0 \in D$ (thus $\mathfrak{R} = \bigcup_{r,R>0} \mathfrak{R}_{R,r}$ with $\mathfrak{R}_{R,r}$ as on p. 23), and let $T > 0$. Then we say that

(F) the map $[0,T] \ni t \to D_t \in \mathfrak{R}$ satisfies (F), or satisfies the moment inequality, if, for each $t \in [0,T]$,

(61) $$\int_{D_t} \varphi - \int_{D_0} \varphi \geq 2\pi t \cdot \varphi(0)$$

for every function $\varphi \in H^2(\mathbb{R}^2)$ which is subharmonic in D_t .

Remarks: By definition (see e.g. [20], Sect. 30) a subharmonic function is required to be upper semicontinuous and to have the sub-meanvalue property with respect to small discs.

The choice of the test class in (F), $H^2(\mathbb{R}^2)$, is perhaps not the most natural one, but it is a choice for which the proof of our only theorem involving (F) (Theorem 10, (F) $\Leftrightarrow$ (C)) works fairly well.

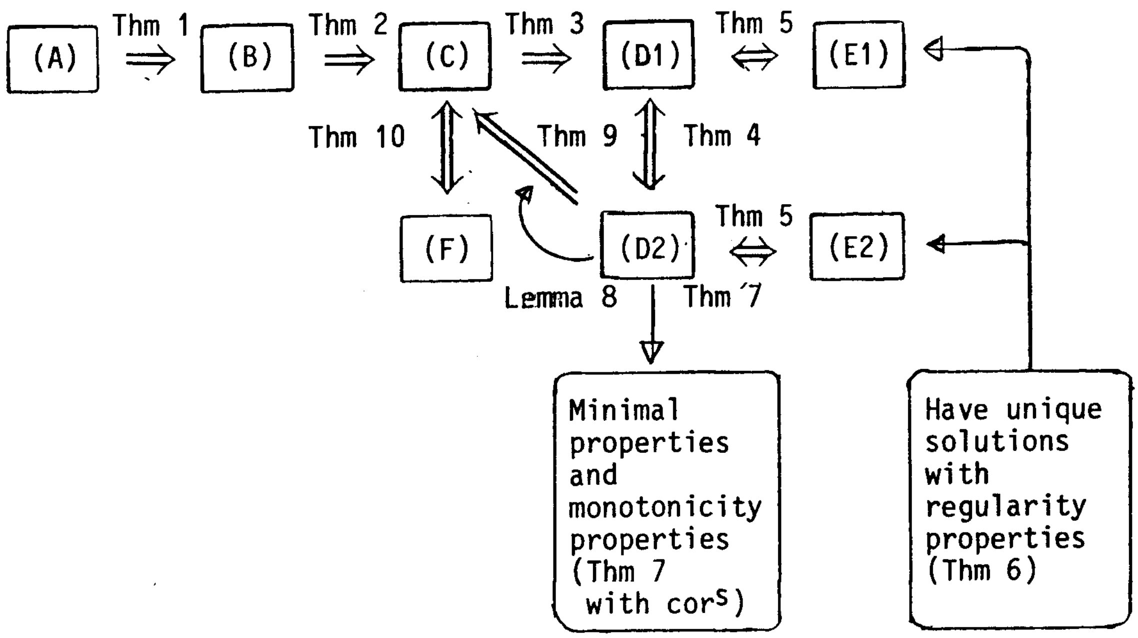
By choosing $\varphi = \pm \operatorname{Re} z^n$ and $\pm \operatorname{Im} z^n$ $(n \geq 0)$ in a neighbourhood of $\overline{D}_0 \cup \overline{D}_t$ we see that (F) implies the moment property:

$$\begin{cases} |D_t| = |D_0| + 2\pi t & (n=0) \quad \text{and} \\ \int_{D_t} z^n = \int_{D_0} z^n & \text{for} \quad n \geq 1 . \end{cases}$$

III. RELATIONS BETWEEN THE MOVING BOUNDARY CONDITIONS, AND EXISTENCE AND UNICITY RESULTS

We shall now set up the relations between the moving boundary conditions (A) - (F) in Section II and thereby also prove existence and unicity of solutions of (C) - (F) (for (A) and (B) only unicity). We will also prove some monotonicity and comparison theorems.

The diagramme below illustrates the contents of this sections.



Thus the conditions (C) - (F) are all equivalent, but we are not able to infer (A) or (B) from them. The existence and unicity of solutions to (C) follows from the elementary fact that (E1) and (E2) have unique solutions.

a. <u>(A) ⇒ (B)</u>

<u>Theorem 1</u>: Suppose the map $(a,b) \ni t \to D_t \in \mathcal{S}$ satisfies (A). Then it also satisfies (B).

<u>Proof</u>: All we have to prove is that the formula

(1) $$\frac{d}{dt} \int_{D_t} \varphi \, dx\, dy = -\int_{\partial D_t} \varphi \frac{\partial g_{D_t}}{\partial n} \, ds$$

holds for all $\varphi \in C_c^\infty(\mathbb{R}^2)$ and that the right member of it is a continuous function of t. Let x,y be the coordinate variables in $\mathbb{R}^2$ and let

(2) $$\begin{cases} \xi = \xi(s,t) \\ \eta = \eta(s,t) \end{cases}$$

denote the components of $\zeta(s,t) \in \mathbb{R}^2$ (p.21-22). Then (A)(iii) becomes

(3) $$\begin{cases} \dfrac{\partial \xi(s,t)}{\partial t} = -\dfrac{\partial g_{D_t}}{\partial x}(\zeta(s,t)) \\[2ex] \dfrac{\partial \eta(s,t)}{\partial t} = -\dfrac{\partial g_{D_t}}{\partial y}(\zeta(s,t)) \end{cases} .$$

Thus the right member of (1) is

$$-\int_{\partial D_t} \varphi \frac{\partial g_{D_t}}{\partial n} \, ds = -\int_{\partial D_t} \varphi \cdot \left(\frac{\partial g_{D_t}}{\partial x} dy - \frac{\partial g_{D_t}}{\partial y} dx\right) = \int_{\partial D_t} \varphi \cdot \left(\frac{\partial \xi}{\partial t} dy - \frac{\partial \eta}{\partial t} dx\right)$$

and we only have to prove the formula

(4) $$\frac{d}{dt} \int_{D_t} \varphi \, dx\, dy = \int_{\partial D_t} \varphi \left(\frac{\partial \xi}{\partial t} dy - \frac{\partial \eta}{\partial t} dx\right) .$$

To prove (4) we consider the expression $\varphi \, dx\, dy$ as a C^∞ 2-form on $\mathbb{R}^2$. Since a 2-form on $\mathbb{R}^2$ is automatically closed (i.e. $d(\varphi \, dx\, dy) = 0$), and hence exact ($\mathbb{R}^2$ being simply connected), there is a C^∞ 1-form $\omega = a\,dx + b\,dy$ $(a = a(x,y),\ b = b(x,y))$ on $\mathbb{R}^2$ such that $d\omega = \varphi\,dx\,dy$, i.e. such that $\frac{\partial b}{\partial x} - \frac{\partial a}{\partial y} = \varphi$.

Now, let $\mathbb{T} = \mathbb{R}/\mathbb{Z}$ (the range of the variable s). Then, using Stokes´ formula at the first step, we get

$$\frac{d}{dt}\int_{D_t} \varphi\,dx\,dy = \frac{d}{dt}\int_{\partial D_t} a\,dx + b\,dy =$$

$$= \frac{d}{dt}\int_{\mathbb{T}} \left[a(\zeta(s,t))\,\frac{\partial\xi(s,t)}{\partial s} + b(\zeta(s,t))\,\frac{\partial\eta(s,t)}{\partial s}\right] ds =$$

$$= \int_{\mathbb{T}} \left[a\cdot\frac{\partial^2\xi}{\partial s\partial t} + b\cdot\frac{\partial^2\eta}{\partial s\partial t}\right] ds + \int_{\mathbb{T}} \left[\left(\frac{\partial a}{\partial x}\cdot\frac{\partial\xi}{\partial t} + \frac{\partial a}{\partial y}\cdot\frac{\partial\eta}{\partial t}\right)\frac{\partial\xi}{\partial s} + \left(\frac{\partial b}{\partial x}\cdot\frac{\partial\xi}{\partial t} + \frac{\partial b}{\partial y}\cdot\frac{\partial\eta}{\partial t}\right)\frac{\partial\eta}{\partial s}\right] ds =$$

$$= \int_{\mathbb{T}} \left[a\cdot\frac{\partial^2\xi}{\partial s\partial t} + b\cdot\frac{\partial^2\eta}{\partial s\partial t}\right] ds + \int_{\mathbb{T}} \left(\frac{\partial b}{\partial x} - \frac{\partial a}{\partial y}\right)\left(\frac{\partial\xi}{\partial t}\cdot\frac{\partial\eta}{\partial s} - \frac{\partial\eta}{\partial t}\cdot\frac{\partial\xi}{\partial s}\right) ds +$$

$$+ \int_{\mathbb{T}} \left[\frac{\partial a}{\partial x}\,\frac{\partial\xi}{\partial t}\,\frac{\partial\xi}{\partial s} + \frac{\partial b}{\partial y}\,\frac{\partial\eta}{\partial t}\,\frac{\partial\eta}{\partial s} + \frac{\partial a}{\partial y}\,\frac{\partial\xi}{\partial t}\,\frac{\partial\eta}{\partial s} + \frac{\partial b}{\partial x}\,\frac{\partial\eta}{\partial t}\,\frac{\partial\xi}{\partial s}\right] ds =$$

$$= \int_{\mathbb{T}} \left(\frac{\partial b}{\partial x} - \frac{\partial a}{\partial y}\right)\left(\frac{\partial\xi}{\partial t}\,\frac{\partial\eta}{\partial s} - \frac{\partial\eta}{\partial t}\,\frac{\partial\xi}{\partial s}\right) ds + \int_{\mathbb{T}} \frac{\partial}{\partial s}\left(a\frac{\partial\xi}{\partial t} + b\frac{\partial\eta}{\partial t}\right) ds$$

$$= \int_{\mathbb{T}} \varphi\cdot\left(\frac{\partial\xi}{\partial t}\,\frac{\partial\eta}{\partial s} - \frac{\partial\eta}{\partial t}\,\frac{\partial\xi}{\partial s}\right) ds = \int_{\partial D_t} \varphi\cdot\left(\frac{\partial\xi}{\partial t}\,dy - \frac{\partial\eta}{\partial t}\,dx\right) .$$

The above computation is valid under the given regularity assumptions (the map $\zeta = (\xi,\eta)$ is of class C^2 and $\zeta(\cdot,t)$ is a diffeomorphism $\mathbb{T} \to \partial D_t$).

It also follows from the above computations that the right member of (1) is a continuous function of t and that hence $\int_{\mathbb{T}} \varphi\,dx\,dy$ is continuously differentiable with respect to t .

This proves Theorem 1.

b. (B) $\Rightarrow$ (C)

Theorem 2: Suppose $(a,b) \ni t \to D_t \in \mathcal{S}$ satisfies (B) and choose $r,R,T > 0$ so that $[0,T] \subset (a,b)$, $\mathbb{D}_r \subset\subset D_0$ and $D_T \subset\subset \mathbb{D}_R$, and let $\Omega = \mathbb{D}_R$. Then $D_t \in \mathfrak{R}_{R,r}$ for all $t \in [0,T]$ and $[0,T] \ni t \to D_t \in \mathfrak{R}_{R,r}$ satisfies (C).

Further, the function $u_t \in H_0^1(\Omega)$ occurring in (C) is

(5) $$u_t = \int_0^t \tilde{g}_{D_\tau}\, d\tau$$

(a vector-valued integral), where $\tilde{g}_D$ is the "smoothed out Green's function", defined by

(6) $$\tilde{g}_D(z) = \frac{1}{|\mathbb{D}_r|} \int_{\mathbb{D}_r} g_D(z,\zeta)\, d\sigma_\zeta =$$

$$\begin{cases} g_D(z) + \log|z| - \dfrac{1}{|\mathbb{D}_r|} \displaystyle\int_{\mathbb{D}_r} \log|z-\zeta|\, d\sigma_\zeta & \text{for } z \in \mathbb{D}_r \\ g_D(z) & z \in D \setminus \mathbb{D}_r \\ 0 & z \in \Omega \setminus D \end{cases}$$

Proof: We first have to prove that $D_t \in \mathfrak{R}_{R,r}$ for $t \in [0,T]$, i.e. that $\mathbb{D}_r \subset\subset D_t \subset\subset \mathbb{D}_R$ for $t \in [0,T]$. Since $\mathbb{D}_r \subset\subset D_0$ and $D_T \subset\subset \mathbb{D}_R$ it suffices to prove that $D_\tau \subset D_t$ for $\tau < t$.

Now $\dfrac{\partial g_{D_t}}{\partial n} \leq 0$ on ∂D_t so that formula (II.11) shows that

(7) $$\frac{d}{dt} \int_{D_t} \varphi\, dx\, dy \geq 0$$

for all $\varphi \in C_c^\infty(\mathbb{R}^2)$ such that $\varphi \geq 0$. Thus $\int_{D_t} \varphi\, dx\, dy$ is a non-decreasing function of t, i.e.

(8) $$\int_{D_\tau} \varphi\, dx\, dy \leq \int_{D_t} \varphi\, dx\, dy \qquad \text{for } \tau < t,$$

for all $\varphi \in C_c^\infty(\mathbb{R}^2)$ with $\varphi \geq 0$. It is easy to see (in view of the regularity requirements on ∂D_t) that this implies that $D_\tau \subset D_t$ for $\tau < t$. Thus $D_t \in \mathfrak{R}_{R,r}$ for $t \in [0,T]$ is proved.

Next we have $\tilde{g}_{D_t} \in H_0^1(\Omega)$. In fact $\tilde{g}_{D_t}$ is continuous in all Ω, is continuously differentiable in $\Omega \setminus \partial D_t$, and at ∂D_t $\nabla \tilde{g}_{D_t}$ is bounded due to the regularity assumption for ∂D_t $(D_t \in \mathcal{S})$. Thus, $\tilde{g}_{D_t} \in H^1(\Omega)$, and so $D_t \subset\subset \Omega$, $\tilde{g}_{D_t} = 0$ on $\Omega \setminus D_t$ implies $\tilde{g}_{D_t} \in H_0^1(\Omega)$.

Now let $u_t \in H_0^1(\Omega)$ be defined by (II.13), i.e.

(9) $$\Delta u_t = \chi_{D_t} - \chi_{D_0} - 2\pi t \cdot \frac{1}{|\mathbb{D}_r|} \chi_{\mathbb{D}_r} \qquad (t \in [0,T]).$$

Then in order to complete the proof of Theorem 2, there are three things to prove. First that

(10) $$u_t = \int_0^t \tilde{g}_{D_\tau}\, d\tau ,$$

and then that (II.14) and (II.15) hold.

(10) means by definition that

(11) $$\langle u_t, \rho \rangle = \int_0^t \langle \tilde{g}_{D_\tau}, \rho \rangle \, d\tau$$

for all $\rho \in H^{-1}(\Omega) \cong H_0^1(\Omega)'$. It will follow from the computations to come that the function $[0,T] \ni \tau \to \langle \tilde{g}_{D_\tau}, \rho \rangle \in \mathbb{R}$ is continuous for each $\rho \in H^{-1}(\Omega)$ (see the last piece of the Remark on p. 40 f), so the integral in (11) certainly exists. Since $\Delta : H_0^1(\Omega) \to H^{-1}(\Omega)$ is an isomorphism, (11) can be written

(12) $$\langle u_t, \Delta\varphi \rangle = \int_0^t \langle \tilde{g}_{D_\tau}, \Delta\varphi \rangle \, d\tau \qquad \text{for all } \varphi \in H_0^1(\Omega) ,$$

i.e. using (9)

(13) $\quad \langle \chi_{D_t} - \chi_{D_0} - 2\pi t \cdot \frac{1}{|\mathbb{D}_r|} \chi_{\mathbb{D}_r}, \varphi \rangle = \int_0^t \langle \tilde{g}_{D_\tau}, \Delta\varphi \rangle \, d\tau \quad$ (for all $\varphi \in H_0^1(\Omega)$) .

We first prove (13) for $\varphi \in C_c^\infty(\Omega)$. Green's (second) formula gives (for $\varphi \in C_c^\infty(\Omega)$)

$$- \int_{\partial D_t} \varphi \cdot \frac{\partial g_{D_t}}{\partial n} \, ds = - \int_{\partial D_t} \varphi \cdot \frac{\partial \tilde{g}_{D_t}}{\partial n} \, ds = - \int_{D_t} \varphi \cdot \Delta \tilde{g}_{D_t} + \int_{D_t} \Delta\varphi \cdot \tilde{g}_{D_t} =$$

$$= - \int_{D_t} \varphi \cdot \left(- \frac{2\pi}{|\mathbb{D}_r|} \chi_{\mathbb{D}_r}\right) + \int_\Omega \Delta\varphi \cdot \tilde{g}_{D_t} = \langle \frac{2\pi}{|\mathbb{D}_r|} \chi_{\mathbb{D}_r}, \varphi \rangle + \langle \tilde{g}_{D_t}, \Delta\varphi \rangle .$$

Combining this with (II.11) we get

$$\langle \chi_{D_t} - \chi_{D_0}, \varphi \rangle = \int_{D_t} \varphi - \int_{D_0} \varphi = \int_0^t \left(- \int_{\partial D_\tau} \varphi \frac{\partial g_{D_\tau}}{\partial n} ds \right) d\tau =$$

$$= \langle 2\pi t \cdot \frac{1}{|\mathbb{D}_r|} \chi_{\mathbb{D}_r}, \varphi \rangle + \int_0^t \langle \tilde{g}_{D_\tau}, \Delta\varphi \rangle \, d\tau$$

Thus (13) is proved for $\varphi \in C_c^\infty(\Omega)$. To extend it to all $\varphi \in H_0^1(\Omega)$ we need only to prove that the right member $\int_0^t \langle g_{D_\tau}, \Delta\varphi \rangle d\tau$ of (13) depends continuously with respect to the H_0^1-norm on φ ; for $C_c^\infty(\Omega)$ is dense in $H_0^1(\Omega)$ and the left member of (13) obviously is H_0^1-continuous with respect to φ .

We have

(14) $\quad \left| \int_0^t \langle \tilde{g}_{D_\tau}, \Delta\varphi \rangle d\tau \right| = \left| \int_0^t (\tilde{g}_{D_\tau}, \varphi) d\tau \right| \le \int_0^t |(\tilde{g}_{D_\tau}, \varphi)| d\tau \le \int_0^t \|\tilde{g}_{D_\tau}\|_{H_0^1} \cdot \|\varphi\|_{H_0^1} d\tau =$

$$= \|\varphi\| \cdot \int_0^t \|\tilde{g}_{D_\tau}\| \, d\tau$$

and thus it is enough to prove that

(15) $\int_0^t \|\tilde{g}_{D_\tau}\| d\tau < \infty$.

To prove this we estimate $\|\tilde{g}_{D_t} - \tilde{g}_{D_\tau}\|_{H_0^1(\Omega)}$ for $\tau < t$ */.

$$\|\tilde{g}_{D_t} - \tilde{g}_{D_\tau}\|^2 = \int_\Omega |\nabla(\tilde{g}_{D_t} - \tilde{g}_{D_\tau})|^2 = -\int_\Omega (\tilde{g}_{D_t} - \tilde{g}_{D_\tau})\cdot\Delta(\tilde{g}_{D_t} - \tilde{g}_{D_\tau}) =$$

$$= \int_{\partial D_t} (\tilde{g}_{D_t} - \tilde{g}_{D_\tau}) \frac{\partial \tilde{g}_{D_t}}{\partial n} ds - \int_{\partial D_\tau} (\tilde{g}_{D_t} - \tilde{g}_{D_\tau}) \frac{\partial \tilde{g}_{D_\tau}}{\partial n} ds =$$

$$= - \int_{\partial D_\tau} \tilde{g}_{D_t} \cdot \frac{\partial \tilde{g}_{D_\tau}}{\partial n} ds = - \int_{\partial D_\tau} g_{D_t} \cdot \frac{\partial g_{D_\tau}}{\partial n} ds \leq \max_{\partial D_\tau} g_{D_t} \cdot \left(- \int_{\partial D_\tau} \frac{\partial g_{D_\tau}}{\partial n} ds\right) =$$

$$= 2\pi \cdot \max_{\partial D_\tau} g_{D_t} .$$

Since $g_{D_t} \leq g_{D_T}$ for $t \in [0,T]$ this shows that

$\|\tilde{g}_{D_t} - \tilde{g}_{D_0}\|^2 \leq 2\pi\cdot\max_{\partial D_0} g_{D_T} < \infty$ for all $t \in [0,T]$. In particular $\|\tilde{g}_{D_t}\|$ is bounded for $t \in [0,T]$ and so (15) is true.

<u>Remark</u>: If we knew a priori that the vector-valued integral $\int_0^t \tilde{g}_{D_\tau} d\tau$ did exist, then it would be trivially true that the right member of (13) depends continuously on φ. A sufficient condition for $\int_0^t \tilde{g}_{D_\tau} d\tau$ to exist is that the map $[0,T] \ni t \to \tilde{g}_{D_t} \in H_0^1(\Omega)$ be continuous (see [18], p. 73 f).

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*/We are then using the fact, already proven (p. 37-38) that $D_\tau \subset D_t$ for $\tau < t$.

It seems very reasonable that this is the case but I have no proof for it. It may, however, be seen from the computation on p. 40 at least that $t \to \tilde{g}_{D_t}$ is continuous from the left, i.e. that $\tilde{g}_{D_\tau} \to \tilde{g}_{D_t}$ as $\tau \nearrow t$.

Moreover, it follows from p. 39 that $t \to \tilde{g}_{D_t}$ is weakly continuous, i.e. that $t \to \langle \tilde{g}_{D_t}, \rho \rangle$ is continuous for each $\rho \in H^{-1}(\Omega)$. In fact, from

(16) $$\langle \tilde{g}_{D_t}, \Delta\varphi \rangle = - \int_{\partial D_t} \varphi \frac{\partial g_{D_t}}{\partial n} \, ds - \langle \frac{2\pi}{|\mathbb{D}_r|} \chi_{\mathbb{D}_r}, \varphi \rangle$$

we see that $t \to \langle \tilde{g}_{D_t}, \rho \rangle$ is continuous for a dense set of ρ:s ($\rho = \Delta\varphi$, $\varphi \in C_c^\infty(\Omega)$), since the first term in the right member of (16) is continuous by assumption (p. 22) , and the continuity for all $\rho \in H^{-1}(\Omega)$ then follows automatically from the boundedness of $\|\tilde{g}_{D_t}\|$ $(t \in [0,T])$.
(End of Remark.)

It remains to prove (II.14) and (II.15). (II.14) follows immediately from (10) (now proven), since $\tilde{g}_{D_\tau} \geq 0$.

To prove (II.15) first observe that

$$\int u_t \cdot (1 - \chi_{D_t}) = \langle u_t, 1 - \chi_{D_t} \rangle = \int_0^t \langle \tilde{g}_{D_\tau}, 1 - \chi_{D_t} \rangle d\tau$$

by choosing $\rho = 1 - \chi_{D_t}$ in (11) .

But now

$$\langle \tilde{g}_{D_\tau}, 1 - \chi_{D_t} \rangle = \int_\Omega \tilde{g}_{D_\tau} \cdot (1 - \chi_{D_t}) = \int_{\Omega \setminus D_t} \tilde{g}_{D_\tau} = 0 \qquad \text{for} \quad \tau \in [0,t] ,$$

since $D_\tau \subset D_t$ (p. 37-38) and $\tilde{g}_{D_\tau} = 0$ outside D_τ . Thus

$$\int_0^t \langle \tilde{g}_{D_\tau}, 1 - \chi_{D_t} \rangle d\tau = 0$$

and (II.15) is proven.

This finishes the proof of Theorem 2.

c. (C) $\Rightarrow$ (D)

Let r, R, T, Ω, D_0 be as on p. 30.

Theorem 3: Suppose the map $[0,T] \ni t \to D_t \in \mathfrak{R}_{R,r}$ satisfies (C) and let $u_t, \psi_t, v_t \in H_0^1(\Omega)$ be the functions defined by (II.13), (II.43) and

$v_t = u_t + \psi_t$.

Then $[0,T] \ni t \to u_t \in H_0^1(\Omega)$ satisfies (D1) and $[0,T] \ni t \to v_t \in H_0^1(\Omega)$ satisfies (D2) .

Proof: This was proved already on p. 30 (the step (C) $\Rightarrow$ (D1) just consisted of replacing the equality in (II.13) by an inequality based on $\chi_{D_t} - 1 \leq 0$, and (D1) $\Leftrightarrow$ (D2) is obvious).

Although obvious we formulate the equivalence between (D1) and (D2) as a separate theorem.

Theorem 4: Let $u_t, v_t, \psi_t \in H_0^1(\Omega)$ be related by $v_t = u_t + \psi_t$ (ψ_t not necessarily defined by (II.43)). Then $[0,T] \ni t \to u_t \in H_0^1(\Omega)$ satisfies (D1) if and only if $[0,T] \ni t \to v_t \in H_0^1(\Omega)$ satisfies (D2).

d. $(D) \Leftrightarrow (E)$

Let r, R, T, Ω, D_0 and ψ_t be as on p. 30.

Theorem 5:

(i) the map $[0,T] \ni t \to u_t \in H_0^1(\Omega)$ satisfies (E1) if and only if it satisfies (D1) .

(ii) the map $[0,T] \ni t \to v_t \in H_0^1(\Omega)$ satisfies (E2) if and only if it satisfies (D2) .

Proof:

(i) $\Leftarrow$: Suppose $t \to u_t$ satisfies (D1). Then, by (II.47) and (II.49) ,

$$\int u_t \cdot (\Delta u + \Delta\psi_t) \leq 0 = \int u_t \cdot (\Delta u_t + \Delta\psi_t)$$

for all $u \in H_0^1(\Omega)$ with $\Delta u + \Delta\psi_t \leq 0$. Thus (by subtracting $\int u_t \cdot \Delta\psi_t$)

$$\int u_t \cdot \Delta(u - u_t) \leq 0,$$

or

$$\int \nabla u_t \cdot \nabla(u - u_t) \geq 0 ,$$

for all $u \in H_0^1(\Omega)$ with $\Delta u + \Delta\psi_t \leq 0$. Since (II.57) is (II.48) this shows that $t \to u_t$ satisfies (E1).

(i) $\Rightarrow$: Suppose $t \to u_t$ satisfies (E1). Then

(17) $$\int u_t \cdot (\Delta u + \Delta\psi_t) \leq \int u_t \cdot (\Delta u_t + \Delta\psi_t)$$

for all $u \in H_0^1(\Omega)$ with $\Delta u + \Delta\psi_t \leq 0$. ((17) is just a reformulation of (II.58)). Here the choice $u = -\psi_t$ shows that the right member of (17) is ≥ 0 , while the choice $u = 2(u_t + \psi_t) - \psi_t$ shows that it is ≤ 0. Hence

(18) $\int u_t \cdot (\Delta u_t + \Delta\psi_t) = 0$.

Writing $\rho = \Delta u + \Delta\psi_t$ and using (18), (17) becomes

(19) $\int u_t \cdot \rho \leq 0$

for all $\rho \in H^{-1}(\Omega)$ with $\rho \leq 0$.

This shows that

(20) $u_t \geq 0$.

Now (20), (II.57) and (18) constitute the conditions in (D1) and so we have proven that $t \to u$ satisfies (D1) .

(ii) $\Leftarrow$: Suppose $t \to v_t$ satisfies (D2). Then, by (II.51) and (II.52) $\int (v - \psi_t) \cdot \Delta v_t \leq 0 = \int (v_t - \psi_t) \cdot \Delta v_t$

for all $v \in H_0^1(\Omega)$ with $v \geq \psi_t$. Thus $\int (v - v_t) \cdot \Delta v_t \leq 0$, or

$\int \nabla(v - v_t) \cdot \nabla v_t \geq 0$,

for all $v \in H_0^1(\Omega)$ with $v \geq \psi_t$. Since (II.59) is (II.50) this shows that $t \to v_t$ satisfies (E2) .

(ii) $\Rightarrow$: Suppose $t \to v_t$ satisfies (E2). Then

(21) $\int (v - \psi_t) \cdot \Delta v_t \leq \int (v_t - \psi_t) \cdot \Delta v_t$

for all $v \in H_0^1(\Omega)$ with $v \geq \psi_t$. ((21) is (II.60)). The choice $v = \psi_t$ shows that the right member of (21) is ≥ 0 , while the choice $v = 2(v_t - \psi_t) + \psi_t$ shows that it is ≤ 0. Thus

(22) $\int (v_t - \psi_t) \cdot \Delta v_t = 0$.

Thus, with $\varphi = v - \psi_t$ (21) becomes

(23) $\int \varphi \cdot \Delta v_t \leq 0$

for all $\varphi \in H_0^1(\Omega)$ with $\varphi \geq 0$. This shows that

(24) $\Delta v_t \leq 0$.

Since (II.59), (24) and (22) constitute the conditions in (D2) we have proven that $t \to v_t$ satisfies (D2) .

This finishes the proof of Theorem 5.

e. Existence, uniqueness and regularity of solutions to (E)

Let r, R, T, Ω, D_0 and ψ_t be as on p. 30. Before formulating the next theorem we shall recall some general regularity results.

Let $2 < p < \infty$ and let $\varphi \in H_0^1(\Omega)$. Then the following implications are true:

(25) $\Delta \varphi \in L^p(\Omega) \Rightarrow \varphi \in H^{2,p}(\Omega)$

$\Rightarrow \varphi \in C^{1,\alpha}(\Omega)$ with $\alpha = 1 - \frac{2}{p}$.

Here $\varphi \in C^{1,\alpha}(\Omega)$ means that φ is continuously differentiable in Ω and that its first order partial derivatives are locally Hölder continuous with exponent α $(0 < \alpha < 1)$, i.e.

$$\sup_{\substack{z,\zeta \in K \\ z \neq \zeta}} \frac{\left|\frac{\partial \varphi}{\partial x}(z) - \frac{\partial \varphi}{\partial x}(\zeta)\right|}{|z - \zeta|^\alpha} < + \infty$$

for each compact $K \subset \Omega$, and similarly for the y-derivative $\frac{\partial \varphi}{\partial y}$.

The first implication in (25) is a consequence of the fact that the operator Δ is uniformly elliptic. ([12], Theorem 4.10, p. 34).

The second implication follows from a well-known lemma by Sobolev ([12], Lemma A 9, p.56).

Now from the definition (II.43) of ψ_t we see that $\Delta\psi_t \in L^\infty(\Omega)$. Thus $\Delta\psi_t \in L^p(\Omega)$ for all $p < \infty$ and so, by (25), $\psi_t \in H^{2,p}(\Omega)$ for all $p < \infty$ and $\psi_t \in C^{1,\alpha}(\Omega)$ for all $\alpha < 1$.

Theorem 6 below is essentially a special case of a standard existence, uniqueness and regularity theorem for solutions of variational inequalities, e.g. Théorème I.1 in [2]. More precisely, the part of Theorem 6 which concerns the function v_t is a direct consequence of Théorème I.1 in [2] while those parts concerning u_t follow by then applying Theorems 4 and 5.

Despite of this we will include a proof of Theorem 6 here. We prove existence and uniqueness of solutions because this is very easy to prove and it is nice to be self-contained to such a low price; and we prove regularity of solutions because we have a proof of that part which is simpler and more elementary than the standard proofs and which perhaps is new.

Theorem 6:

(i) The variational inequalities (E1) and (E2) have unique solutions $u_t, v_t \in H_0^1(\Omega)$ $(t \in [0,T])$.
(This depends only on the fact that $\psi_t \in H_0^1(\Omega)$.)

(ii) u_t and v_t are related by $v_t = u_t + \psi_t$.

(iii) $u_t, v_t \in H^{2,p}(\Omega)$ for all $p < \infty$. In particular $u_t, v_t \in C^{1,\alpha}(\Omega)$ for all $\alpha < 1$.
(This part depends on $\psi_t \in H^{2,p}(\Omega)$, all $p < \infty$.)

Proof: (i) follows from standard Hilbert space theory. Consider (E1) for example. Fix $t \in [0,T]$ and let

(26) $$K = \{u \in H_0^1(\Omega) : \Delta u + \Delta\psi_t \leq 0\} .$$

Then K is a closed convex set in $H_0^1(\Omega)$. That it is convex is obvious and that it is closed follows from the fact that the condition $\Delta u + \Delta\psi_t \leq 0$ can be formulated

(27) $$\langle \Delta u + \Delta\psi_t , \varphi\rangle \leq 0 \quad \text{for all} \quad \varphi \in H_0^1(\Omega) \text{ with } \varphi \geq 0 , \quad \text{or}$$

(28) $$(u,\varphi) \geq -(\psi_t,\varphi) \quad \text{for all} \quad \varphi \in H_0^1(\Omega) \text{ with } \varphi \geq 0 .$$

(28) shows that K is the intersection of a set of (weakly and strongly) closed halfspaces and so is closed (weakly and strongly).

The condition in (E1) now becomes (for fixed $t \in [0,T]$)

(29) $$\begin{cases} u_t \in K \quad \text{and} \\ (u - u_t, u_t) \geq 0 \quad \text{for all} \quad u \in K , \end{cases}$$

and (K being closed and convex) this condition is satisfied by a unique $u_t \in H_0^1(\Omega)$, namely that u_t which solves the minimum norm problem

(30) $$\text{Find } u_t \in K \text{ such that } \|u_t\| = \inf_{u \in K} \|u\| .$$

Thus (i) is proved (with a similar argument for (E2)).

(ii) follows by combining Theorem 5 and Theorem 4.

(iii): In view of (ii) it is sufficient to prove that one of the functions u_t and v_t belongs to $H^{2,p}(\Omega)$. We give two alternatives.

Alternative 1: For the fact that $v_t \in H^{2,p}(\Omega)$ (whenever $\psi_t \in H^{2,p}(\Omega)$, $2 < p < \infty$) we may refer to the existing literature on variational inequalities of the kind (E2), for example [2], Théorème I.1.

Alternative 2: We can prove that $u_t \in H^{2,p}(\Omega)$ by a rather simple and elementary argument which also might be of independent interest. It goes as follows:

We want to show that if $\psi \in H_0^1(\Omega) \cap H^{2,p}(\Omega)$ for some $2 < p < \infty$, then the solution $u \in H_0^1(\Omega)$ of

(31) $$\begin{cases} \Delta u + \Delta\psi \leq 0 \ , \\ \int \nabla(u'-u)\cdot\nabla u \geq 0 \quad \text{for all} \quad u' \in H_0^1(\Omega) \\ \text{with} \quad \Delta u' + \Delta\psi \leq 0 \end{cases}$$

also belongs to $H^{2,p}(\Omega)$. ((31) is (E1) for a fixed $t \in [0,T]$.) To this end, consider the following auxiliary variational inequality, in which the conditions on u are more restrictive:

Find $u \in H_0^1(\Omega)$ which satisfies

(32) $$\begin{cases} \min(0,\Delta\psi) \leq \Delta u + \Delta\psi \leq 0 \ , \\ \int \nabla(u'-u)\cdot\nabla u \geq 0 \quad \text{for all} \quad u' \in H_0^1(\Omega) \\ \text{with} \quad \min(0,\Delta\psi) \leq \Delta u' + \Delta\psi \leq 0 \ . \end{cases}$$

Since, just as for (E1), the constraints on u (and u') above delimit a closed convex subset of $H_0^1(\Omega)$, (32) has a unique solution $u \in H_0^1(\Omega)$. Now this solution a priori belongs to $H^{2,p}(\Omega)$ since the conditions on u are of the form $f_1 \leq \Delta u \leq f_2$ with $f_1, f_2 \in L^p(\Omega)$. Therefore if we can show that the solution of u of (32) is the same as the solution of (31) we are done. And to show this we need only show that u satisfies (D1), i.e.

$$\begin{cases} u \geq 0 & (33) \\ \Delta u + \Delta\psi \leq 0 & (34) \\ \int u\cdot(\Delta u + \Delta\psi) = 0 & (35) \end{cases}$$

since these conditions characterize the solution of (31) (Theorem 5 (i)).

Thus let $u \in H_0^1(\Omega)$ be the solution of (32) and we shall prove that u satisfies (33) - (35) above. First we get from (32) that

(36) $$\int u\cdot(\Delta u' + \Delta\psi) \leq \int u\cdot(\Delta u + \Delta\psi)$$

for all $u' \in H_0^1(\Omega)$ with $\min(0,\Delta\psi) \leq \Delta u' + \Delta\psi \leq 0$, or

(37) $$\int u\cdot\rho \leq \int u\cdot(\Delta u + \Delta\psi)$$

for all $\rho \in H^{-1}(\Omega)$ with $\min(0,\Delta\psi) \leq \rho \leq 0$.

First choose $\rho = 0$ in (37). This gives

(38) $$\int u\cdot(\Delta u + \Delta\psi) \geq 0 \ .$$

Then choose

(39) $$\rho = \begin{cases} \min(0,\Delta\psi) & \text{on } N \\ \Delta u + \Delta\psi & \text{on } \Omega \setminus N \end{cases}$$

where

(40) $$N = \{z \in \Omega : u(z) < 0\}$$

(Observe that since $u \in H^{2,p}(\Omega)$, u is a continuous function, so that N is a well-defined open set in Ω .) We get

(41) $$\int_N u\cdot\min(0,\Delta\psi) \leq \int_N u\cdot(\Delta u + \Delta\psi) \ , \text{ or}$$

(42) $$\int_N u\cdot(\Delta u + \Delta\psi - \min(0,\Delta\psi)) \geq 0$$

Since $\Delta u + \Delta\psi - \min(0,\Delta\psi) \geq 0$ and $u < 0$ on N (42) forces $\Delta u + \Delta\psi - \min(0,\Delta\psi) = 0$ on N. This shows that

$\Delta u \leq 0$ on N.

But now $u = 0$ on $\partial N \cup \partial\Omega$ (by the definition of N, and since u is continuous and belongs to $H_0^1(\Omega)$). Therefore $\Delta u \leq 0$ on N implies $u \geq 0$ on N. Comparing with the definition of N we must conclude that N is the empty set. Thus

$u \geq 0$.

Thus (33) is proven. (34) we know from the beginning (it is part of (32)), and combining (33) and (34) with (38) yields (35).

This ends the proof that the solution of (31) belongs to $H^{2,p}(\Omega)$, and completes the Alternative 2 program.

This also finishes the proof of Theorem 6.

<u>Remark</u>: Observe that, although it is trivial that (D1) and (D2) are equivalent via the relation $v_t = u_t + \psi_t$ (as stated in Theorem 4), it is not a complete triviality that (E1) and (E2) are equivalent under the same relation, i.e. statement (ii) of Theorem 6 is not completely obvious. In terms of the minimum norm problems associated with the variational equalities (E1) and (E2) it says for example that the following two problems have the same solution ($= v_t = u_t + \psi_t$) :

Minimize $\|v - \psi_t\|$ when $\Delta v \leq 0$ $(v \in H_0^1(\Omega))$

Minimize $\|v\|$ when $v \geq \psi_t$ $(v \in H_0^1(\Omega))$.

f. <u>(D) ⇒ (C)</u>.

By the Theorems 4, 5 and 6 we know that (D1) and (D2) have unique solutions u_t and v_t $(t \in [0,T])$, that these are continuously differentiable functions and are related by $v_t = u_t + \psi_t$. Our goal now is to prove that these solutions give rise to a solution of (C). As a preparation for this result (Theorem 9) we need two lemmas, Lemma 7 and Lemma 8.

<u>Lemma 7</u>:

(i) Let $u_t \in H_0^1(\Omega)$ be the solution of (D1). Then $u_t \leq u$ for all $u \in H_0^1(\Omega)$ which satisfy $\Delta u + \Delta\psi_t \leq 0$ and $u \geq 0$.

(ii) Let $v_t \in H_0^1(\Omega)$ be the solution of (D2). Then $v_t \leq v$ for all $v \in H_0^1(\Omega)$ which satisfy $v \geq \psi_t$ and $\Delta v \leq 0$.

<u>Remark</u>: (i) says that in the class of functions that satisfy the two first conditions, (II.47) and (II.48), in (D1) there is a smallest function, namely that function which also satisfies the third condition, (II.49). Similarly for (ii).

(ii) of Lemma 7 is (more or less) a special case of Lemma 1.1 in [13], and (i) is a consequence of (ii). Due to the importance of Lemma 7 for us we will, however, include a proof here. In this proof the function ψ_t will be assumed to be that function defined by (II.43), or at least be some function in $H^{2,p}(\Omega)$ (for some $p < \infty$), although Lemma 7 is true for arbitrary $\psi_t \in H_0^1(\Omega)$.

<u>Proof of Lemma 7</u>: Since we know that the functions u_t and v_t in (i) and (ii) respectively are related by $v_t = u_t + \psi_t$ (Theorem 4), and since the relation $v = u + \psi_t$ obviously sets up a one-to-one correspondence between the functions u satisfying the conditions in (i) and those functions v satisfying the conditions in (ii), it suffices to prove one of (i) and (ii). We prove (ii).

With v_t and v as in the statement of the lemma, put $w = v - v_t$, and we shall prove that $w \geq 0$. Let

(43) $$I(v_t) = \{z \in \Omega : v_t(z) = \psi_t(z)\} .$$

This is a well-defined closed set in Ω since, by assumption, $\psi_t \in H^{2,p}(\Omega)$ and so (Theorem 6) ψ_t and v_t are continuous functions.

Denote by $\operatorname{supp}\Delta v_t$ the support of Δv_t as a distribution on Ω (so, by definition, $\operatorname{supp}\Delta v_t$ is a relatively closed set in Ω). Then, $v_t - \psi_t$ being a non-negative continuous function which is >0 in $\Omega \setminus I(v_t)$, it follows from (II.51) and (II.52) that

(44) $$\operatorname{supp}\Delta v_t \subset I(v_t) .$$

(44) shows that

(45) $$\Delta w = \Delta v \leq 0 \quad \text{in} \quad \Omega \setminus I(v_t) .$$

Everywhere in Ω we have

(46) $$w = (v - \psi_t) - (v_t - \psi_t) \geq -(v_t - \psi_t) .$$

Since $-(v_t - \psi_t)$ is a continuous function which vanishes on $I(v_t)$ and on $\partial\Omega$ (46) shows that (in some sense)

(47) $$w \geq 0 \quad \text{on} \quad \partial(\Omega \setminus I(v_t)) \subset \partial\Omega \cup I(v_t) .$$

The argument will now be that (45) together with (47) imply $w \geq 0$ in $\Omega \setminus I(v_t)$ (by the minimum principle for superharmonic functions) and this then gives the desired conclusion, $w \geq 0$ in all Ω .

However, some care is needed in applying the minimum principle since w not need be a nice function but is just an element of $H_0^1(\Omega)$. We can proceed for example as follows.

Choose an arbitrary $\varepsilon > 0$. Then there is a neighbourhood N of $I(v_t) \cup \partial\Omega$ in Ω such that

(48) $$\varepsilon - (v_t - \psi_t) \geq 0 \quad \text{in} \quad N .$$

Since $w + \varepsilon \geq \varepsilon - (v_t - \psi_t)$ everywhere (by (46)) we have

(49) $$w + \varepsilon \geq 0 \quad \text{in} \quad N$$

(in the sense of distributions). By (45)

(50) $$\Delta(w+\varepsilon) \leq 0 \quad \text{in} \quad \Omega \setminus I(v_t) ,$$

and in particular in a neighbourhood of the compact set $\Omega \setminus N$ ($\subset \Omega \setminus I(v_t)$).

(49) and (59) now show that

(51) $$w + \varepsilon \geq 0$$

in (a neighbourhood of $\Omega \setminus N$) $\cup$ $N = \Omega$, for example by applying the ordinary minimum principle for C^2 superharmonic functions to some suitable family of regularizations of $w + \varepsilon$ (say to $(w+\varepsilon) * h_\rho$, and letting $\rho \searrow 0$, where $h_\rho(z) = \rho^{-2}h(z/\rho)$, where $h \in C_c^\infty(\mathbb{R}^2)$, $h \geq 0$ and $\int h = 1$).

Since $\varepsilon > 0$ in (51) was arbitrary, we conclude that $w \geq 0$ in Ω, and Lemma 7 is proven.

Corollary 7.1: Let $\psi, \psi' \in H_0^1(\Omega) \cap H^{2,p}(\Omega)$ (for some $p < \infty$), and let u, u' and v, v' be the solutions of (D1) and (D2) corresponding to the choices $\psi_t = \psi, \psi'$.

(i) If $\Delta\psi \leq \Delta\psi'$ then $u \leq u'$.

(ii) If $\psi \geq \psi'$ (or if $\Delta\psi \leq \Delta\psi'$) then $v \geq v'$.

(iii) (= (i) and (ii) combined)
If $\Delta\psi \leq \Delta\psi'$ then

$$u' - (\psi - \psi') \leq u \leq u' \quad \text{and} \quad v' + (\psi - \psi') \geq v \geq v' .$$

Proof:

(i) We have $\Delta u' + \Delta\psi \leq \Delta u' + \Delta\psi' \leq 0$ and $u' \geq 0$. Thus (i) of the lemma gives (with $\psi_t = \psi$, $u_t = u$) $u \leq u'$.

(ii) We have $v \geq \psi \geq \psi'$ and $\Delta v \leq 0$. Thus (ii) of the lemma gives (with $\psi_t = \psi'$, $v_t = v'$) $v' \leq v$. ($\Delta\psi \leq \Delta\psi'$ implies $\psi \geq \psi'$ as $\psi, \psi' \in H_0^1(\Omega)$ by a weak maximum principle. See [20], Corollary 28.10 for example.)

(iii) (i) gives $u + \psi \leq u' + \psi$, i.e. $v \leq v' + (\psi - \psi')$, which together with (ii) is the second pair of inequalities in (iii). The first pair is then obtained by subtracting ψ .

Corollary 7.2: Let r, R, T, Ω, D_0 and ψ_t be as on p. 30, let $[0,T] \ni t \to u_t \in H_0^1(\Omega)$ be the solution of (D1) and $[0,T] \ni t \to v_t \in H_0^1(\Omega)$ the solution of (D2). Further let $\theta \in H_0^1(\Omega)$ be the function defined by $\Delta\theta = -\frac{1}{|\mathbb{D}_r|}\chi_{\mathbb{D}_r}$.

Then, for $\tau < t$,

$$u_t - 2\pi(t-\tau)\cdot\theta \leq u_\tau \leq u_t \quad \text{and}$$

$$v_t + 2\pi(t-\tau)\cdot\theta \geq v_\tau \geq v_t \quad .$$

Proof: From

$$\Delta\psi_t = \chi_{D_0} - 1 + 2\pi t\cdot\frac{1}{|\mathbb{D}_r|}\chi_{\mathbb{D}_r} =$$

$$= \chi_{D_0} - 1 - 2\pi t\cdot\Delta\theta$$

we get

$$\Delta\psi_\tau \leq \Delta\psi_t \qquad \text{for} \quad \tau < t$$

and

$$\psi_\tau - \psi_t = 2\pi(t-\tau)\theta \ .$$

Now Corollary 7.2 follows from (iii) of Corollary 7.1 .

Corollary 7.3: Let r, R, T, Ω be as on p. 30, let D_0, D_0' satisfy $\mathbb{D}_r \subset\subset D_0 \subset D_0' \subset\subset \mathbb{D}_R$ and let ψ_t, ψ_t' be defined by (II.43) with respect to D_0 and D_0' respectively. Let u_t, u_t' and v_t, v_t' be the corresponding solutions of (D1) and (D2). Then

$$u_t \leq u_t' \quad \text{and} \quad v_t \geq v_t'$$

for all $t \in [0,T]$.

Proof: Since $\Delta\psi_t \leq \Delta\psi_t'$ for all $t \in [0,T]$ the corollary follows immediately from (i) and (ii) of Corollary 7.1 .

Lemma 8: With r, R, T, Ω, D_0 and ψ_t as on p. 30, let

(52) $$[0,T] \ni t \to v_t \in H_0^1(\Omega)$$

be the solution of (D2) and put

(53) $$I(v_t) = \{z \in \Omega : v_t(z) = \psi_t(z)\} \quad ,$$

(54) $$D_t = D_0 \cup (\Omega \setminus I(v_t)) \; .$$

Then

(55) $$\operatorname{supp}\Delta v_t \subset (\Omega \setminus D_0) \cap I(v_t) \quad \text{and}$$

(56) $$D_t \subset \Omega \setminus \operatorname{supp}\Delta v_t \quad ,$$

where in both inclusions the difference-sets have measure zero. Moreover,

(57) $$\Delta v_t = -\chi_{(\Omega\setminus D_0)\cap I(v_t)} = \chi_{D_t} - 1 \; .$$

Proof: Recall (p. 45-46 and Theorem 6) that ψ_t, $v_t \in H^{2,p}(\Omega)$ for all $p < \infty$. Thus ψ_t, v_t are continuous functions, $I(v_t)$ is a well-defined closed set in Ω and D_t is an open set.

Observe next that the definition (54) of D_t is consistent for $t = 0$ since the solution of (D2) for $t = 0$ is $v_0 = \psi_0$ (in view of $\Delta\psi_0 \leq 0$), and so $I(v_0) = \Omega$.

Now ψ_t, $v_t \in H^{2,p}(\Omega)$ implies that $\Delta\psi_t$ and Δv_t are L^p-functions. We claim that $v_t = \psi_t$ on $I(v_t)$ implies

(58) $$\Delta v_t = \Delta\psi_t \quad \text{a.e. on} \quad I(v_t)$$

(58) is a consequence of the following lemma, which is non-trivial and which will not be proven here.

Lemma: Suppose $u \in H^{2,p}(\Omega)$ $(1 \leq p \leq \infty)$ and that $u = 0$ on a closed set $I \subset \Omega$. Then all partial derivates of order ≤ 2 vanish almost everywhere on I, in particular $\Delta u = 0$ a.e. on I.

(By "derivative" we always mean distribution derivative. For a function in $H^{2,p}$ every such derivative of order ≤ 2 have a representative in form of an L^p-function, and to say that it vanishes a.e. on a set means to say that any one, and then all, of its representatives vanishes a.e. on that set.)

For a proof of the above lemma we may refer to [12], Lemma A.4, p. 53 . Two applications of that lemma yield our lemma.

Applications of the lemma to $u = v_t - \psi_t$ settles (58). Now Lemma 8 will follow by combining (58) with the following facts:

(59) $$\operatorname{supp}\Delta v_t \subset I(v_t) \quad ,$$

(60) $$\Delta v_t \leq 0 \quad ,$$

(61) $$\Delta\psi_t = \begin{cases} 2\pi t \cdot \frac{1}{|\mathbb{D}_r|} \chi_{\mathbb{D}_r} \geq 0 & \text{on } \mathbb{D}_r \\ 0 & D_0 \setminus \mathbb{D}_r \\ -1 & \Omega \setminus D_0 \end{cases}$$

(59) was proven on p. 52, (60) is (II.51) and (61) is (II.43) (together with $\mathbb{D}_r \subset D_0$).

(58) and (60) imply that almost everywhere on $I(v_t)$ we have

(62) $$\Delta\psi_t = \Delta v_t \leq 0 ,$$

hence by (61)

(63) $$\Delta\psi_t = - \chi_{\Omega \setminus D_o} \qquad \text{(a.e. on } I(v_t))$$

hence

(64) $$\Delta v_t = - \chi_{\Omega \setminus D_0} \qquad \text{(a.e. on } I(v_t))$$

In view of (59), (64) shows that in all Ω

(65) $$\Delta v_t = - \chi_{\Omega \setminus D_o} \cdot \chi_{I(v_t)} = - \chi_{(\Omega \setminus D_0) \cap I(v_t)} .$$

Since $(\Omega \setminus D_0) \cap I(v_t)$ is relatively closed in Ω, (65) shows that

(66) $$\operatorname{supp}\Delta v_t \subset (\Omega \setminus D_o) \cap I(v_t) .$$

It also follows that the difference-set in (66) has measure zero. (Generally, if E is a closed set in $\mathbb{R}^n$ then $\operatorname{supp}\chi_E \subset E$ and $E \setminus \operatorname{supp}\chi_E$ has n-dimensional Lebesgue measure zero.)

Thus (55) is proven. (56) is obtained from (55) by taking complements (and using the definition of D_t), and (57) is (65) above together with the definition of D_t.

This proves Lemma 8.

Theorem 9: With r, R, T, Ω, D_0 and ψ_t as on p. 30, let

(67) $$[0,T] \ni t \to u_t \in H_0^1(\Omega) \quad \text{and}$$

(68) $$[0,T] \ni t \to v_t \in H_0^1(\Omega)$$

be the solutions of (D1) and (D2) respectively (so that $v_t = u_t + \psi_t$). Define

(69) $$D_t = D_0 \cup (\Omega \smallsetminus I(v_t)) = D_0 \cup \{z \in \Omega : u_t(z) > 0\}$$

for $t \in (0,T]$. Then, if R is large enough or T small enough (it suffices that $R^2 > 2T + \rho^2$, if $D_0 \subset \mathbb{D}_\rho$), the map

(70) $$[0,T] \ni t \to D_t \in \mathcal{R}_{R,r}$$

is well-defined and satisfies (C). Further, the function "u_t" appearing in (C) is identical with the u_t $(= v_t - \psi_t)$ above .

Proof: First we must prove that $D_t \in \mathcal{R}_{R,r}$ for all $t \in [0,T]$, i.e. that $\mathbb{D}_r \subset\subset D_t \subset\subset \mathbb{D}_R$ $(t \in [0,T])$.

$\mathbb{D}_r \subset\subset D_t$ follows from $\mathbb{D}_r \subset\subset D_0$ and $D_0 \subset D_t$.

To prove $D_t \subset\subset \mathbb{D}_R$ we shall apply Corollary 7.3. Choose ρ such that $D_0 \subset \mathbb{D}_\rho \subset\subset \mathbb{D}_R$ and let $D_0' = \mathbb{D}_\rho$. Then $\mathbb{D}_r \subset\subset D_0 \subset D_0' \subset\subset \mathbb{D}_R$, and with ψ_t', u_t', v_t' defined as in Corollary 7.3 we have by Corollary 7.3 $u_t \le u_t'$ $(t \in [0,T])$, or $v_t - \psi_t \le v_t' - \psi_t'$. This shows that

(71) $$I'(v_t') \subset I(v_t) ,$$

where $I'(v_t') = \{z \in \Omega : v_t'(z) = \psi_t'(z)\}$.

Now we claim that

(72) $$\Omega \smallsetminus I'(v_t') = \mathbb{D}_{\rho_t} \qquad \text{for } t \in (0,T] ,$$

where $\rho_t > 0$ is defined by

(73) $$|\mathbb{D}_{\rho_t}| = 2\pi t + |\mathbb{D}_\rho| \quad , \quad \text{i.e.}$$

(74) $$\rho_t^2 = 2t + \rho^2 \quad .$$

Once (72) is proven $D_t \subset\subset \mathbb{D}_R$ will follow (for $t \in (0,T]$ and assuming that $\rho_T^2 = 2T + \rho^2 < R^2$) since (71) and (72) then give

(75) $$D_t = D_0 \cup (\Omega \setminus I(v_t)) \subset D_0 \cup (\Omega \setminus I'(v_t')) = D_0 \cup \mathbb{D}_{\rho_t} = \mathbb{D}_{\rho_t} \subset \mathbb{D}_{\rho_T} \subset\subset \mathbb{D}_R \ .$$

To prove (72) is equivalent to proving that

(76) $$\{z \in \Omega : u_t'(z) > 0\} = \mathbb{D}_{\rho_t} \ ,$$

and this can be proved simply by computing the function u_t' explicitly. In fact it is found that u_t' is the function in $H_0^1(\Omega)$ for which

(77) $$\Delta u_t' = \chi_{\mathbb{D}_{\rho_t}} - \chi_{D_0'} - 2\pi t \cdot \frac{1}{|\mathbb{D}_r|} \chi_{\mathbb{D}_r} \ ,$$

and (76) follows from (77). ((77) is suggested by the fact that the map $t \to \mathbb{D}_{\rho_t}$ is a solution to (A) and (B), and hence to (C), with $\mathbb{D}_{\rho_0} = \mathbb{D}_\rho = D_0'$ as initial domain and that therefore by Theorem 3 $u_t' = \int_0^t \tilde{g}_{\mathbb{D}_{\rho_\tau}} \, d\tau$.
It is also easy to check directly that $u_t' \in H_0^1(\Omega)$ defined by (77) solves (D1) with $\psi_t' \in H_0^1(\Omega)$ defined by $\Delta\psi_t = \chi_{D_0'} - 1 + 2\pi t \cdot \frac{1}{|\mathbb{D}_r|} \chi_{\mathbb{D}_r}$ and $D_0' = \mathbb{D}_\rho$.)

Thus (72) is proven, and so we have proven that $D_t \in \mathcal{R}_{R,r}$ for $t \in [0,T]$ if $2T + \rho^2 < R^2$, $D_0 \subset \mathbb{D}_\rho$.

It remains to prove (II.13) - (II.15) of (C).*/ Lemma 8 (57) shows that

(77) $$\Delta u_t + \Delta\psi_t = \Delta v_t = \chi_{D_t} - 1$$

which in view of (II.43) is (II.13). (II.14) is (II.47), and (II.15) is (II.49) combined with (77) above.

This completes the proof of Theorem 9.

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*/With u_t given by (67).

Corollary 9.1: Let r, R, T, Ω and D_0 be as on p. 30 with R large enough ($R^2 > 2T + \rho^2$ suffices, if $D_0 \subset \mathbb{D}_\rho$). Then there exists a solution

(78) $$[0,T] \ni t \to D_t \in \mathcal{R}_{R,r}$$

of (C). This solution is unique up to null-sets, i.e. χ_{D_t} is unique as a distribution for each $t \in [o,T]$. Moreover, let $u_t \in H_0^1(\Omega)$ be the function appearing in (C). Then u_t is unique as an element of $H_0^1(\Omega)$, and in (78) above D_t can be chosen to be

(79) $$D_t = D_0 \cup \{z \in \Omega : u_t(z) > 0\}$$

for all $t \in [0,T]$. (In (79) u_t refers to the continuous representative of u_t.)

Proof: The existence of the solution (78) follows immediately from Theorem 9 (combined with the fact that there exists a solution of (D1)). Theorem 9 also shows that D_t can be chosen as in (79) for $t \in (0,T]$. For $t = 0$ (79) is true since $u_t = 0$ in that case (this follows from (II.13) for example). As to the unicity, suppose we have two solutions of (78), say $t \to D_t$ and $t \to D_t'$ (where $D_0' = D_0$), and let u_t and $u_t' \in H_0^1(\Omega)$ be the corresponding solutions of (II.13). Then u_t and u_t' both solve (D1) (for the same ψ_t) by Theorem 3. Thus $u_t = u_t'$ since the solution of (D1) is unique (Theorems 5 and 6). Thus, also by (II.13), $\chi_{D_t} = \chi_{D_t'}$ as distributions, or almost everywhere. This shows the unicity part of the corollary.

Corollary 9.2: Let

(80) $$[0,T] \ni t \to D_t \in \mathcal{R}_{R,r}$$

satisfy (C). Then, for $\tau < t$, $D_\tau \subset D_t$ except for a null-set (i.e. $D_\tau \setminus D_t$ has measure zero, or $\chi_{D_\tau} \leq \chi_{D_t}$ in the distribution sense).

Proof: Let u_t and v_t be the solutions of (D1) and (D2) produced by the solution (80) of (C) (as in Theorem 3). Then the uniqueness part of Theorem 9 shows that, by changing D_t with a set of measure zero if necessary, we can assume that D_t is

$$D_t = D_0 \cup (\Omega \setminus I(v_t)) =$$

$$= D_0 \cup \{z \in \Omega : u_t(z) > 0\} .$$

Now, by Corollary 7.2 $u_\tau \leq u_t$ for $\tau < t$, showing that $D_\tau \subset D_t$ for $\tau < t$ as was to be proved.

Corollary 9.3: Let

$[0,T] \ni t \to D_t \in \mathcal{R}_{R,r}$ and

$[0,T] \ni t \to D_t' \in \mathcal{R}_{R,r}$

be two solutions of (C), and suppose that $D_0 \subset D_0'$. Then $D_t \subset D_t'$ (except for null-sets) for all $t \in [0,T]$.

Proof: The proof is similar to that of Corollary 9.2, but with Corollary 7.3 used in the last argument instead of Corollary 7.2.

Corollary 9.4: (Unicity of classical solutions): Suppose

$(a,b) \ni t \to D_t, D_t' \in \mathcal{S}$

are two solutions of (A) or (B), and suppose that for some $t_0 \in (a,b)$ $D_{t_0} = D_{t_0}'$. Then $D_t = D_t'$ for all $t \geq t_0$ $(t \in (a,b))$.

Proof: We may assume that $t_0 = 0$. By applying Theorem 2 (and Theorem 1 in case of (A)) we obtain for suitable choices of r, R, T > 0 (T can be chosen arbitrarily close to b) two solutions

$[0,T] \ni t \to D_t, D_t' \in \mathcal{R}_{R,r}$

of (C). The unicity part of Corollary 9.1 combined with the regularity assumptions on D_t and D_t' then gives that $D_t = D_t'$ for $t \in [0,T]$. This implies the corollary.

g. <u>(C) $\Leftrightarrow$ (F)</u>

<u>Theorem 10</u>: A map

(81) $[0,T] \ni t \to D_t \in \mathcal{R}_{R,r}$

satisfies (C) if and only if it satisfies (F).

<u>Proof</u>:

$\Rightarrow$: Suppose (81) satisfies (C). Since $D_t \subset\subset \Omega = \mathbb{D}_R$ for all $t \in [0,T]$ it is enough to prove (II.61) for all $\varphi \in H_0^1(\Omega) \cap H^2(\mathbb{R}^2)$ which are subharmonic in D_t.

Let $u_t \in H_0^1(\Omega)$ be the function defined by (II.13). Then $u_t \geq 0$, and $u_t = 0$ a.e. an $\Omega \setminus D_t$ (by (II.14) and (II.15)). Moreover, u_t is continuous (since $\Delta u_t \in L^\infty(\Omega)$), in particular bounded (as $u_t = 0$ on $\partial\Omega$).

Now, let $\varphi \in H_0^1(\Omega) \cap H^2(\mathbb{R}^2)$ be subharmonic in D_t. then $\Delta\varphi \geq 0$ in D_t in the sense of distributions. Moreover, since $\Delta\varphi \in L^2(\Omega)$, the above properties of u_t show that $u_t \cdot \Delta\varphi \in L^2(\Omega)$, $u_t \cdot \Delta\varphi = 0$ a.e. on $\Omega \setminus D_t$ and hence $\int_{\Omega \setminus D_t} u_t \cdot \Delta\varphi = 0$. Using these facts and also (II.13) and the submeanvalue property of φ in $\mathbb{D}_r$ we get

$$\int_{D_t} \varphi - \int_{D_0} \varphi = \langle \chi_{D_t} - \chi_{D_0}, \varphi \rangle =$$

$$= \langle \Delta u_t + 2\pi t \cdot \frac{1}{|\mathbb{D}_r|} \chi_{\mathbb{D}_r}, \varphi \rangle =$$

$$= \langle u_t, \Delta\varphi \rangle + 2\pi t \cdot \frac{1}{|\mathbb{D}_r|} \langle \chi_{\mathbb{D}_r}, \varphi \rangle =$$

$$= \int_\Omega u_t \cdot \Delta\varphi + 2\pi t \cdot \frac{1}{|\mathbb{D}_r|} \int_{\mathbb{D}_r} \varphi =$$

$$= \int_{D_t} u_t \cdot \Delta\varphi + 2\pi t \cdot \frac{1}{|\mathbb{D}_r|} \int_{\mathbb{D}_r} \varphi \geq$$

$$\geq 2\pi t \cdot \frac{1}{|\mathbb{D}_r|} \int_{\mathbb{D}_r} \varphi \geq 2\pi t \cdot \varphi(0) .$$

Thus (81) satisfies (F) .

$\Leftarrow$: Suppose (81) satisfies (F).
Thus

(82) $$\int_{D_t}\varphi - \int_{D_0}\varphi \geq 2\pi t \cdot \varphi(0)$$

for every $\varphi \in H^2(\mathbb{R}^2)$ which is subharmonic in D_t . As we shall presently see this implies that

(83) $$\int_{D_t}\varphi - \int_{D_0}\varphi \geq 2\pi t \cdot \frac{1}{|\mathbb{D}_r|}\int_{\mathbb{D}_r}\varphi$$

for all $\varphi \in H_0^1(\Omega) \cap H^2(\Omega)$ which are subharmonic in D_t .

Let us assume (83) for a moment and define $u_t \in H_0^1(\Omega)$ by (II.13). Then (83) can be written

(84) $$\langle \Delta u_t, \varphi \rangle \geq 0$$

for all $\varphi \in H_0^1(\Omega) \cap H^2(\Omega)$ with $\Delta\varphi \geq 0$ in D_t. Since Δ maps $H_0^1(\Omega) \cap H^2(\Omega)$ (bijectively) onto $L^2(\Omega)$, (84) is the same as (with $\rho = \Delta\varphi$)

(85) $$\langle u_t, \rho \rangle \geq 0$$

for all $\rho \in L^2(\Omega)$ with $\rho \geq 0$ in D_t .

Since all non-negative $\rho \in L^2(\Omega)$ are allowed in (85) we have

(86) $$u_t \geq 0 \quad (\text{in } \Omega) .$$

The choice $\rho = \chi_{D_t} - 1$ is also allowed in (85). This gives

(87) $$\int u_t \cdot (\chi_{D_t} - 1) = \langle u_t, \chi_{D_t} - 1 \rangle \geq 0 ,$$

and so, since the integrand in (87) is non-positive,

(88) $$\int u_t \cdot (\chi_{D_t} - 1) = 0 .$$

This proves that the function $u_t \in H_0^1(\Omega)$ defined by (II.13) satisfies (II.14) and (II.15) (= (86) and (88)). Thus the map (81) satisfies (C), and the proof is complete as soon as we have deduced (83) from (82).

For that purpose observe first that the test class $H^2(\mathbb{R}^2)$ for (82) can be replaced by $H^2(\Omega)$ since the restriction mapping $H^2(\mathbb{R}^2) \to H^2(\Omega)$ is onto ([20], Theorem 26.7). Now let $\varphi \in H_0^1(\Omega) \cap H^2(\Omega)$ be given with $\Delta\varphi \geq 0$ in D_t and we shall prove (83). Since φ is continuous (as $\varphi \in H^2(\Omega)$) there is a continuous function $\tilde{\varphi}$ in Ω which coincides with φ outside $\mathbb{D}_r$ and which is harmonic in $\mathbb{D}_r$. This function is also subharmonic in D_t but unfortunately it does not belong to $H^2(\Omega)$ since $\Delta\tilde{\varphi}$ will be a measure having a singular part on $\partial\mathbb{D}_r$, hence $\Delta\tilde{\varphi} \notin L^2(\Omega)$). Suppose nevertheless that (82) were applicable to $\tilde{\varphi}$. Then we would obtain, using the mean-value property for harmonic functions and the fact that $\tilde{\varphi} \geq \varphi$ in $\mathbb{D}_r$ (and everywhere),

(89) $$\int_{D_t}\varphi - \int_{D_0}\varphi = \int_{D_t}\tilde{\varphi} - \int_{D_0}\tilde{\varphi} \geq 2\pi t \cdot \tilde{\varphi}(0) = 2\pi t \cdot \frac{1}{|\mathbb{D}_r|}\int_{\mathbb{D}_r}\tilde{\varphi} \geq 2\pi t \cdot \frac{1}{|\mathbb{D}_r|}\int_{\mathbb{D}_r}\varphi .$$

Thus (83) would be proven.

To actually prove (83) it is enough to prove that the function $\tilde{\varphi}$ above can be approximated uniformly by functions $\tilde{\varphi}_\varepsilon$ $(\varepsilon \searrow 0)$ to which (82) is applicable. For with $\tilde{\varphi}_\varepsilon$ in place of $\tilde{\varphi}$ in (89) the first inequality there will be valid (by (82)) and in the remaining equalities and inequality (which are valid for $\tilde{\varphi}$) there will be errors which tend to zero as $\varepsilon \to 0$. Thus (83) will follow.

It remains to construct $\tilde{\varphi}_\varepsilon$. Consider $\Delta\tilde{\varphi}$. It can be decomposed as $\Delta\tilde{\varphi} = \rho + \mu$, where μ is the singular part of $\Delta\tilde{\varphi}$, supported by $\partial\mathbb{D}_r$, and $\rho = \Delta\tilde{\varphi} - \mu$. What we have to do is to smooth out μ. Thus let $\{h_\varepsilon\}_{\varepsilon>0}$ be a family of positive mollifiers, say of the kind $h_\varepsilon(z) = \varepsilon^{-2}h(z/\varepsilon)$, where $h \in C_c^\infty(\mathbb{R}^2)$ $h \geq 0$ and $\int h = 1$, let $\tilde{\varphi}_1 \in H_0^1(\Omega)$ be defined by

$$\Delta\tilde{\varphi}_1 = \rho$$

(hence $\tilde{\varphi}_1 \in H^2(\Omega)$, as $\rho \in L^2(\Omega)$), and let

$\tilde{\varphi}_2 = \tilde{\varphi} - \tilde{\varphi}_1$.

Extend both $\tilde{\varphi}_1$ and $\tilde{\varphi}_2$ to all $\mathbb{R}^2$ by putting them zero outside Ω . Then $\tilde{\varphi}_1$ will be continuous (on $\mathbb{R}^2$), hence also $\tilde{\varphi}_2$ will be so.

Now define

$\tilde{\varphi}_\varepsilon = \tilde{\varphi}_1 + \tilde{\varphi}_2 * h_\varepsilon$.

Since $\tilde{\varphi}_2$ is continuous $\tilde{\varphi}_2 * h_\varepsilon \to \tilde{\varphi}_2$ uniformly as $\varepsilon \to 0$, and so $\tilde{\varphi}_\varepsilon \to \tilde{\varphi}$ uniformly. Further $\tilde{\varphi}_\varepsilon \in H^2(\Omega)$ since $\tilde{\varphi}_1 \in H^2(\Omega)$ and $\tilde{\varphi}_2 * h_\varepsilon \in C^\infty(\mathbb{R}^2)$. Finally we have $\Delta\tilde{\varphi}_\varepsilon = \rho + \mu * h_\varepsilon \geq 0$ in D_t since $\rho \geq 0$ in D_t and $\mu \geq 0$.

Thus $\tilde{\varphi}_\varepsilon$, $\varepsilon > 0$, have the required properties: they approximate $\tilde{\varphi}$ uniformly as $\varepsilon \to 0$ and (82) applies to them.

Thus (83) is proved and the proof of Theorem 10 is complete.

IV. SUMMARIZING RESULTS AND APPLICATIONS

a. Main theorems

We now want to summarize the essential result of Section III in a way which is free from some of the technicalities of Section III (e.g. occurrence of the parameters r,R,T). Our main theorem is Theorem 13. Theorem 12 is a preliminary theorem. We begin with a lemma.

Lemma 11: Let

(1) $[0,T] \ni t \to D_t \in \mathcal{R}_{R,r}$ and

(2) $[0,T'] \in t \to D_t' \in \mathcal{R}_{R',r'}$

be two solutions of (C) (with parameter values r,R,T and r',R',T' respectively), and suppose that $D_0 = D_0'$. Then, modulo null-sets, $D_t = D_t'$ for $t \in [0, \min(T,T')]$. Moreover, $u_t = u_t'$ in $\mathbb{D}_{\min(R,R')} \setminus \overline{\mathbb{D}}_{\max(r,r')}$ for $t \in [0, \min(T,T')]$.
(u_t, u_t' are the functions defined by (II.13) in (C).)

Proof: We may suppose that the triplets (r,R,T) and (r',R',T') differ only in one component, since the general case then is obtained as a combination of these pure cases.

Case 1: $r \neq r'$ */, $R = R'$, $T = T'$. We may assume that $r \leq r'$. Put $\Omega = \mathbb{D}_R$ and let $w \in H_0^1(\Omega)$ be the solution of

(3) $$\Delta w = -2\pi\left(\frac{1}{|\mathbb{D}_r|}\chi_{\mathbb{D}_r} - \frac{1}{|\mathbb{D}_{r'}|}\chi_{\mathbb{D}_{r'}}\right).$$

Then it is not hard to see that $w \geq 0$ in Ω and $w = 0$ outside $\mathbb{D}_{r'}$.

*/To mean "not necessarily r=r' ".

Define for $t \in [0,T]$

(4) $u''_t = u'_t + t \cdot w$,

(5) $D''_t = D'_t$.

We want to show that $t \to D''_t$ (together with u''_t) satisfies (C) for the parameter triple (r,R,T). Whenever this is done it follows from the unicity part of Corollary 9.1 that $D''_t = D_t$ */ (except for a null-set) and $u''_t = u_t$ which, in view of the properties of w, proves the lemma in Case 1.

Using the properties of w we get (for $t \in [0,T]$) $u''_t \in H^1_0(\mathbb{D}_R)$,

(6) $u''_t \geq u'_t \geq 0$,

(7) $\int u''_t \cdot (1 - \chi_{D''_t}) = \int u'_t \cdot (1 - \chi_{D'_t}) + t \int w \cdot (1 - \chi_{D'_t}) = 0$

(since $w = 0$ outside $D'_t \supset \mathbb{D}_{r'}$) and

(8) $\chi_{D''_t} - \chi_{D''_0} = \Delta u''_t - t \cdot \Delta w + 2\pi t \cdot \frac{1}{|\mathbb{D}_{r'}|} \chi_{\mathbb{D}_{r'}} = \Delta u''_t + 2\pi t \cdot \frac{1}{|\mathbb{D}_r|} \chi_{\mathbb{D}_r}$.

(6), (7) and (8) show that $t \to D''_t$, u''_t satisfies (C) for r,R,T as we wanted to prove.

Case 2: $r = r'$, $R \neq R'$, $T = T'$. We may suppose that $R' \leq R$.
Define, for $t \in [0,T]$,

$$u''_t = \begin{cases} u'_t & \text{in } \mathbb{D}_{R'} \\ 0 & \text{in } \mathbb{D}_R \setminus \mathbb{D}_{R'} \end{cases} ,$$

$D''_t = D'_t$.

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*/ Observe that $D''_0 = D_0$

Then $u''_t \in H^1_0(\mathbb{D}_R)$ (since $u'_t \in H^1_0(\mathbb{D}_{R'})$) and it is immediately verified that $t \to D''_t$, u''_t satisfies (C) for r,R,T. The unicity statement of Corollary 9.1 shows that $D''_t = D_t$ (a.e.) and $u''_t = u_t$ which is the required conclusion in this case.

Case 3: $r = r'$, $R = R'$, $T \neq T'$.
The required conclusion follows trivially from Corollary 9.1 in this case.

The proof of Lemma 11 is complete.

Theorem 12: Given an open bounded set $D_0 \subset \mathbb{R}^2$ with $0 \in D_0$ (i.e. given a $D_0 \in \mathfrak{R}$ (p.33)) there is a map

(9) $$[0,\infty) \ni t \to D_t \in \mathfrak{R}$$

such that for each $T > 0$, for each $r > 0$ with $\mathbb{D}_r \subset\subset D_0$ and for each $R > 0$ sufficiently large ($R > \sqrt{2T + \rho^2}$ suffices, if $D_0 \subset \mathbb{D}_\rho$) the restriction of (9) to $[0,T]$ satisfies (C) for the parameter values r,R,T. The map (9) is unique with these properties, where "unique" means that each D_t is unique up to a set of measure zero.

Moreover, (9) has the following properties:

(i) It satisfies the moment inequality, (F). (Strictly speaking, as (F) is formulated, the restriction of (9) to $[0,T]$ satisfies (F) for each $T > 0$.)

(ii) Modulo null-sets $D_\tau \subset D_t$ for $\tau < t$.

(iii) If $[0,\infty) \ni t \to D'_t \in \mathfrak{R}$ has the properties of (9) with respect to another initial domain $D'_0 \in \mathfrak{R}$, and $D_0 \subset D'_0$ then $D_t \subset D'_t$ modulo null-sets for all $t \in [0,\infty)$.

Proof: Corollary 9.1 gives a lot of solutions of (C) (for the given D_0 and corresponding to the various allowable choices of r,R and T) and Lemma 11 shows that these melt together into a global solution (9) with the asserted property. The unicity of (9) follows also immediately from Corollary 9.1, and the three additional properties (i), (ii) and (iii) follow from Theorem 10, Corollary 9.2 and Corollary 9.3 respectively. This proves the theorem.

Let us finally give a formulation of Theorem 12 which is self-contained (i.e. does not refer to (C) etc.), in which the function $\frac{1}{|\mathbb{D}_r|}\chi_{\mathbb{D}_r}$ is replaced by what it approximates, the Dirac measure δ at the origin, and which is formulated in such a way that the domains D_t really become unique.

Theorem 13: Given an open bounded set $D_0 \subset \mathbb{R}^2$ with $0 \in D_0$ (i.e. given a $D_0 \in \mathfrak{R}$) there exists a unique map

(10) $$[0,\infty) \ni t \to D_t \in \mathfrak{R}$$

with the following property:

For each $t \in [0,\infty)$ there is a distribution u_t with compact support in $\mathbb{R}^2$ such that

(11) $$\chi_{D_t} - \chi_{D_0} = \Delta u_t + 2\pi t\cdot\delta \ ,$$

(12) $$u_t \geq 0 \qquad \text{and}$$

(13) $$D_t = D_0 \cup \{z \in \mathbb{R}^2 : u_t(z) > 0\} \ ,$$

where (11) shows that u_t has a representation in form of a function, continuous outside 0, and (13) refers to any such representative.

The map (10) has the following additional properties:

(i) $\int_{D_t}\varphi - \int_{D_0}\varphi \geq 2\pi t\cdot\varphi(0)$

for every $\varphi \in H^2(\mathbb{R}^2)$ which is subharmonic in D_t .

(ii) $D_\tau \subset D_t$ for $\tau < t$.

(iii) If $[0,\infty) \ni t \to D'_t \in \mathfrak{R}$ is another map with the properties of (10) for another $D'_0 \in \mathfrak{R}$ and if

$D_0 \subset D'_0$ then

$D_t \subset D'_t$ for all $t \in [0,\infty)$.

Proof: Essentially, the main part of the theorem follows by combining Corollary 9.1 with Lemma 11 just as in the proof of Theorem 12. We need only modify the functions u_t in Corollary 9.1 a little. Thus, in the situation of Corollary 9.1, add to u_t the function w_t defined by

$$w_t(z) = -t\cdot\left(\log|z| - \frac{1}{|\mathbb{D}_r|}\int_{\mathbb{D}_r} \log|z-\zeta|\,d\sigma_\zeta\right).$$

This means that

$$\Delta w_t = -2\pi t\cdot\left(\delta - \frac{1}{|\mathbb{D}_r|}\chi_{\mathbb{D}_r}\right),$$

$w_t \geq 0$ everywhere (in $\mathbb{R}^2$) and

$w_t = 0$ outside $\mathbb{D}_r$.

Then it is easily seen that the solutions of (C) provided by Corollary 9.1 fit together into a solution (10) satisfying (11) - (13) for the new u_t. The unicity and the three additional properties of (10) also follow easily (cf. Theorem 12).

Corollary 13.1: Let $M_0, M_1, M_3, \ldots$ and c be given complex numbers with M_0 and c real and positive and suppose that there is a bounded domain $D \subset \mathbb{C}$ containing the origin such that

$$\int_D z^n dx\,dy = M_n \qquad \text{for all } n \geq 0.$$

Then there exists a domain D' (bounded and containing D) such that

$\int_{D'} dx\,dy = M_0 + c$ and

$\int_{D'} z^n\,dx\,dy = M_n$ for all $n \geq 1$.

Proof: Apply the theorem with $D_0 = D$. Then $D' = D_t$ for $t = \frac{c}{2\pi}$ has the required property as is seen by choosing $\varphi = \pm \operatorname{Re} z^n$ and $\pm \operatorname{Im} z^n$ in D' (and defining φ outside D' in such a way that $\varphi \in H^2(\mathbb{R}^2)$) in the moment inequality (i) .

b. Additional properties

We shall give a few further properties of the map (10) in Theorem 13. Thus the background situation will now be that in Theorem 13:

$D_0 \in \mathcal{R}$ is a given domain and

(14) $[0,\infty) \ni t \to D_t \in \mathcal{R}$

is the map uniquely determined by the following property: there exist distributions u_t ($t \in [0,\infty)$) with compact support in the $\mathbb{C}$*/ such that

(15) $\chi_{D_t} - \chi_{D_0} = \Delta u_t + 2\pi t \cdot \delta$

(16) $u_t \geq 0$

(17) $D_t = D_0 \cup \{z \in \mathbb{C} : u_t(z) > 0\}$.

The distribution u_t will always be represented by the pointwise function (also called u_t) completely determined by

(18) $u_t(z) \equiv 0$ for $t = 0$ and

(19) $u_t(z)$ is continuous outside $z = 0$

(20) $u_t(0) = +\infty$

*/ We are going to use some complex variable theory in this section, so we now replace $\mathbb{R}^2$ by $\mathbb{C}$.

for $t > 0$. (18) is possible since $u_t = 0$ for $t = 0$, (19) is possible since $\Delta u_t \in L^\infty$ outside 0 by (15), and (20) is natural since $u_t(z) \sim -t \cdot \log|z|$ near the origin. In particular (17) refers to this representative of u_t.

With this pointwise definition u_t is a subharmonic function outside 0 and is a superharmonic function in D_0.

Now Theorem 13 gives us three properties of the map (14):

(i) It satisfies the moment inequality,

(ii) D_t is an increasing function of t,

(iii) D_t is increasing as a function of D_0 for fixed t.

One expected property of (14) which would be nice to prove can be vaguely stated as follows:

(iv) D_t becomes nicer with increasing t.

(iv) could be made precise for example in the following way: Suppose D_0 is connected but otherwise allowed to be very irregular (to have infinite connectivity e.g.). Then one expects that

(a) $D_0 \subset\subset D_t$ for all $t > 0$.

(b) D_t has finite connectivity for all $t > 0$.

(c) D_t is simply connected for all sufficiently large t.

(d) Asymptotically as $t \to \infty$ the shape of D_t approaches that of a circular disc.

(e) ∂D_t consists of analytic curves for all $t > 0$.

We have not been able to prove any complete form of (iv). What we shall prove is only that assuming (a) and (b) are true, then e) holds (Theorem 15 below) */. We begin with a lemma which among other things shows to what extent the term D_0 in (17) really is necessary.

Lemma 14: Let

$U_t = \{z \in \mathbb{C} : u_t(z) > 0\}$

Then, for $t > 0$,

(i) U_t is connected.

(ii) If N is a component of D_0 then either $N \subset U_t$ or $N \cap U_t = \emptyset$.

(iii) D_t is the union of U_t and those components of D_0 which do not meet U_t.

(iv) If D_0 is connected then $D_t = U_t$ and D_t is connected.

Proof: (i) It is enough to prove that each component of U_t contains 0. Thus let V be a component of U_t. Then $\partial V \subset \partial U_t \subset \mathbb{C} \setminus U_t$, so $u_t = 0$ on ∂V. Since $u_t > 0$ in V this shows that u_t cannot be a subharmonic function in V. But u_t is a subharmonic function outside 0. Thus $0 \in V$ as we wanted to show.

(ii) In D_0, and in particular in N, u_t is a superharmonic function. Therefore, since N is connected and $u_t \geq 0$, if u_t attains the value 0 in N it must be constantly equal to 0 in N. Thus either $u_t > 0$ in N or $u_t \equiv 0$ in N , proving (ii).

(iii) is an immediate consequence of (ii) and the defintion of D_t, $D_t = D_0 \cup U_t$.

(iv) Since $0 \in D_0 \cap U_t$ (i) and (ii) show that $D_0 \subset U_t$ (D_0 being connected). This gives immediately (iv).

*/ Sakai har proved that a) is true, under the hypothesis that D_0 is a "domain with quasi-smooth boundary", meaning essentially that ∂D_0 is of class C^1 but with certain types of corners allowed (finitely many). See [19], Theorem 3.7. Sakai also has results on b) to e) for solutions (in his sense) of the Hele Shaw flow problem. Cf. p. 15 in the introduction (of the present paper).

Theorem 15: Suppose that, for some fixed $t > 0$, D_t is finitely connected and $D_0 \subset\subset D_t$. Then ∂D_t is a finite disjoint union of analytic curves*/ and isolated points.

Proof: Let $U_t = \{z \in \mathbb{C} : u_t(z) > 0\}$. Then $D_0 \subset\subset D_t = D_0 \cup U_t$ shows that $\partial D_0 \subset U_t$. This implies that each component of D_0 intersects U_t, and so, by (ii) of Lemma 14, $D_0 \subset U_t$. Thus

$$D_t = U_t \ .$$

In particular D_t is connected.

Now let γ be a component of $\partial D_t = \partial U_t$ and we shall show that γ is an analytic curve or a point. Let $\hat{\mathbb{C}} = \mathbb{C} \cup \{\infty\}$ denote the Riemann sphere. Since U_t is connected there is exactly one component of $\hat{\mathbb{C}} \setminus \gamma$ which contains U_t (namely that component which contains 0).

Let V denote that component. Then it is easy to see that $\partial V = \gamma$. Since γ is connected this also shows that V is simply connected.

Put

$$W = D_t \setminus \overline{D}_0 = U_t \setminus \overline{D}_0 \subset V \tag{21}$$

and define

$$S(z) = \overline{z} - 4\frac{\partial u_t}{\partial z} \tag{22}$$

for $z \in W \cup \gamma$. Due to the assumption that D_t is finitely connected D_t is a neighbourhood of γ in V (the other components of ∂D_t cannot cluster at γ). Since $\overline{D}_0$ is a compact subset of D_t (also by assumption) it follows that also W is a full neighbourhood of γ in V.

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*/ By "analytic curve" we mean the following: a subset of $\mathbb{C}$ is an analytic curve if it is the image of (say) $\partial\mathbb{D}$ under some non-constant function holomorphic in a neighbourhood of $\partial\mathbb{D}$. Thus an analytic curve is allowed to intersect itself and to have cusps and other singularities. We say that the analytic curve is non-singular if the function above can be chosen to be univalent in a neighbourhood of $\partial\mathbb{D}$.

As u_t is continuously differentiable outside the origin $S(z)$ is a continuous function on $W \cup \gamma$. On $\gamma \subset \mathbb{C} \setminus U_t$ u_t attains its minimum ($u_t = 0$). Therefore

$\frac{\partial u_t}{\partial z} = 0$ on γ , so

(23) $$S(z) = \bar{z} \quad \text{on} \quad \gamma \ .$$

In W $S(z)$ is holomorphic since, by (15),

(24) $$\frac{\partial S}{\partial \bar{z}} = 1 - \Delta u_t = 1 - \chi_{D_t} + \chi_{D_0} + 2\pi t \delta = 0 \qquad \text{in} \quad W \ .$$

Now it is known that the existence of a function with the properties of $S(z)$ above gives the desired conclusion for γ. To be precise, if γ just consists of one point we are done. Otherwise (since V is simply connected and $\partial V = \gamma$) V can be mapped conformally onto $\mathbb{D}$. Let $f : \mathbb{D} \to V$ be the inverse map .

Then $S(f(\zeta))$ is holomorphic in the neighbourhood $f^{-1}(W)$ of $\partial\mathbb{D}$ in $\mathbb{D}$ and (23) shows that

(25) $$S(f(\zeta)) - \overline{f(\zeta)} \to 0 \quad \text{as} \quad \zeta \to \partial\mathbb{D}$$

$$(\zeta \in \mathbb{D})$$

It can be seen that (25) implies that $f(\zeta)$ extends analytically across $\partial\mathbb{D}$ by defining

(26) $$f(\zeta) = \overline{S(f(1/\bar{\zeta}))}$$

for ζ in a neighbourhood of $\partial\mathbb{D}$ in $\mathbb{C} \setminus \overline{\mathbb{D}}$.

Moreover it is seen that

(27) $$f(\partial\mathbb{D}) = \gamma \ .$$

This shows that γ is an analytic curve (as defined on p. 74) and the theorem is proven.

Remarks:

1 It might seem that Theorem 15 is not of much value since it presupposes knowledge a priori about the unknown domain D_t before drawing conclusions about it. It would be desirable to have a theorem with hypotheses only on D_0 and with the conclusion that ∂D_t is analytic. Then it should be enough to assume about D_0 that it is connected (so that $D_t = U_t$ for $t > 0$; Lemma 14(iv)). However, since the hypotheses on D_t in Theorem 15 seem very plausible (except perhaps in very pathological situations), Theorem 15 at least makes the assertion that ∂D_t is analytic very probable.

Let us mention two situations in which the hypothesis $D_0 \subset\subset D_t$ can be dispensed with. Firstly, whenever t is large enough $D_0 \subset\subset D_t$ is automatically fulfilled since by applying Theorem 13 (iii) to the situation $D_0' \subset D_0$, where D_0' is some disc centered at the origin, we have $D_t' \subset D_t$ for all t, and clearly for t large enough $D_0 \subset\subset D_t'$. (If $D_0' = \mathbb{D}_\rho$ then $D_t' = \mathbb{D}_{\rho_t}$ with $\rho_t^2 = \rho^2 + 2t$.)

Secondly, suppose (in place of $D_0 \subset\subset D_t$) that D_0 is connected and that ∂D_0 is a finite disjoint union of non-singular analytic curves. The first hypothesis implies that $D_t = U_t$ and the second one implies (and can be replaced by) the following: there is a function $S_0(z)$ defined and continuous on $(D_0 \setminus K) \cup \partial D_0$ where K is a compact subset of D_0, holomorphic in $D_0 \setminus K$ and with

(28) $$S_0(z) = \bar{z} \quad \text{on} \quad \partial D_0 .$$

Then, in the proof of Theorem 15 we change the definitions (21) and (22) of W and $S(z)$ to

(29) $$W = D_t \setminus (K \cup \{0\}) \subset V \quad \text{and}$$

(30) $$S(z) = \bar{z} - 4 \frac{\partial u_t}{\partial z} + \chi_{D_0}(z) \cdot (S_0(z) - \bar{z}) \qquad (z \in W \cup \gamma) .$$

In (30) it is assumed that $S_0(z)$ is extended to $W \cup \gamma$ in some way, say by $S_0(z) = \bar{z}$ for $z \in (W \cup \gamma) \setminus \overline{D}_0$. Then $S(z)$ is continuous on $W \cup \gamma$, holomorphic in W (since (15) shows that $\frac{\partial S}{\partial \bar{z}} = 0$ in $W \setminus \partial D_0$ and ∂D_0 is a nice curve) and $S(z) = \bar{z}$ on γ.

The rest of the proof of Theorem 15 works as before, and so the conclusion of the theorem holds with the changed hypotheses.

2 If γ is any non-singular analytic arc there is a function $S(z)$, the Schwarz function for γ, defined and holomorphic in a neighbourhood of γ such that $S(z) = \bar{z}$ on γ. See [4]. The interpretation of $S(z)$ is that the anticonformal map $z \to z^* = \overline{S(z)}$ is the reflection in γ.

It is clear that the function $S(z)$ in the proof of Theorem 15 is the Schwarz function for ∂D_t at the pieces of ∂D_t where it is non-singular. This gives an interpretation of u_t near the boundary. Namely, by (22), if $z \in D_t$, z near to ∂D_t, the reflection of z in ∂D_t is

$$z^* = z - 4\frac{\partial u_t}{\partial \bar{z}}(z) .$$

Thus,

$$-\text{grad } u_t(z) = -2\frac{\partial u_t}{\partial \bar{z}}(z) = \frac{1}{2}(z^* - z) = \frac{1}{2} \cdot \text{(the vector from } z \text{ to its}$$

reflected point z^*).

3 Theorem 15 is similar to theorems on the regularity of the coincidence set for variational inequalities of the kind (E2). There are such theorems having as conclusion that the boundary of the coincidence set is an analytic curve, e.g. Theorem 4.3 in [13] (see also §4 in [11]). These theorems cannot, however, be immediately applied to our problem since they always require the obstacle function (our ψ_t) to be real analytic, a hypothesis which is not satisfied for us (along with other hypotheses on the obstacle function which are not satisfied).

The last lines above shed some light on the hypothesis $D_0 \subset\subset D_t$ in Theorem 15. Namely, although ψ_t is not real analytic in all Ω it is so outside $\overline{D}_0$ and the hypothesis $D_0 \subset\subset D_t$ can be viewed as a way to guarantee a priori that ∂D_t avoids the set where ψ_t is not real analytic.

c. Applications to quadrature domains

The next theorem concerns a class of domains called quadrature domains. We say that $D \subset \mathbb{C}$ is a quadrature domain if there exist points $z_1,\dots,z_n$ in D and complex numbers $a_{j,k}$, where $1 \leq j \leq n$, $0 \leq k \leq r_j - 1$, $r_j \geq 1$, such that

(31) $$\int_D f\, d\sigma = \sum_{j=1}^{n} \sum_{k=0}^{r_j-1} a_{j,k}\, f^{(k)}(z_j)$$

for every function f which is analytic and integrable in D.

Quadrature domains and quadrature identities (i.e. identities of the kind (31) above) are treated in [1] and [9], from which we shall cite some results. Let D be a bounded domain in $\mathbb{C}$. Then

(i) D is a quadrature domain if and only if there is a function $S(z)$ meromorphic in D and continuously extendible to $\overline{D}$ such that

(32) $$S(z) = \overline{z} \quad \text{for} \quad z \in \partial D$$

([1], Lemma 2.3).

(ii) If D is simply connected then D is a quadrature domain if and only if some (and then every) conformal map of the unit disc $\mathbb{D}$ onto D is a rational function ([1], Theorem 1).

(iii) If D is a quadrature domain then ∂D is a complete algebraic curve (i.e. there exists a real polynomial $P(x,y)$ such that

(33) $$\partial D = \{z = x + iy : P(x,y) = 0\}$$

except for a finite set (the set in the right member of (33) may contain isolated points not in ∂D)) ([1], Theorem 3 and [9], Theorem 3.4).

It follows from (ii) that the quadrature domains make up a dense subclass (in any reasonable topology) of all bounded simply connected domains. This is also true for domains of higher connectivity (to some extent) ([9], Theorem 3.3).

Now return to the situation on p. 71 and suppose that D_0 there is a quadrature domain, say satisfies (31). Then, for $t > 0$, the moment inequality (Theorem 13) shows that

(34) $$\int_{D_t} f\,d\sigma = \int_{D_0} f\,d\sigma + 2\pi t \cdot f(0)$$

$$= \sum_{j=1}^{n} \sum_{k=0}^{r_j-1} a_{j,k} f^{(k)}(z_j) + 2\pi t \cdot f(0)$$

for all $f \in H^2(\mathbb{R}^2)$ */ which are analytic in D_t. (Apply the moment inequality to $\pm \operatorname{Re} f$ and $\pm \operatorname{Im} f$.) This nearly shows that D_t is a quadrature domain. We must only extend the validity of (34) to all $f \in L^1(D_t)$ which are analytic in D_t. Instead of doing so, however, we shall base our proof that D_t is a quadrature domain on the characterization (i) (p.78).

Theorem 16: Suppose D_0 is a quadrature domain. Then, for each $t > 0$, D_t is a quadrature domain. In particular ∂D_t is an algebraic curve.

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*/ We temporarily admit complex-valued functions in $H^2(\mathbb{R}^2)$.

Proof: Let $S_0(z)$ be the function given by (i) on p. 78 for $D = D_0$. The theorem is proven as soon as we have established the existence of a similar function $S_t(z)$ for $D = D_t$.

Define, for $z \in \overline{D}_t$,

(35) $$S_t(z) = \overline{z} - 4\frac{\partial u_t}{\partial z} + \chi_{D_0}(z)\cdot(S_0(z) - \overline{z}) ,$$

where we have extended $S_0(z)$ (continuously) to all $\mathbb{C}$ by $S_0(z) = \overline{z}$ for all $z \in \mathbb{C} \setminus D_0$.

Then $S_t(z)$ is continuous, except at the origin and at the poles of $S_0(z)$. On ∂D_t the last three terms in (35) vanish, so

(36) $$S_t(z) = \overline{z} \quad \text{on } \partial D_t .$$

In $D_t \setminus \overline{D}_0$ and in D_0, except at the origin and at the poles of $S_0(z)$, $\frac{\partial S_t}{\partial \overline{z}} = 0$ (by using (15)). Thus, $S_t(z)$ is meromorphic in $(D_t \setminus \overline{D}_0) \cup D_0$. Since $S_t(z)$ is continuous in a neighbourhood of ∂D_0 (which is an algebraic curve) and $(D_t \setminus \overline{D}_0) \cup D_0 \cup \partial D_0 \supset D_t$ it follows that $S_t(z)$ actually is meromorphic in all D_t. Thus $S_t(z)$ has all the required properties and the theorem is proven.

It is clear that the quadrature identities for D_0 and D_t in Theorem 16 are related as follows.
If that of D_0 is

(37) $$\int_{D_0} f\,d\sigma = \sum_{j=1}^{n} \sum_{k=0}^{r_j-1} a_{j,k} f^{(k)}(z_j)$$

then that of D_t is

(38) $$\int_{D_t} f\,d\sigma = \sum_{j=1}^{n} \sum_{k=0}^{r_j-1} a_{j,k} f^{(k)}(z_j) + b_1 f(z_1') ,$$

where $b_1 = 2\pi t$ and $z_1' = 0$.

Let us now combine Theorem 16 with the existence part of Theorem 13. This yields that if D_0 is a bounded quadrature domain containing the origin with the quadrature identity (37), then given any $b_1 > 0$ there exists a quadrature domain D_t with the quadrature identity (38), where $z_1' = 0$. Since the origin is not in any way a distinguished point in the context of quadrature identities it is clear that the two hypotheses $0 \in D_0$ and $z_0' = 0$ can be replaced by the single hypothesis $z_0' \in D_0$. Actually this hypothesis can be relaxed to $z_1' \notin \partial D_0$ since if $z_1' \notin \overline{D}_0$ we may apply the previous reasoning with D_0 replaced by $D_0 \cup \mathbb{D}(z_1';\varepsilon)$ for $\varepsilon > 0$ so small that $D_0 \cap \mathbb{D}(z_1';\varepsilon) = \emptyset$ and $\pi\varepsilon^2 \leq b_1$; $D_0 \cup \mathbb{D}(z_1';\varepsilon)$ is then a quadrature domain with $z_1' \in D_0 \cup \mathbb{D}(z_1';\varepsilon)$ and the quadrature identity

$$\int_{D_0 \cup \mathbb{D}(z_1';\varepsilon)} f\, d\sigma = \sum_{j=1}^{n} \sum_{k=0}^{r_j-1} a_{j,k} f^{(k)}(z_j) + \pi\varepsilon^2 \cdot f(z_1') .$$

The above reasoning may be repeated finitely many times for finitely many points $z_1', \ldots, z_m'$ with $z_j' \notin \partial D_0$ if we just take care to make the modification of D_0 indicated in the last lines above at once and simultaneously for all those z_j' for which $z_j' \notin \overline{D}_0$ (to avoid that at some step some z_j' lies on the boundary of the current domain at that step). Thus we have proved the following corollary of Theorem 16.*/

<u>Corollary 16.1</u>: Let D be bounded domain in $\mathbb{C}$ admitting the quadrature identity

$$\int_D f\, d\sigma = \sum_{j=1}^{n} \sum_{k=0}^{r_j-1} a_{j,k} f^{(k)}(z_j) \qquad (f \in L_a^1(D))^{**/}.$$

Then given finitely many points $z_1', \ldots, z_m' \in \mathbb{C}$, $z_j' \notin \partial D$ and arbitrary positive numbers $b_1, \ldots, b_m$ there exists a bounded domain D' with

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*/More general results, along the same lines as this corollary, are obtained in [19] .

**/$L_a^1(D)$ denotes the space of integrable analytic functions in D.

$$\int_{D'} f\, d\sigma = \sum_{j=1}^{n} \sum_{k=0}^{r_j-1} a_{j,k}\, f^{(k)}(z_j) + \sum_{j=1}^{m} b_j\, f(z'_j)$$

for all $f \in L^1_a(D')$.

In particular (choosing $D = \emptyset$ above), given $z'_1,\ldots,z'_m \in \mathbb{C}$, $b_1,\ldots,b_m > 0$ there exists a bounded domain D' with

$$\int_{D'} f\, d\sigma = \sum_{j=1}^{m} b_j\, f(z'_j)$$

for all $f \in L^1_a(D')$.

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