ON QUADRATURE DOMAINS, GRAVIEQUIVALENCE AND BALAYAGE

by

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Prologue.

First of all I would like to thank the organizing committee for inviting me to this conference on inverse problems for potential fields. I am particularly indepted to Dimiter Zidarov whose interesting and original work in geophysics I have learnt about during the meeting and to Ogny Kounchev who (among a lot of other things) recognized the connection between the work by me and some of my collegues on quadrature domains and the work of the Bulgarian school of geophysics, and who also explained part of Zidarov's work to me (and the other people in the meeting).

The present paper is an informal summary of some results obtained by Makato Sakai and myself, mainly in [Gu] and [Gu–Sa]. Many results will be stated here without complete assumptions concerning e.g. regularity. For complete statements and for proofs we refer to [Gu], [Gu–Sa].

Some notations and conventions.

If $\Omega \subset \mathbb{R}^N$ then $\Omega^c = \mathbb{R}^N \setminus \Omega$ and $\chi_\Omega(x) = 1$ if $x \in \Omega, = 0$ if $x \notin \Omega$. By a mass distribution we mean, mathematically, a signed measure with compact support in $\mathbb{R}^N$. (Thus we allow negative masses.) Usually our mass distributions will be tacitly assumed to have a density function which is not too bad (e.g. which is bounded). The Newtonian potential of the mass distribution μ is (if $N \geq 3$)

$$U^\mu(x) = \text{const.} \cdot \int \frac{d\mu(y)}{|x-y|^{N-2}},$$

so that $-\Delta U^\mu = \mu$. A **body** is a bounded open set (with reasonable boundary) considered as a mass distribution of density one. A **solid body** is a body without cavities, i.e. with a connected complement. If Ω is a body we write U^Ω in place of U^{χ_Ω}.

1. On graviequivalent bodies and quadrature domains.

Let us start by considering the following inverse problem of potential theory.

(P) Do there exist two different solid bodies Ω_1 and Ω_2 in $\mathbb{R}^N$ ($N \geq 2$) such that their Newtonian potentials coincide everywhere on the complement of their union, i.e.

$$U^{\Omega_1} = U^{\Omega_2} \quad \text{on} \quad (\Omega_1 \cup \Omega_2)^c? \tag{1.1}$$

This problem is classical, e.g. P. S. Novikov was working on it already in the 1930's, but it seems still to be open. Many partial results are known, e.g. that the answer is "no" if both Ω_1 and Ω_2 are assumed starshaped or if one of them is a ball. See [Is], [Za] for history and references.

Now assume that (1.1) holds, with Ω_1, Ω_2 solid or not. Then it can easily be seen that there exists a common mass distribution μ concentrated on $\Omega_1 \cap \Omega_2$ such that

$$\begin{cases} U^{\Omega_1} = U^\mu & \text{on} \quad \Omega_1^c, \\ U^{\Omega_2} = U^\mu & \text{on} \quad \Omega_2^c. \end{cases} \tag{1.2}$$

(See fig. 1.) Conversely, it is obvious that if (1.2) holds for some μ then also (1.1) holds.

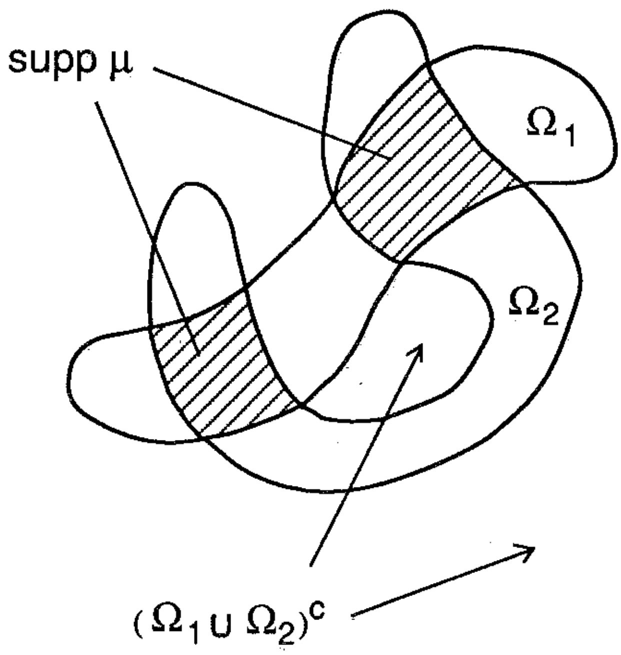

Fig. 1

Thus the inverse problem (P) breaks down into a family of uniqueness problems, one problem for each mass distribution μ:

(Q) Do there exist more than one solid body Ω graviequivalent to μ in the sense that

$$U^{\Omega} = U^{\mu} \qquad \text{on} \quad \Omega^c? \tag{1.3}$$

For many specific choices of μ (e.g. μ a point mass) this question can be answered to be "no", but in general the answer is not known. A more general problem is:

For any μ find all bodies graviequivalent to μ in the sense (1.3).

Example 1: If $\mu = \delta$ (point mass) then the appropriate ball is the only body (solid or not) satisfying (1.3).

Example 2: If μ is a uniform mass distribution on the unit sphere $S^{N-1} = \{|x| = 1\}$ with total mass slightly greater than the volume of the unit ball then there are two bodies Ω satisfying (1.3): one is the appropriate ball and the other is a certain shell domain around S^{N-1} (fig. 2).

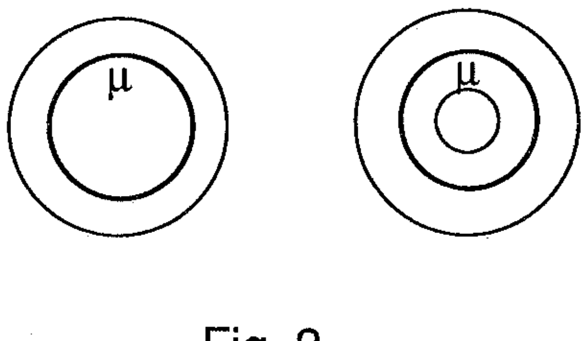

Fig. 2

When we think of it there are actually many natural notions of graviequivalence. If μ and ν are any two mass distributions then the weakest resonable notion of graviequivalence is the one which says that μ and ν are graviequivalent if

$$U^\mu = U^\nu \quad \text{far away.} \tag{1.4}$$

This is a symmetric relation. With $\mu = \chi_{\Omega_1}$, $\nu = \chi_{\Omega_2}$ (1.4) is weaker than (1.1) and there do exist different solid bodies which are equivalent in the sense (1.4) [Sa 1], [Za]. (1.1) too is a symmetric relation but it turns out that it is not an equivalence relation.

Here we shall be concerned with less symmetric situations, namely equivalence between a general mass distribution μ and one of the form $\nu = \chi_\Omega$, i.e. a body. Then (1.4) makes sense as well as (the stronger) (1.3), but we also have the intermediate one

$$\nabla U^\Omega = \nabla U^\mu \qquad \text{on} \quad \Omega^c \tag{1.5}$$

(coinciding gravitational *fields* outside Ω) as well as that which requires that

$$U^\Omega = U^\mu \qquad \text{on} \quad \Omega^c \qquad \text{and} \tag{1.6}$$

$$U^\Omega \leq U^\mu \qquad \text{in} \quad \mathbb{R}^N. \tag{1.7}$$

(1.6)–(1.7) is the strongest and most asymmetrical of all the above notions of graviequivalence. The inequality (1.7) might look unmotivated at first sight but it occurs naturally in connection with balayage (see § 3) and it also guarantees uniqueness of Ω given μ.

The following notation, due to Sakai, is convenient: given μ we write

$$\begin{aligned} \Omega \in Q(\mu, AL^1) \qquad & \text{if} \quad (1.5) \text{ holds,} \\ \Omega \in Q(\mu, HL^1) \qquad & \text{if} \quad (1.3) \text{ holds,} \\ \Omega \in Q(\mu, SL^1) \qquad & \text{if} \quad (1.6)\text{–}(1.7) \text{ hold.} \end{aligned}$$

Thus the $Q(\mu, \cdot)$ are families of bodies graviequivalent to μ in various senses,

$$Q(\mu, SL^1) \subset Q(\mu, HL^1) \subset Q(\mu, AL^1)$$

and (Q) is the question whether $Q(\mu, HL^1)$ can contain more than one solid body. In this question HL^1 cen be replaced by AL^1, because if Ω is solid then $\Omega \in Q(\mu, AL^1)$ if and only if $\Omega \in Q(\mu, HL^1)$.

An equivalent way of expressing (1.3) is by saying that $\mu = 0$ outside Ω and

$$\int_\Omega \varphi dx = \int \varphi d\mu \tag{1.8}$$

for every integrable harmonic function φ in Ω ($\varphi \in HL^1(\Omega)$). (1.8) is an example of what we call a **quadrature identity** (for harmonic functions) and Ω is then called a **quadrature domain**. This explains the letter Q in the notation $Q(\mu, HL^1)$. For $Q(\mu, AL^1)$ and

$Q(\mu, SL^1)$ there are similar reformulations with S standing for "subharmonic" and, if $N = 2$, A for "analytic". For further information about quadrature domains, see [Gu], [Sa 2], [Sa 3], [Sh].

Example 3: With μ as in Example 2 and setting

$$\Omega(t) = \{x \in \mathbb{R}^N : t < \omega_N |x|^N < t + \int d\mu\},$$

where ω_N denotes the volume of the unit ball, we have (identifying $\Omega(0)$ with $\Omega(0) \cup \{0\}$)

$$\begin{aligned} Q(\mu, AL^1) &= \{\Omega(t) : 0 \le t < \omega_N\}, \\ Q(\mu, HL^1) &= \{\Omega(0), \Omega(t_0)\}, \\ Q(\mu, SL^1) &= \{\Omega(t_0)\} \end{aligned}$$

for a certain value of t_0 $(0 < t_0 < \omega_N)$.

2. The Zidarov bubbling process.

There is a natural balayage process associated with our notions of graviequivalence (in particular with the strongest one, (1.6)–(1.7)). This process (as well as other related ones) has been developed, from numerical and physical points of view, by Zidarov [Zi] who calls it "(partial) graviequivalent mass scattering". Later it has been developed by e.g. Kounchev [Ko] and (independently) Sakai and myself [Sa 2], [Sa 3], [Gu], [Gu – Sa]. Inspired by Kounchev's vivid description of the process during the conference we shall here use his terminology "(Zidarov) bubbling" for it.

In mathematical terms the Zidarov bubbling process is a projection operator $F : M \to M$, where M denotes the set of mass distributions. It replaces a given mass distribution by the nearest one (in the energy norm) which has density at most one. In lucky cases, but not always, the result is a body.

We give two equivalent definitions of F. Let $\mu \in M$.

Definition 1: $F(\mu) = \nu$ where ν solves $\text{Min} \|\mu - \nu\|_{\text{energy}} : \nu \in M, \nu \le 1$.

Definition 2: $F(\mu) = -\Delta V^\mu$ where V^μ is the largest function satisfying $V^\mu \le U^\mu$, $-\Delta V^\mu \le 1$.

The function V^μ in the second definition will coincide with U^μ far away. Hence it is a potential, namely that of $F(\mu)$: $V^\mu = U^{F(\mu)}$. Therefore the two inequalities in the following complementarity system for $F(\mu)$ follow directly from Definition 2.

$$F(\mu) \le 1,$$

$$U^{F(\mu)} \le U^\mu, \tag{2.1}$$

$$F(\mu) = 1 \quad \text{on} \quad \{U^{F(\mu)} < U^\mu\}. \tag{2.2}$$

The third, complementarity, condition follows easily from the maximality of V^μ; if it did not hold V^μ could be made larger in $\{V^\mu < U^\mu\}$. Cf. fig. 3.

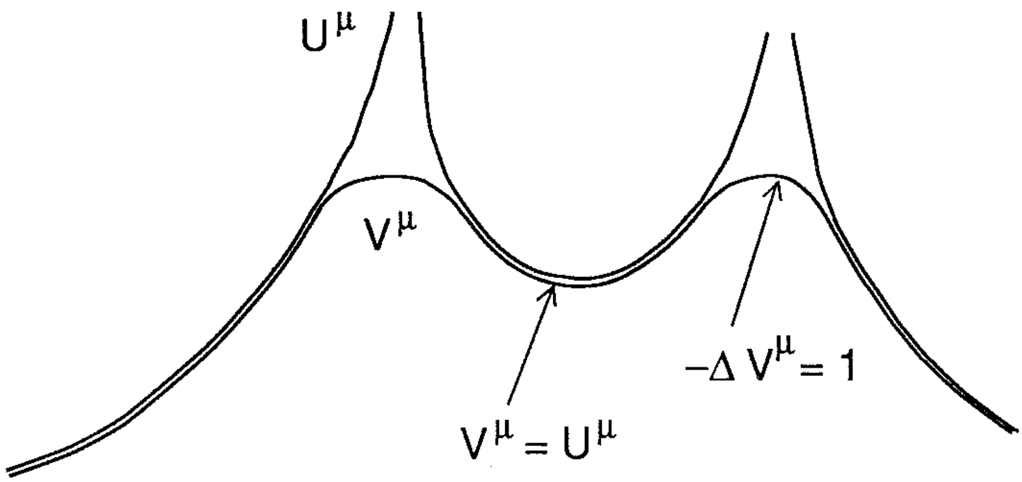

Fig. 3

Set

(2.3) $$\Omega = \Omega(\mu) = \{F(\mu) = 1\} = \text{the "saturated" part of } \mathbb{R}^N.$$

Then

$$U^{F(\mu)} = U^\mu \qquad \text{outside } \Omega(\mu)$$

by (2.1), (2.2). This shows to what extent $F(\mu)$ is graviequivalent to μ.

Besides $\Omega = \Omega(\mu)$ there is in general also an unsaturated part of space, where $F(\mu) < 1$ but $F(\mu) \neq 0$. The general form of $F(\mu)$ is

(2.4) $$F(\mu) = \chi_\Omega + \mu\chi_{\Omega^c}$$

(fig. 4). In "good" cases however (e.g. if $\mu \geq 1$ on supp μ) (2.4) takes the pure form

(2.5) $$F(\mu) = \chi_\Omega,$$

and this is usually the desired result of applying F. To relate F to our previous notions of graviequivalence we have

$$\begin{cases} Q(\mu, SL^1) = \{\Omega\} & \text{if (2.5) holds,} \\ Q(\mu, SL^1) = \varnothing & \text{otherwise.} \end{cases}$$

In particular $Q(\mu, SL^1)$ contains at most one element.

Some further properties of F worth mentioning here are

$$\begin{aligned} F(F(\mu_1) + \mu_2) &= F(\mu_1 + \mu_2) \qquad \text{if} \qquad \mu_2 \geq 0, \\ F(\mu_1) &\leq F(\mu_2) \qquad \text{if} \qquad \mu_1 \leq \mu_2. \end{aligned}$$

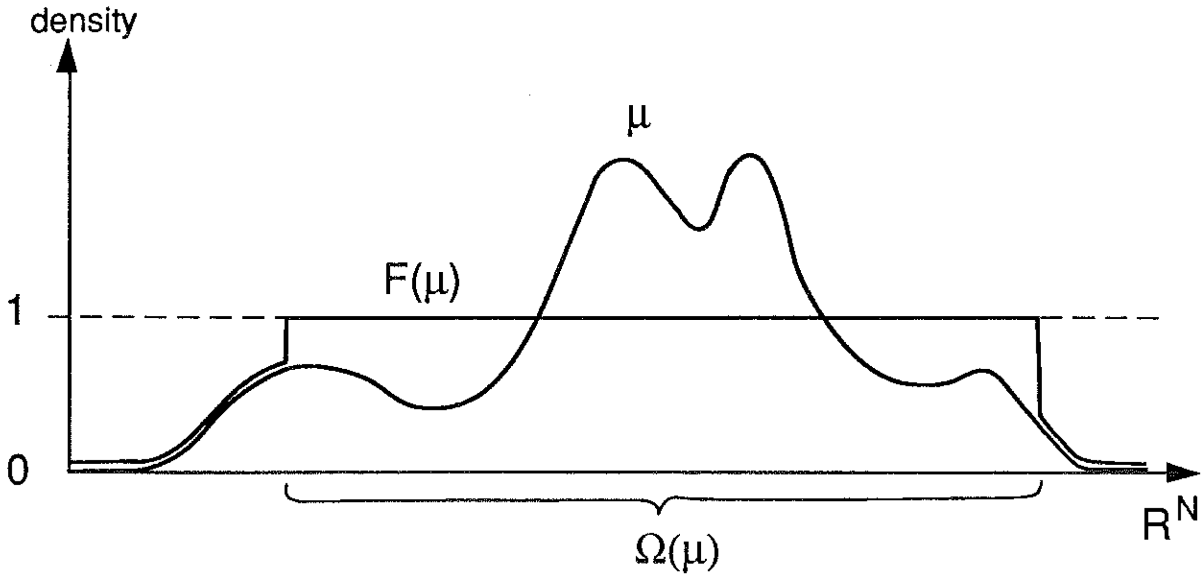

Fig. 4

3. A partial order.

The following partial order $\prec$ among mass distributions and bodies turns out to be useful.

Definition:

$$\mu_1 \prec \mu_2 \quad \text{if} \quad U^{\mu_1} \geq U^{\mu_2} \quad \text{in} \quad \mathbb{R}^N,$$
$$\Omega_1 \prec \Omega_2 \quad \text{if} \quad U^{\Omega_1} \geq U^{\Omega_2} \quad \text{in} \quad \mathbb{R}^N.$$

My experience is that among mass distribution which are graviequivalent in the sense (1.4) $\mu_1 \prec \mu_2$ means in some way that μ_1 is "more concentrated" than μ_2 (or Ω_1 is "more solid" than Ω_2 in the case $\Omega_1 \prec \Omega_2$). In Example 3, e.g., we have $\Omega(t_1) \prec \Omega(t_2)$ if and only if $t_1 \leq t_2$.

Some simple results about $\prec$ are:

- Suppose $\Omega_1, \Omega_2 \in Q(\mu, HL^1)$. Then

(3.1) if $\Omega_1 \prec \Omega_2$ then $\partial\Omega_1 \subset \overline{\Omega}_2$ (the closure);

(3.2) if $\Omega_1 \not\prec \Omega_2$ then there exists $x \in \Omega_2 \backslash \Omega_1$ such that $U^{\Omega_2}(x) > U^{\mu}(x)$, $\nabla U^{\Omega_2}(x) = \nabla U^{\mu}(x)$.

If $\Omega_2 \in Q(\mu, SL^1)$ then, by (1.7), (3.2) cannot occur. Thus

(3.3) • If $\Omega_1 \in Q(\mu, HL^1), \Omega_2 \in Q(\mu, SL^1)$ then $\Omega_1 \prec \Omega_2$.

So, among all bodies graviequivalent to μ in the sense (1.3) the one (if any) obtained by Zidarov bubbling is the least solid one and the boundary of any of the other ones is contained in its closure. Even if $F(\mu)$ is not of the form (2.5), so that $Q(\mu, SL^1)$ is empty, one has $\Omega_1 \prec F(\mu)$ and $\partial\Omega_1 \subset \overline{\Omega(\mu)}$ for any $\Omega_1 \in Q(\mu, HL^1)$. Note in this connection that in terms of $\prec$ the second definition of F can be written

$$F(\mu) = \inf \{\nu \in M : \mu \prec \nu, \nu \leq 1\},$$

where the infimum is taken with respect to $\prec$.

A consequence of (3.1) is:

- If $\Omega \in Q(\mu, HL^1)$ is solid then Ω is minimal in $Q(\mu, HL^1)$ with respect to $\prec$. (The same is true with AL^1 in place of HL^1.)

  Combining this with (3.3) gives:

- If the body $\Omega \in Q(\mu, SL^1)$ obtained by Zidarov bubbling of μ turns out to be solid, then it is the only body graviequivalent to μ in the sense (1.3), i.e. $Q(\mu, HL^1) = \{\Omega\}$.

Example: If $\mu \geq 0$ and has support in a hyperplane then it follows from the result in § 4 that $\Omega \in Q(\mu, SL^1)$ exists and is solid. Thus the answer of (Q) is "no" for such μ.

An interesting operation which can be performed within $Q(\mu, AL^1)$ and $Q(\mu, HL^1)$ is that of the least upper bound with respect to $\prec$: if $\Omega_1, \Omega_2 \in Q(\mu, AL^1)$ then there exists $\Omega = \Omega_1 \vee \Omega_2 \in Q(\mu, AL^1)$ such that $\Omega_1 \prec \Omega, \Omega_2 \prec \Omega$ and such that $\Omega \prec \Omega'$ for any Ω' with the same properties. Moreover, if Ω_1, Ω_2 are in $Q(\mu, HL^1)$ then so is Ω.

For $Q(\mu, AL^1)$ we also have the following: if $\Omega_0, \Omega_1 \in Q(\mu, AL^1)$ with $\Omega_0 \prec \Omega_1$ then there exists a chain $\Omega(t) \in Q(\mu, AL^1)$, $0 \leq t \leq 1$, joining Ω_0 and Ω_1 (i.e. $\Omega(0) = \Omega_0$, $\Omega(1) = \Omega_1$ and $\Omega(t_1) \prec \Omega(t_2)$ when $t_1 \leq t_2$). Cf. Example 3. Combining this with the preceding result shows that the family $Q(\mu, AL^1)$ always is connected (in particular it is a "continuous" family): any two Ω_1, $\Omega_2 \in Q(\mu, AL^1)$ can be continuously deformed into each other within $Q(\mu, AL^1)$, namely via $\Omega_1 \vee \Omega_2$. This result is particularly interesting in connection with the question (equivalent to (Q)) whether $Q(\mu, AL^1)$ can contain two different solid bodies.

4. On the geometry of the bubbling process.

Let μ be a positive mass distribution and assume that supp μ is contained in (e.g.) the lower half-space $H = \{x_N \leq 0\}$ $(x = (x_1, \ldots, x_N))$. Then one can show [Gu – Sa] that the part of $F(\mu)$ which bubbles up above H is a body which is the subgraph of a real analytic function. In other words $F(\mu)|_{H^c} = \chi_D$ where $D = \Omega(\mu) \backslash H$ is of the form $D = \{x : 0 < x_N < \varphi(x_1, \ldots, x_{N-1})\}$ for some real analytic function φ. Moreover $\widetilde{D} \subset \Omega(\mu)$ where $\widetilde{D}$ denotes the reflexion of D in $x_N = 0$. (Fig. 5.)

If, given $\mu \geq 0$, we apply this result to all half-spaces containing supp μ we obtain a lot of interesting information about $F(\mu)$ outside the convex hull K of supp μ, e.g.:

- $\partial\Omega(\mu) \backslash K$ is real analytic;
- $\Omega(\mu) \cup K$ is solid (no cavities);
- for any $x \in \partial\Omega(\mu) \backslash K$ the inward normal N_x of $\partial\Omega(\mu)$ at x intersects K (if $N = 2$ and supp μ is connected it even intersects supp μ). (Fig. 6.)

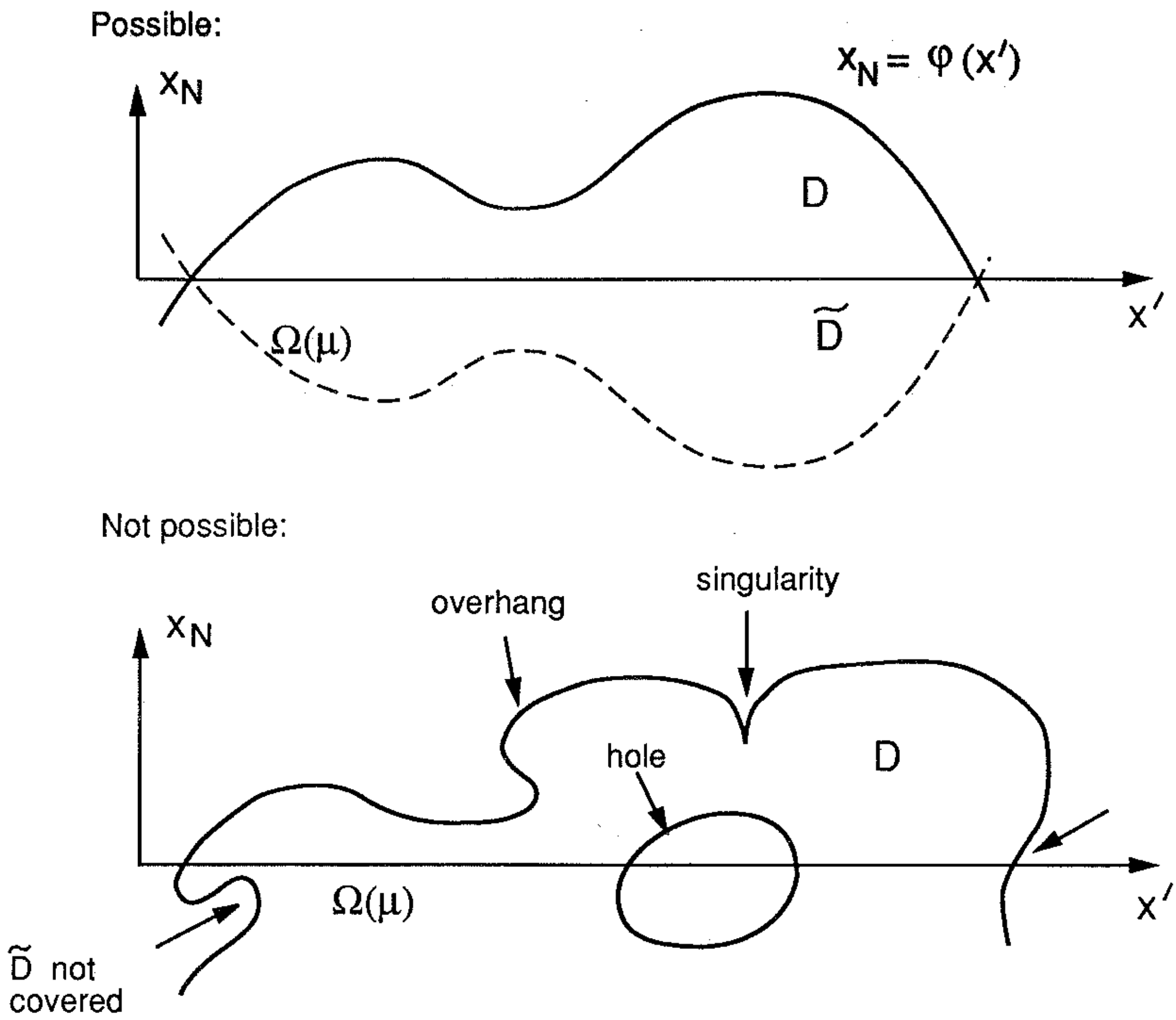
Possible:
xN
xN = φ (x′)
D
x′
Ω(μ)
D̃
Not possible:
singularity
overhang
xN
hole
D
x′
Ω(μ)
D̃ not
covered

Fig. 5

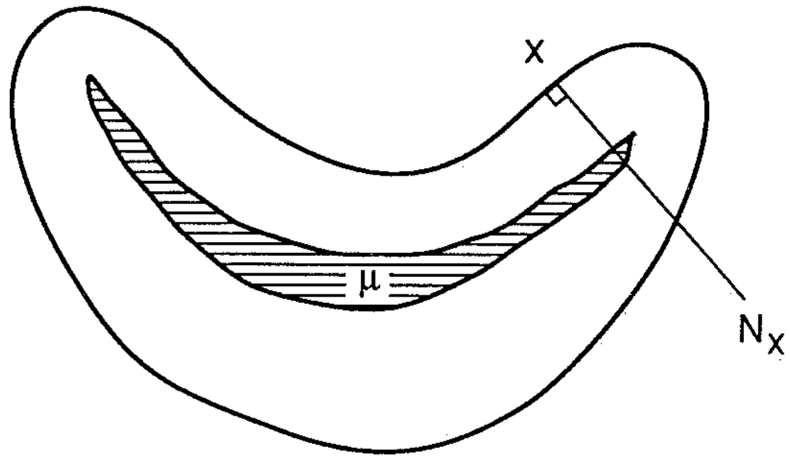
x
μ
Nx

Fig. 6

One can also use these results in the reverse direction: given a body Ω one may ask for mass distributions $\mu \geq 0$ which generate Ω in the sense that $F(\mu) = \chi_\Omega$ ($\Omega \in Q(\mu, SL^1)$) (such a μ is called a "mother body" or "maternal body" by Zidarov, at least if supp μ is minimal in some sense). We then obtain information of the type indicated in fig. 7, namely saying that supp μ necessarily has to enter certain parts of Ω.

Any mother body
has to enter all
the shaded areas.

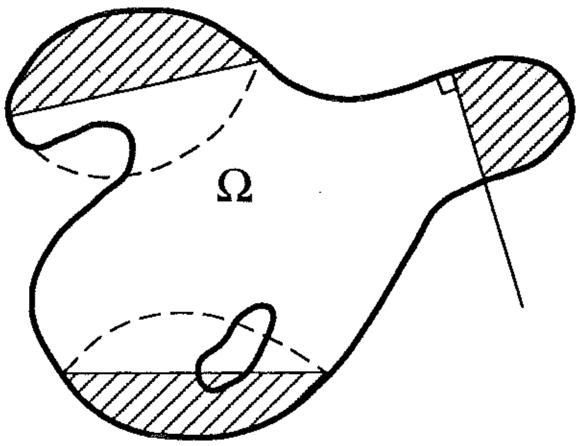

Fig. 7

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