SINGULAR AND SPECIAL POINTS ON QUADRATURE DOMAINS FROM AN ALGEBRAIC GEOMETRIC POINT OF VIEW

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0. Introduction

Inspired by an observation of Harold S. Shapiro [17] concerning the number of singular points on the boundary and certain "special" points in the interior of quadrature domains for analytic functions (see §1 for definitions) we make here a systematic investigation of such points from an algebraic geometric point of view (the boundaries of these quadrature domains are known to be algebraic). Using the so-called genus formula we thereby obtain an upper bound for the number of these singular and special points in terms of the order of the quadrature identity (Theorem 2.1). This, which is our main result, contains Shapiro's result as a special case. M. Sakai [15] has, using other methods, obtained complementary results which, among other things, give lower bounds for the same quantities.

We also give some examples, one of which, as a by-product, gives rise to an explicit solution of a certain moving boundary problem studied in [9]. This example also shows that there exist different but conformally equivalent quadrature domains which admit the same quadrature identity (for analytic functions).

Some notation

$|E|$ = area of E for $E \subset \mathbf{C}$.

$[\Omega]$: see (1.3).

$\mathrm{D}(a; r) = \{z \in \mathbf{C} : |z - a| < r\}$.

$\mathbf{P} = \mathbf{C} \cup \{\infty\}$: the Riemann sphere.

$\mathrm{AL}^1(\Omega)$, $\mathrm{SL}^1(\Omega)$: the classes of, respectively, analytic and subharmonic functions in L^1 (Ω, area measure).

If $\Omega \subset \mathbf{C}$ is a bounded domain and μ is a distribution of the form (1.1) or an L^∞-function concentrated on Ω then

$$\Omega \in Q(\mu, \mathrm{AL}^1) \text{ means that } \int_\Omega f dxdy = \mu(f) \text{ for all } f \in \mathrm{AL}^1(\Omega).$$

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Provided that μ moreover is a positive measure

$$\Omega \in Q(\mu, \mathrm{SL}^1) \text{ means that } \int_\Omega f dxdy \geqq \mu(f) \text{ for all } f \in \mathrm{SL}^1(\Omega).$$

1. Preliminaries

Let Ω be a bounded domain in the complex plane and denote the class of integrable (with respect to area measure) analytic functions in Ω by $\mathrm{AL}^1(\Omega)$. Further, let μ be a functional (distribution) of the form

$$\mu(f) = \sum_{k=1}^{m} \sum_{j=0}^{n_k - 1} c_{kj} f^{(j)}(z_k) \tag{1.1}$$

($f \in \mathrm{AL}^1(\Omega)$) where $z_1, \ldots, z_m \in \Omega$, $c_{kj} \in \mathbf{C}$, $m \geqq 1$, $n_k \geqq 1$. If the identity (*quadrature identity*)

$$\int_\Omega f dxdy = \mu(f) \tag{1.2}$$

holds for all $f \in \mathrm{AL}^1(\Omega)$ then Ω is called a *quadrature domain* (for the class of analytic functions) for μ. Following [13] we then write $\Omega \in Q(\mu, \mathrm{AL}^1)$ (i.e. $Q(\mu, \mathrm{AL}^1)$ denotes the class of (bounded) quadrature domains for μ). We refer to [1], [4], [6], [7], [13], [14] for more background on quadrature identities.

In the expression (1.1) we can clearly assume that the z_k are distinct and that $c_{k,n_k-1} \neq 0$ ($k = 1, \ldots, m$). Then the integer $n = n_1 + \cdots + n_m$ is called the *order* of μ (or of the identity (1.2)). We shall now recall some known facts about quadrature domains for analytic functions.

Assume $\Omega \in Q(\mu, \mathrm{AL}^1)$ with μ as in (1.1) and of order n. It is shown in [1] that $\partial\Omega$ is a subset of an algebraic curve. From this it easily follows that int clos Ω is a domain bounded by finitely many continua and that it coincides with

$$[\Omega] = \{z \in \mathbf{C} : \exists r > 0 \text{ such that } |\mathbf{D}(z; r) \setminus \Omega| = 0\}, \tag{1.3}$$

the areal completion of Ω. Since

$$\Omega \subset [\Omega], \qquad |[\Omega]| = |\Omega|$$

$[\Omega]$ satisfies the same quadrature identity (1.2) as Ω (i.e. $[\Omega] \in Q(\mu, \mathrm{AL}^1)$).

To $[\Omega]$ the theory in [7] can be applied. Thus, if W is any plane domain bounded by finitely many smooth analytic curves and conformally equivalent to $[\Omega]$ and $\phi : W \to [\Omega]$ is any conformal map then ϕ extends to a meromorphic function on the (Schottky) double $\hat{W} = W \cup \partial W \cup \tilde{W}$ of W (i.e. the compact Riemann surface obtained by completing W with a back-side $\tilde{W}$, provided with the opposite conformal structure and glued with W along ∂W; see e.g. [3]). On $\hat{W}$ there is a natural anticonformal involution $\zeta \to \tilde{\zeta}$: if $\zeta \in W$ (resp. $\tilde{W}$) then $\tilde{\zeta}$ is the corresponding point on $\tilde{W}$ (resp. W), if $\zeta \in \partial W$ then $\tilde{\zeta} = \zeta$. If $\zeta_1, \ldots, \zeta_m \in W$ are the points mapped onto $z_1, \ldots, z_m \in \Omega$ respectively by ϕ then ϕ has poles of orders n_k at $\tilde{\zeta}_k$ $(k = 1, \ldots, m)$ and no other singularities.

In particular, the order of ϕ (i.e. the number of times it takes almost every value in $\mathbf{P}$) equals n, the order of the quadrature identity. Another meromorphic function, ϕ^*, of order n on $\hat{W}$ can be defined by

$$\phi^*(\zeta) = \overline{\phi(\tilde{\zeta})} \qquad (\zeta \in \hat{W}).$$

It is shown in [7] that ϕ and ϕ^* generate the field of meromorphic functions on $\hat{W}$ and that they are related by a polynomial relation of the form

$$P(\phi, \phi^*) \equiv 0, \tag{1.4}$$

where

$$P(z, w) = \sum_{k,j=0}^{n} a_{kj} z^k w^j, \tag{1.5}$$

n is the same as above, $a_{kj} = \bar{a}_{jk}$ and $a_{nn} \neq 0$. Moreover $P(z, w)$ is irreducible (in $\mathbf{C}(z)[w]$ for example) and is uniquely determined after the normalization $a_{nn} = 1$ (henceforth assumed).

The hermitean nature of the coefficient matrix (a_{kj}) means that $P(z, \bar{z})$ is a real-valued function $(z \in \mathbf{C})$. Set

$$\begin{aligned}
U &= \{z \in \mathbf{C} : P(z, \bar{z}) < 0\},\\
V &= \{z \in \mathbf{C} : P(z, \bar{z}) = 0\},\\
E &= \{z \in V : \exists\, r > 0 \text{ such that } \mathbf{D}(z; r) \cap V = \{z\}\}\\
&= \{\text{isolated points in } V\},\\
B &= \left\{z \in V : \frac{\partial}{\partial \bar{z}} P(z, \bar{z}) = 0\right\}.
\end{aligned}$$

The points in E turn out to be the same as Shapiro's ([17]) "special points". Also, in the terminology of Sakai [13], $E = E([\Omega]; \mu, AL^1)$. These statements follow from Lemma 1.1 below.

In our context it is natural to call a point $(z, w) \in \mathbf{C}^2$ *real* if $w = \bar{z}$. Thus V can be identified with the real locus of the algebraic curve $P(z, w) = 0$ and B with its set of singular points in the real. (Observe that

$$\left.\frac{\partial}{\partial z} P(z, \bar{z}) = \overline{\frac{\partial}{\partial \bar{z}} P(z, \bar{z})}.\right)$$

Since $P(z, w)$ is irreducible there are only finitely many singular points. In particular, B is finite. Finally notice that $E \subset B$.

If ν is a distribution with compact support in $\mathbf{C}$ we denote its Cauchy-transform by $\hat{\nu}$:

$$\hat{\nu}(z) = \left\langle \nu(\zeta), \frac{1}{\zeta - z} \right\rangle$$

whenever it is defined. Thus $\partial\hat{\nu}/\partial\bar{z} = -\pi\nu$ and $\hat{\nu}(z) \to 0$ as $|z| \to \infty$. Since, as a function of ζ, $1/(\zeta - z) \in \mathrm{AL}^1(\Omega)$ for every $z \in \mathbf{C} \setminus \Omega$ and the linear combinations of these functions are dense in $\mathrm{AL}^1(\Omega)$ [5] we see that $\Omega \in Q(\mu, \mathrm{AL}^1)$ is equivalent to that

$$\hat{\chi}_\Omega = \hat{\mu} \qquad \text{in } \mathbf{C} \setminus \Omega. \tag{1.6}$$

Since

$$\frac{\partial}{\partial \bar{z}} \hat{\chi}_\Omega = \frac{\partial}{\partial \bar{z}} \hat{\chi}_{[\Omega]} = -\pi\chi_{[\Omega]},$$

$\hat{\chi}_\Omega(z) + \pi\bar{z}$ is holomorphic in $[\Omega]$ and we can define a meromorphic function S in $[\Omega]$ by

$$S(z) = \bar{z} + \frac{1}{\pi}(\hat{\chi}_\Omega(z) - \hat{\mu}(z)). \tag{1.7}$$

By (1.6) S extends continuously to clos Ω with

$$S(z) = \bar{z} \qquad \text{on } \partial\Omega \supset \partial[\Omega]. \tag{1.8}$$

Thus S is the so-called Schwarz function for $\partial[\Omega]$ (or $\partial\Omega$) [6].

The function $S(\phi(\zeta))$ is meromorphic in W and it coincides with ϕ^* on ∂W since, for $\zeta \in \partial W$, $\phi(\zeta) \in \partial[\Omega]$ and $\phi^*(\zeta) = \overline{\phi(\zeta)}$. Therefore

$$S(\phi(\zeta)) = \phi^*(\zeta) \qquad (\zeta \in W) \tag{1.9}$$

identically, and

$$P(z, S(z)) = 0 \qquad (z \in [\Omega]) \tag{1.10}$$

by (1.4). (1.10) shows that $S(z)$ is an algebraic function and hence has analytic extensions (possibly with branch points) across $\partial[\Omega]$. (1.10) also shows, by (1.8), that $P(z, \bar{z}) = 0$ on $\partial[\Omega]$ (even on $\partial\Omega$), i.e. that $\partial[\Omega] \subset V$.

Lemma 1.1. (a) $V \setminus E = \phi(\partial W) = \partial[\Omega]$.

(b) $$E = \{\phi(\zeta) : \zeta \in W, \phi(\zeta) = \phi(\bar{\zeta})\} = \{z \in [\Omega] : S(z) = \bar{z}\}$$
$$= \{z \in [\Omega] : \hat{\chi}_\Omega(z) = \hat{\mu}(z)\}$$
$$= V \cap [\Omega].$$

(c) $[\Omega] = U \cup E$. *Also: for every subset F of E $U \cup F$ is a quadrature domain for μ (and Ω is one of them) and no domain strictly included in U or strictly including $U \cup E$ is a quadrature domain for μ.*

Proof. (a) was proved in [7].

(b) The third equality follows directly from the definition of S. We now prove

$$E \subset \{\phi(\zeta) : \zeta \in W, \phi(\zeta) = \phi(\bar{\zeta})\} \tag{1.11}$$
$$\subset \{z \in [\Omega] : S(z) = \bar{z}\} \subset V \cap [\Omega] \subset E.$$

Suppose $z \in E$. Since $P(z, \bar{z}) = 0$ there exists at least one point $\zeta \in \hat{W}$ such that $(\phi(\zeta), \phi^*(\zeta)) = (z, \bar{z})$ (cf. §3), i.e. such that $\phi(\zeta) = \phi(\bar{\zeta}) = z$. If $\zeta \in \partial W$ then z is not isolated in V since neighbouring $\zeta_j \in \partial W$ give rise to neighbouring $z_j = \phi(\zeta_j) \in \phi(\partial W) \subset V$. Thus either ζ or $\bar{\zeta}$ belongs to W which proves the first inclusion in (1.11).

The second inclusion in (1.11) follows directly from (1.9).

If $z \in [\Omega]$, $S(z) = \bar{z}$ then $P(z, \bar{z}) = P(z, S(z)) = 0$ by (1.10) so that $z \in V$, proving the third inclusion. Finally, $V \cap [\Omega] \subset E$ follows from (a). This proves (b).

(c) $E \subset [\Omega]$ is clear by (b). Since the singular set B is finite and $\partial[\Omega] \setminus B = V \setminus B$, $P(z, \bar{z})$ always changes sign across $\partial[\Omega]$. On the other hand $P(z, \bar{z})$ never changes sign in $[\Omega]$ since $[\Omega] \cap V = E$ is finite. Since finally $P(z, \bar{z}) > 0$ for $|z|$ large (by the normalization $a_{nn} = 1$) one concludes that $\{z \in \mathbf{C} : P(z, \bar{z}) < 0\} = [\Omega] \setminus E$, i.e. $[\Omega] = U \cup E$.

From (1.6) and (b) we see that if F is any subset of $[\Omega]$ then $[\Omega] \setminus F \in Q(\mu, \mathrm{AL}^1)$ if and only if $F \subset E$. Also any domain strictly including $[\Omega]$ has larger area than $[\Omega]$ and hence cannot be in $Q(\mu, \mathrm{AL}^1)$ (by choosing $f = 1$ in (1.2)). End of proof.

Finally in this section we shall introduce a convenient notation. Let Ω_1, Ω_2 be bounded domains in $\mathbf{C}$. Then, by [13], [14], there exists an essentially unique bounded open set $\Omega \supset \Omega_1 \cup \Omega_2$ such that

$$\int_{\Omega_1} f + \int_{\Omega_2} f \leqq \int_{\Omega} f \tag{1.12}$$

for all $f \in \mathrm{SL}^1(\Omega)$, the class of all subharmonic functions in $L^1(\Omega)$. To make Ω uniquely determined we require Ω to be minimal with the above properties. We set

$$\Omega_1 + \Omega_2 = \Omega$$

for Ω so defined. (Ω may e.g. be obtained as $\Omega = \{z \in D : u(z) > 0\}$ where $D \subset \mathbf{C}$ is a sufficiently large disc and $u \in H_0^1(D)$ (Sobolev space) minimizes $\int_D |\nabla u|^2 dxdy$ under the constraint $\Delta u + \chi_{\Omega_1} + \chi_{\Omega_2} \leqq 1$. Then $\Delta u + \chi_{\Omega_1} + \chi_{\Omega_2} = \chi_\Omega$ for the minimizing u.)

If $\Omega_1 \cap \Omega_2 = \varnothing$ then $\Omega_1 + \Omega_2$ simply equals $\Omega_1 \cup \Omega_2$. If $\Omega_1 \cap \Omega_2 \neq \varnothing$ then $\Omega_1 + \Omega_2$ is connected and is the minimum domain in $Q((\chi_{\Omega_1} + \chi_{\Omega_2})dxdy, \mathrm{SL}^1)$ (see [13], [14]).

In both cases it is obvious that if $\Omega_j \in Q(\mu_j, \mathrm{AL}^1)$, i.e.

$$\int_{\Omega_j} f = \mu_j(f) \qquad \text{for all } f \in \mathrm{AL}^1(\Omega_j) \qquad (j = 1, 2),$$

then

$$\int_{\Omega_1 + \Omega_2} f = (\mu_1 + \mu_2)(f) \qquad \text{for all } f \in \mathrm{AL}^1(\Omega_1 + \Omega_2),$$

so that $\Omega_1 + \Omega_2 \in Q(\mu_1 + \mu_2, \mathrm{AL}^1)$ whenever $\Omega_1 + \Omega_2$ is connected.

Obviously the above construction generalizes to define $\Omega_1 + \cdots + \Omega_N$ for arbitrary $N \geqq 2$. We then have $\Omega_1 + \Omega_2 + \Omega_3 = (\Omega_1 + \Omega_2) + \Omega_3$, etc. (e.g. by [13, Proposition 3.10]). (The construction also generalizes without changes to higher dimensions.)

2. Statement of the main result

Let Ω be a bounded quadrature domain as in §1, namely $\Omega \in Q(\mu, \mathrm{AL}^1)$ with μ of order n, given by (1.1). For simplicity of notation we now assume that Ω is maximal, i.e. that $\Omega = [\Omega]$. Retain the remaining assumptions and notations of §1. In particular $\phi : W \to \Omega$ is a conformal map, ϕ meromorphic on $\hat{W}$ and $P(z, w)$ is the polynomial (1.5) associated with Ω. (Clearly $P(z, w)$ depends only on Ω, i.e. is independent of the choice of W and ϕ when Ω is fixed.)

Let C denote the set of *cusp points* and D the set of *double points* of $\partial\Omega$. This means that

$$C = \{z \in \partial\Omega : z = \phi(\zeta) \text{ for some } \zeta \in \partial W \text{ with } \phi'(\zeta) = 0\},$$

$$D = \{z \in \partial\Omega : z = \phi(\zeta_1) = \phi(\zeta_2) \text{ for two different } \zeta_j \in \partial W\}.$$

Recall that E and B were defined in §1. Set

$$c = \operatorname{card} C, \quad d = \operatorname{card} D, \quad e = \operatorname{card} E,$$

$$p = (\text{connectivity of } \Omega) - 1$$
$$= \text{genus of } \hat{W}$$

(card = "cardinality of"). Our main result is the following.

Theorem 2.1. *We have $B = C \cup D \cup E$ and*

$$p + \sum_{z \in B} \delta_z \leqq (n-1)^2, \tag{2.1}$$

where δ_z are certain integers $\geqq 1$ satisfying (i)–(iii) *below. Moreover, the difference between the two members in* (2.1) *is an even number.*

(i) *For $z \in C$, $\delta_z = \frac{1}{2}(r-1)$, where r is the least odd number with $c_r \neq 0$ in the Puiseaux-series*

$$S(z+t) = \bar{z} + \sum_{k=2}^{\infty} c_k t^{k/2} \qquad (t \in \mathbf{C} \text{ small}).$$

(z is a branch point of S when $z \in C$.)

(ii) *For $z \in D$, δ_z is the least number r for which $S_1^{(r)}(z) \neq S_2^{(r)}(z)$ (the rth derivatives), where S_1 and S_2 denote the two branches of the Schwarz function at z. This gives in particular $\delta_z \geqq 2$.*

(iii) *For $z \in E$, $\delta_z = 1$ if $|S'(z)| \neq 1$, $\delta_z \geqq 2$ if $|S'(z)| = 1$. In the latter case we have more precisely: the function $S^*(\zeta) = \overline{S(\bar{\zeta})}$ has an inverse $T = S^{*-1}$ near $\zeta = \bar{z}$ and δ_z is the least number r for which $T^{(r)}(z) \neq S^{(r)}(z)$.*

Corollary. $p + c + 2d + e \leqq (n-1)^2$.

Remark 2.1. From Sakai [15] one derives the complementary inequality

$$c + e \geqq p + n - 1.$$

The proof of the theorem is contained in §3 below.

3. Classification of singularities

In this section we classify the singularities of the algebraic curve $P(z, w) = 0$ and relate them to the singularities of $\partial\Omega$. From this we obtain Theorem 2.1 above by applying the so-called genus formula for algebraic curves.

With notations and assumptions as in §2, let $Q(t, z, w)$ be the homogenization of $P(z, w)$, i.e.

$$Q(t, z, w) = t^{2n} P\left(\frac{z}{t}, \frac{w}{t}\right) = \sum_{k,j=0}^{n} a_{kj} t^{2n-k-j} z^k w^j.$$

Let

$$\operatorname{loc} Q = \{(t : z : w) \in \mathbf{P}_2(\mathbf{C}) : Q(t, z, w) = 0\},$$

$$\operatorname{loc} P = \{(z, w) \in \mathbf{C}^2 : P(z, w) = 0\} \cong \{(1 : z : w) \in \mathbf{P}_2(\mathbf{C}) : Q(1, z, w) = 0\}.$$

Thus $\operatorname{loc} Q$ is the completion of $\operatorname{loc} P$ in projective space $\mathbf{P}_2(\mathbf{C})$. Set

$$A = \{\zeta \in \hat{W} : \phi^* \text{ has a pole at } \zeta\},$$

$$\tilde{A} = \{\tilde{\zeta} \in \hat{W} : \zeta \in A\} = \{\zeta \in \hat{W} : \phi \text{ has a pole at } \zeta\}.$$

Then $A \subset W, \tilde{A} \subset \tilde{W}$.

Consider the map

$$\Phi : \hat{W} \setminus (A \cup \tilde{A}) \to \operatorname{loc} P \tag{3.1}$$

given by $\zeta \to (\phi(\zeta), \phi^*(\zeta))$. By regarding $\operatorname{loc} P$ as a subset of $\mathbf{P}_2(\mathbf{C})$ and by setting

$$\Phi(\zeta) = \begin{cases} (0 : 0 : 1) & \text{for } \zeta \in A, \\ (0 : 1 : 0) & \text{for } \zeta \in \tilde{A}, \end{cases}$$

Φ extends continuously to a map

$$\Phi : \hat{W} \to \operatorname{loc} Q. \tag{3.2}$$

Since Q is irreducible and $\hat{W}$ is compact, (3.2) is surjective and it follows that also (3.1) is surjective. Further, due to the univalence of $\phi|_W$, Φ is "essentially one-to-one", i.e. $\Phi^{-1}(\{a\})$ consists of more than one point just for finitely many $a \in \operatorname{loc} Q$. (Examples of such points $a \in \operatorname{loc} Q$ are $(0 : 0 : 1)$ and $(0 : 1 : 0)$ if $m > 1$.)

We first classify the points in $\hat{W}$ with respect to their behaviour under Φ. There are three cases to distinguish for a point $\zeta \in \hat{W}$.

(I) $\zeta \notin A \cup \tilde{A}$ and $\phi'(\zeta)$ and $\phi^{*\prime}(\zeta)$ are not both equal to zero. Then the branch of $\operatorname{loc} Q$ which is parametrized by Φ near ζ is nonsingular at $\Phi(\zeta)$ (which does not exclude that $\Phi(\zeta)$ is a singular point of $\operatorname{loc} Q$ since another branch may be crossing at $\Phi(\zeta)$).

(II) $\zeta \notin A \cup \tilde{A}$ and $\phi'(\zeta) = \phi^{*\prime}(\zeta) = 0$. Since $\phi|_W$ (and hence $\phi^*|_{\tilde{W}}$) is univalent, this can only occur for $\zeta \in \partial W$. Then $z = \phi(\zeta) \in \phi(\partial W) = \partial\Omega$ is a cusp point of $\partial\Omega$ ($z \in C$). Also, $\phi''(\zeta)$ (and $\phi^{*\prime\prime}(\zeta)$) must be non-zero due to the univalence.

It follows that, in terms of a local parameter t on $\hat{W}$ with $t = 0$ at ζ,

$$\phi(\zeta + t) = z + a_2 t^2 + a_3 t^3 + \cdots,$$

$$\phi^*(\zeta + t) = \bar{z} + b_2 t^2 + b_3 t^3 + \cdots,$$

with $a_2, b_2 \neq 0$. Since $a_2 \neq 0$ we can define a new parameter τ near ζ by

$$\tau = t\sqrt{a_2 + a_3 t + \cdots}$$

(one of the branches chosen). Then

$$\phi(\zeta + \tau) = z + \tau^2, \tag{3.3}$$

$$\phi^*(\zeta + \tau) = \bar{z} + \sum_{k=2}^{\infty} c_k \tau^k, \tag{3.4}$$

for suitable $c_k \in \mathbf{C}$. Here $c_k \neq 0$ for at least one odd value of k, for otherwise we would have a contradiction to the fact that Φ is essentially one-to-one. In terms of the Schwarz function, (3.3), (3.4) becomes, with $\sigma = \tau^2$ and using (1.9),

$$S(z + \sigma) = \bar{z} + \sum_{k=2}^{\infty} c_k \sigma^{k/2} \qquad \text{for } \sigma \text{ small.}$$

Following [12] we shall use the following terminology for cusps. Suppose $(z, w) \in \operatorname{loc} P$ and consider one irreducible branch of $\operatorname{loc} P$ through (z, w). After an affine change of cordinates in $\mathbf{C}^2$ ((z, w)-space) this branch can be parametrized in terms of some local variable t on $\hat{W}$ by $t \to (t^m, t^n + c_{n+1} t^{n+1} + \cdots)$, where m is as small as possible and $n > m$. If $m = 1$ then the branch is non-singular (smooth) at (z, w), if $m > 1$ it has a cusp of multiplicity m. If $n = m + 1$ the cusp is called simple (otherwise it is of "higher order"). The terminology "n/m-cusp" will also be used.

Thus (3.3), (3.4) parametrizes a cusp of multiplicity two and the cusp is simple if and only if $c_3 \neq 0$. (Observe that the possible τ^2-term in (3.4) disappears by subtracting a constant multiple of (3.3) from (3.4).)

(III) $\zeta \in A \cup \tilde{A}$. Assume $\zeta \in A$ for example. Then $\phi(\zeta)$ is one of the points $z_1, \ldots, z_m$ in (1.1), say z_k. Then ϕ^* has a pole of order n_k at ζ and ϕ is regular and univalent at ζ. This means that if t is a local variable on W near ζ, with $t = 0$ at ζ, then $\Phi(\zeta + t) = (1 : \phi(\zeta + t) : \phi^*(\zeta + t))$ behaves like $(1 : z_k + ct : t^{-n_k}) = (t^{n_k} : z_k t^{n_k} + ct^{n_k+1} : 1)$ near $t = 0$, where $c = \phi'(\zeta) \neq 0$. Thus Φ parametrizes, in a neighbourhood of ζ, a simple cusp of multiplicity n_k of $\operatorname{loc} Q$ at $(0 : 0 : 1)$. Similarly for $\zeta \in \tilde{A}$.

Next we classify the points in $\operatorname{loc} Q = \operatorname{loc} P \cup \{(0:0:1), (0:1:0)\}$. For $(z, w) \in \operatorname{loc} P$ we distinguish two cases.

(A) $(z, w) = \Phi(\zeta)$ for only one point $\zeta \in \hat{W} \setminus (A \cup \tilde{A})$. If (I) above holds for ζ, then (z, w) is a non-singular point of $\operatorname{loc} P$. If (II) holds, then $(z, w) = (z, \bar{z})$ (since $\phi^*(\zeta) = \overline{\phi(\zeta)}$ for $\zeta \in \partial W$) with $z \in \partial\Omega$ a cusp point ($z \in C$). Moreover, by (3.3), (3.4), (z, w) is a cusp of multiplicity two on $\operatorname{loc} P$ and therefore (z, w) is a singular point of multiplicity two of $\operatorname{loc} P$. (The multiplicity of a singular point (z, w) of $\operatorname{loc} P$ is defined as the degree of the lowest order terms actually occurring in the Taylor expansion of $P(z, w)$ at (z, w).)

(B) $(z, w) = \Phi(\zeta_1) = \Phi(\zeta_2)$ for (at least) two different points $\zeta_1, \zeta_2 \in \hat{W} \setminus (A \cup \tilde{A})$. The conditions on ζ_1, ζ_2 can be written

$$\text{(3.5)} \qquad \begin{cases} \phi(\zeta_1) = \phi(\zeta_2) = z, \\ \phi(\tilde{\zeta}_1) = \phi(\tilde{\zeta}_2) = \bar{w}. \end{cases}$$

Because of the univalence of $\phi\,|_W$, (3.5) can occur only if

(1) $\zeta_1 \in W$, $\zeta_2 \in \tilde{W}$ or conversely

or

(2) $\zeta_1, \zeta_2 \in \partial W$.

In both cases it follows from the univalence of $\phi\,|_W$ that we cannot have $(z, w) = \Phi(\zeta_3)$ for a third point $\zeta_3 \in \hat{W} \setminus (A \cup \tilde{A})$ and that (I) (not (II)) holds for both of ζ_1 and ζ_2. Therefore (z, w) is an intersection between exactly two smooth branches of loc P. These branches have the same tangent directions at (z, w) if and only if

$$\text{(3.6)} \qquad \frac{\phi^{*\prime}(\zeta_1)}{\phi'(\zeta_1)} = \frac{\phi^{*\prime}(\zeta_2)}{\phi'(\zeta_2)}.$$

Consider case (1). Since either ζ_1 or ζ_2 belongs to W we have $z \in \Omega$. Similarly $\bar{w} \in \Omega$. Let us assume that $\zeta_1 \in W$, $\zeta_2 \in \tilde{W}$. We have two subcases.

(a) $\zeta_2 = \tilde{\zeta}_1$. Then $(z, w) = (z, \bar{z})$. Thus $z \in V$. Since moreover $z \notin \phi(\partial W)$ we have $z \in E$ by Lemma 1.1. Using (1.9) the criterion (3.6) for tangency becomes $S'(z) = 1/\overline{S'(z)}$, i.e. $|S'(z)| = 1$.

(b) $\zeta_2 \neq \tilde{\zeta}_1$. Then $\bar{w} \neq z$ due to the univalence of $\phi\,|_W$. It follows that $z \notin V$ (and $\bar{w} \notin V$); for if $P(z, \bar{z}) = 0$ there would be some $\zeta \in \hat{W} \setminus (A \cup \tilde{A})$ with $\phi(\zeta) = z$, $\phi(\tilde{\zeta}) = z$ and if say $\zeta \in W$ (recall that $z \notin \phi(\partial W)$), the univalence of $\phi\,|_W$ would give $\zeta = \zeta_1$ and the contradiction $z = \phi(\tilde{\zeta}) = \phi(\tilde{\zeta}_1) = \bar{w}$.

Thus in subcase (b) (z, w) is a singular point of loc P which does not show up in the real locus V. (This is also true for the singular points $(0:0:1)$ and $(0:1:0)$ of loc Q.) What can be said of the non-real singular points of loc P (or loc Q) is that they appear in pairs because of the symmetry of loc P: corresponding to the involution $\zeta \to \tilde{\zeta}$ on $\hat{W}$ there is the involution $(z, w) \to (\bar{w}, \bar{z})$ on loc P (extending to loc Q) and by definition, the real points of loc P are exactly the fixed points of this involution. If (z, w) is a singular point, $(\bar{w}, \bar{z})$ is clearly a singular point of the same type (multiplicity etc.).

Consider finally case (2). It is immediate that $(z, w) = (z, \bar{z})$ with z a double point of $\partial\Omega$ ($z \in D$). Also, it is straighforward to check that the two branches of loc P through (z, w) always have the same tangent directions in this case.

(C) Finally we investigate the points $(0:0:1)$ and $(0:1:0)$. Developing $Q(t, z, w)$ around $(0:0:1)$ for example we get (setting $w = 1$)

$$\text{(3.7)} \qquad \begin{aligned} Q(t, z, 1) &= \sum_{k,j=0}^{n} a_{kj} t^{2n-k-j} z^k \\ &= \sum_{k=0}^{n} a_{kn} t^{n-k} z^k + \sum_{k=0}^{n} a_{k,n-1} t^{n-k+1} z^k + \cdots, \end{aligned}$$

TABLE 1

Type of pre-image in $\hat{W}$	Points in loc Q				
	Type in classification	Appearance in the real	Multi-plicity	Type of singularity in $\mathbf{P}_2(\mathbf{C})$	
(I)	(A)	$V\backslash B$ or non-real	1	non-singular	
(II)	(A)	C	2	cusp of multiplicity 2	
(I)	(B) 1a	E	2	intersection between two smooth branches	tangency iff $\lvert S'(z)\rvert = 1$
(I)	(B) 1b	non-real	2	intersection between two smooth branches	
(I)	(B) 2	D	2	intersection between two smooth branches	always tangency
(III)	(C)	non-real	n	m simple cusps of multiplicities $n_1, \ldots, n_m$ with distinct tangent directions	

which shows that $(0:0:1)$ always is a singular point of multiplicity n (recall that $a_{nn} = 1$). (This also follows from (III) above: we had m branches through $(0:0:1)$, one for each z_k, the kth branch had a simple cusp of multiplicity n_k at $(0:0:1)$ and $\Sigma_{k=1}^{m} n_k = n$. Compare below.)

It is shown in [7] that the terms in (1.5) which are of degree n and $n-1$ in w are explicity related to the data (1.1) of μ by

$$\text{(3.8)}\qquad \begin{aligned} \frac{\sum_{k=0}^{n} a_{k,n-1} z^k}{\sum_{k=0}^{n} a_{kn} z^k} &= -\frac{1}{\pi} \sum_{k=1}^{m} \sum_{j=0}^{n_k-1} \frac{j!\,c_{kj}}{(z-z_k)^{j+1}} + \text{constant} \\ &= \frac{1}{\pi}\, \hat{\mu}(z) + \text{constant}. \end{aligned}$$

(See also Remark 3.1 below.) Thus $\Sigma_{k=0}^{n} a_{kn} z^k = \Pi_{k=1}^{m} (z - z_k)^{n_k}$ so that we get, for the lowest order terms in (3.7),

$$\sum_{k=0}^{n} a_{kn} t^{n-k} z^k = \prod_{k=1}^{m} (z - z_k t)^{n_k}.$$

This shows that at $(0:0:1)$ we have m distinct tangent directions, one for each z_k, the kth direction having multiplicity n_k.

This finishes the classification of the points in loc Q. The results are summarized in Table 1. Observe that $B = C \cup D \cup E$ is a particular consequence of the classification.

In the first two columns (I), (A) etc. refer to the cases in the classification above. In the third column, if e.g. $(1:z:w)\in \operatorname{loc} Q$ is the point in question then "non-real" means that $w \neq \bar{z}$ while e.g. C means that $w=\bar{z}$ and $z\in C$.

In general, if $\operatorname{loc} Q$ is an irreducible algebraic curve of degree d in $\mathbf{P}_2(\mathbf{C})$ the genus formula says that

$$g+\sum_q \delta_q=\frac{(d-1)(d-2)}{2},$$

where g is the (geometric) genus of $\operatorname{loc} Q$, q is summed over all singular points of $\operatorname{loc} Q$ and δ_q are certain integers $\geqq 1$. (See below.)

In our case we have $d=2n$ and $g=\text{genus}\,(\hat{W})=p$. We shall now use the classification above to say as much as possible about the δ_q.

In general

$$\delta_q \geqq \frac{n_q(n_q-1)}{2} \tag{3.9}$$

where n_q is the multiplicity of the singularity at q. Moreover, equality in (3.9) holds if and only if (i) all the branches of $\operatorname{loc} Q$ through q have different tangent directions *and* (ii) each branch either is smooth or has a simple cusp at q. See [12].

From the above we immediately obtain (in our case)

$$\delta_{(0:0:1)}=\delta_{(0:1:0)}=\frac{n(n-1)}{2}.$$

For $z\in B$ we write $\delta_z=\delta_{(z,\bar{z})}$. Then $\delta_z\geqq 1$ if $z\in C$, $\delta_z\geqq 2$ if $z\in D$ (by tangency) and, if $z\in E$, $\delta_z\geqq 1$ with equality if and only if $|S'(z)|\neq 1$ (the non-tangency condition).

The more detailed assertions about δ_z in Theorem 2.1 follow by applying the technique in [12, p. 116ff]. Assume $z\in E$ for example. Then the two branches of $\operatorname{loc} P$ passing through $(z,\bar{z})$ can be parametrized by

$$\zeta \to (\zeta, S(\zeta))$$

(ζ close to z) and

$$\zeta \to (S^*(\zeta), \zeta)$$

(ζ close to $\bar{z}$) respectively. If $|S'(z)|\neq 1$ these branches have different tangent directions and we have $\delta_z=1$ by the previous paragraph.

If $|S'(z)|=1$ (or, more generally, if $S'(z)\neq 0$) $S^*(\zeta)$ has an inverse T near $\zeta=\bar{z}$ and the second parametrization can by replaced by

$$\zeta \to (\zeta, T(\zeta))$$

(ζ close to z). Now set

$$R(x, y) = (y - S(x))(y - T(x)) = y^2 - (S(x) + T(x))y + S(x)T(x)$$

($(x, y) \in \mathbf{C}^2$ close to $(z, \bar{z})$). Thus $R(\zeta, S(\zeta)) = R(\zeta, T(\zeta)) \equiv 0$ so that, close to $(z, \bar{z})$, $R(x, y) = 0$ is an equation for $\operatorname{loc} P$ of a special form (in a translated coordinate system, with the origin at $(z, \bar{z})$, it becomes a Weierstrass polynomial in y). The meromorphic differentials

$$\frac{d\zeta}{\dfrac{\partial R}{\partial y}(\zeta, S(\zeta))} \quad \text{and} \quad \frac{d\zeta}{\dfrac{\partial R}{\partial y}(\zeta, T(\zeta))}$$

have poles of certain orders (namely $= r$ in the theorem) at $\zeta = z$ and, by [12], δ_z equals one-half times the sum of these orders. This gives (iii) in the theorem.

(ii) is treated similarly with $S(\zeta)$, $T(\zeta)$ above replaced by $S_1(\zeta)$, $S_2(\zeta)$. For (i) one can use explicit formulas [11], [12] for δ for cusps or one can use the above technique slightly modified. In the latter case one can exploit the parametrization (3.3), (3.4) of $\operatorname{loc} P$ near $(z, \bar{z})$ and define

$$\begin{aligned} R(x, y) &= (y - \phi^*(\sqrt{x - z}))\cdot(y - \phi^*(-\sqrt{x - z})) \\ &= y^2 - (\phi^*(\sqrt{x - z}) + \phi^*(-\sqrt{x - z}))y + \phi^*(\sqrt{x - z})\phi^*(-\sqrt{x - z}). \end{aligned}$$

Then $R(x, y)$ is well-defined (for (x, y) close to $(z, \bar{z})$) if $\sqrt{x - z}$ and $-\sqrt{x - z}$ are the two square-roots of $x - z$, and $R(\phi(\tau), \phi^*(\tau)) = 0$. By [12] δ_z equals one-half times the order of the pole at $\tau = 0$ of the differential

$$\frac{d\phi(\tau)}{\dfrac{\partial R}{\partial y}(\phi(\tau), \phi^*(\tau))}.$$

This gives (i).

Substituting into the genus formula gives

$$p + 2\,\frac{n(n-1)}{2} + \sum_{z \in B} \delta_z + \sum_q \delta_q = \frac{(2n-1)(2n-2)}{2}$$

where q is summed over all non-real singularities in $\mathbf{C}^2$. Since these occur in pairs ((z, w) and $(\bar{w}, \bar{z})$), $\Sigma\, \delta_q$ is an even number and we obtain (2.1). By this Theorem 2.1 is proved.

Remark. It may be worth explaining why (3.8) holds. Writing $P(z, w) = \Sigma_{j=0}^{n} P_j(z) w^j$ and using that (by (1.7)) $-(1/\pi)\hat{\mu}(z)$ simply is the singular part of the Schwarz function $S(z)$, (3.8) can be written

$$S(z) + \frac{P_{n-1}(z)}{P_n(z)} = \text{holomorphic (in } \Omega).$$

Let $S_1(z), \ldots, S_n(z)$ be the branches in Ω of the algebraic function $w = w(z)$ defined by $P(z, w) = 0$. Then $S(z)$ is one of them, say $S(z) = S_1(z)$. Thus $S_1(z)$ is in particular single-valued in Ω while $S_2(z), \ldots, S_n(z)$ may have branch-points and get mixed up with each other. Since $(w - S_1(z)) \cdot \cdots \cdot (w - S_n(z)) \equiv P(z, w)/P_n(z)$,

$$-\frac{P_{n-1}(z)}{P_n(z)} = S(z) + S_2(z) + \cdots + S_n(z).$$

From this it follows that $h(z) = S_2(z) + \cdots + S_n(z)$ is single-valued in Ω. Moreover, $h(z)$ has no poles in Ω because $S_1(z), \ldots, S_n(z)$ have altogether at most n poles in Ω (since ϕ, or ϕ^*, has only n poles) and already $S_1(z) = S(z)$ has n poles. Thus $h(z)$ is holomorphic in Ω, proving (3.8).

Using

$$\sum_{1 \leq k < j \leq n} S_k(z) S_j(z) = \frac{P_{n-2}(z)}{P_n(z)}, \ldots, S_1(z) \cdot \cdots \cdot S_n(z) = (-1)^n \frac{P_0(z)}{P_n(z)}$$

one obtains, by recursion, further relations for $S(z)$, namely

$$S(z)^2 + \frac{P_{n-1}(z)}{P_n(z)} S(z) + \frac{P_{n-2}(z)}{P_n(z)} = \text{holomorphic},$$

$$S(z)^3 + \frac{P_{n-1}(z)}{P_n(z)} S(z)^2 + \frac{P_{n-2}(z)}{P_n(z)} S(z) + \frac{P_{n-3}(z)}{P_n(z)} = \text{holomorphic (in } \Omega)$$

etc. If $S(z)$ is known explicitly these relations can be used to determine $P_{n-2}, P_{n-3}, \ldots, P_0$.

Finally we wish to remark that if $\Omega_j \in Q(\mu_j, \mathrm{AL}_1)$ ($j = 1, \ldots, N$) and $P_j(z, w)$ is the polynomial associated with Ω_j then, provided the μ_j have mutually disjoint supports, the polynomial $P(z, w)$ associated with $\Omega = \Omega_1 + \cdots + \Omega_N$ (see §1) is of the form

$$P(z, w) = P_1(z, w) \cdot \cdots \cdot P_N(z, w) + R(z, w), \tag{3.10}$$

where $R(z, w)$ is a polynomial of the form

$$R(z, w) = \sum_{k, j=0}^{n-2} b_{kj} z^k w^j \tag{3.11}$$

($b_{kj} = \bar{b}_{jk}$) and n is the order of $\mu_1 + \cdots + \mu_N$ (= the sum of the orders of the μ_j). This follows by a straightforward computation using (3.8). Actually (3.10), (3.11) hold whenever $\Omega \in Q(\mu_1 + \cdots + \mu_N, \mathrm{AL}^1)$ (not necessarily of the form $\Omega_1 + \cdots + \Omega_N$).

4. Examples

We shall exemplify Theorem 2.1 for certain small values of n.

For $n = 1$ the theorem gives that $p = 0$ and that B is empty. Also it is a classical result (see [1] for references) that the only quadrature domains for $n = 1$ are the circular discs.

For $n = 2$, (2.1) becomes $p + \Sigma\, \delta_z \leqq 1$. Since the difference is to be an even number we actually must have equality. Since moreover it is known that $p = 0$ when $n = 2$ [4], [7] (this also follows by using Remark 2.1) we get $\Sigma\, \delta_z = 1$. It follows that $B = C \cup D \cup E$ consists of exactly one point which either is a $\frac{3}{2}$-cusp or a point in E. This result (essentially) was first obtained by Shapiro [17]. Also, both cases can occur (examples are given in [17]). Further studies of the case $n = 2$ can be found in [15].

For $n \geqq 3$ we shall consider quadrature domains $\Omega \in Q(\mu, \mathrm{AL}^1)$ where $\mu = \mu_r$ is the measure

$$\mu_r(f) = \pi r^2 \sum_{k=1}^{n} f(z_k),$$

$r > 0$, $z_k = \omega^k$, $\omega = e^{2\pi i/n}$. Observe that μ_r has certain symmetry properties: it is invariant under rotations $z \to \omega^k z$ and under reflexions in the lines $e^{k\pi i/n}\mathbf{R}$ ($k \in \mathbf{Z}$). We shall only consider domains Ω which also are invariant under these transformations. Also, for simplicity of notation we shall assume that $\Omega = [\Omega]$.

The principal example of an $\Omega \in Q(\mu_r, \mathrm{AL}^1)$ as above is, if $r > \sin(\pi/n)$,

$$\Omega = \mathbf{D}(z_1; r) + \cdots + \mathbf{D}(z_n; r)$$

($+$ defined in §1). If $r \leqq \sin(\pi/n)$ this Ω is disconnected.

In the sequel we consider a fixed domain $\Omega \in Q(\mu_r, \mathrm{AL}^1)$. Let $P(z, w) = \Sigma\, a_{kj} z^k w^j$ be its associated polynomial (as in (1.4), (1.5)). Then $P(\omega z, \overline{\omega z}) \equiv P(z, \bar{z})$ by the assumed symmetry of Ω and it follows that a_{kj} can be non-zero only when $\omega^{k-j} = 1$, i.e. when $k - j \in n\mathbf{Z}$. Since μ has order n this shows that P is of the form

$$\begin{aligned} P(z, w) = a_0 + a_1 zw + \cdots + a_{n-2}(zw)^{n-2} \\ + a_{n-1}(zw)^{n-1} + (zw)^n + a_{n0} z^n + a_{0n} w^n \end{aligned}$$

($a_j = a_{jj} \in \mathbf{R}$). Here a_{n-1}, a_{n0}, a_{0n} can be determined using (3.8). This gives

$$\begin{aligned} P(z, w) = a_0 + a_1 zw + \cdots + a_{n-2}(zw)^{n-2} \\ - nr^2(zw)^{n-1} + (zw)^n - z^n - w^n. \end{aligned} \tag{4.1}$$

Let u be $-1/2\pi$ times the logarithmic potential of $\chi_\Omega - \mu_r$ so that

$$\Delta u = \chi_\Omega - \mu_r,$$

$$\frac{\partial u}{\partial z} = -\frac{1}{4\pi}(\hat{\chi}_\Omega - \hat{\mu}_r)$$

in $\mathbf{C}$. Thus, at each point in $\mathbf{C}$, $\hat{\chi}_\Omega = \hat{\mu}_r$ if and only if $\partial u/\partial z = 0$. Hence, by (1.6), u is constant in each component of $\mathbf{C}\setminus\Omega$. In the exterior component this constant is zero because $u \to 0$ at infinity. Further, by Lemma 1.1

$$E = \{z \in \Omega : \partial u/\partial z = 0\},$$

i.e. E simply consists of the stationary points of u in Ω. Since $E = V \cap \Omega$ (Lemma 1.1) this shows that, for $z \in \Omega$,

$$P(z, \bar{z}) = 0 \qquad \text{if and only if } \frac{\partial u}{\partial z} = 0.$$

The last relation enables us to obtain some information about the coefficients $a_0, \ldots, a_{n-2}$ in (4.1) (for $n = 3$ we will obtain complete information).

Suppose first that $0 \in \Omega$. Then it follows from the symmetry that u is stationary at $z = 0$ (hence $0 \in E$) with the development

$$u(z) = u(0) + \tfrac{1}{4}|z|^2 + O(|z|^3)$$

($z \to 0$). (Observe that $\Delta u = 1$ near the origin.) In particular, u is strictly increasing on each radius emanating from the origin (in some neighbourhood of the origin).

Consider now u on the radii

$$N_k = \left\{z \in \mathbf{C}\setminus\{0\} : \arg z = \frac{\pi}{n} + k\,\frac{2\pi}{n}\right\},$$

$k = 1, \ldots, n$. Since $0 \in \Omega$ there is a (unique) point $\zeta_k \in \partial\Omega \cap N_k$ such that the segment $[0, \zeta_k)$ belongs to Ω. There are two cases to distinguish: either ζ_k is a cusp point of $\partial\Omega$ ($\zeta_k \in C$), or ζ_k is a smooth point or a double point of $\partial\Omega$ ($\zeta_k \in (V\setminus B) \cup D$). In the latter case the tangent of $\partial\Omega$ at ζ_k is perpendicular to N_k and therefore, since u solves a Cauchy problem $\nabla u = 0$ on $\partial\Omega$, $\Delta u = 1$ in Ω in a neighbourhood of ζ_k, u must be strictly decreasing (as a function of $|z|$) on $[0, \zeta_k)$ in some neighbourhood of ζ_k. (In fact one obtains

$$u(z) = u(\zeta_k) + \tfrac{1}{2}|z - \zeta_k|^2 + O(|z - \zeta_k|^3) \qquad \text{for } z \in [0, \zeta_k)$$

close to ζ_k.)

Combining this with the information about u near the origin it follows that if ζ_k is not a cusp point of $\partial\Omega$ then $u\,|_{N_k}$ must have an interior local maximum on $[0, \zeta_k)$. Since, by symmetry, the derivate of u in the direction perpendicular to N_k

always is zero (on N_k) this local maximum is a stationary point of u. Thus $(0, \zeta_k)$ contains at least one point in E (if $\zeta_k \notin C$).

To summarize the case $0 \in \Omega$ we have at least $n + 1$ points in B, namely $0 \in E$ and either n points (ζ_k) in C or another n points in E.

Consider now the case $0 \notin \Omega$. Then $\Omega \cap N_k$ contains a segment (ζ_k, η_k) with $\zeta_k, \eta_k \in \partial\Omega$. An analysis similar to the above shows that either at least one of ζ_k and η_k is a cusp point of $\partial\Omega$ or the segment (ζ_k, η_k) contains a point in E. Thus we have at least n points in C or at least n points in E. Moreover $p \geqq 1$ in this case.

Let us now stick to the case $n = 3$. By Theorem 2.1, $p + \Sigma\, \delta_z \leqq 4$ and by the above analysis, $p + \Sigma\, \delta_z \geqq 4$. Hence we have equality and it follows (using the above analysis) that we have the following four possibilities.

$$(4.2)\qquad \begin{cases} \text{(i)} & \begin{matrix} p = 0, & c = 3 \\ e = 1, & d = 0, \end{matrix} \\[2ex] \text{(ii)} & \begin{matrix} p = 0, & c = 0, \\ e = 4, & d = 0, \end{matrix} \\[2ex] \text{(iii)} & \begin{matrix} p = 1, & c = 3, \\ e = 0, & d = 0, \end{matrix} \\[2ex] \text{(iv)} & \begin{matrix} p = 1, & c = 0, \\ e = 3, & d = 0. \end{matrix} \end{cases}$$

In particular $D = \varnothing$ in all cases. Also $\delta_z = 1$ when it is non-zero, and, except for the three possible cusps (necessarily $\frac{3}{2}$-cusps), $\partial\Omega$ is smooth. We shall see later that all four cases above can occur.

For $n = 3$, (4.1) becomes

$$(4.3)\qquad P(z, w) = a_0 + a_1 zw - 3r^2 z^2 w^2 + z^3 w^3 - z^3 - w^3.$$

To use the obtained information about special points on N_k we study $P(z, \bar{z})$ on N_k. Thus set

$$(4.4)\qquad \begin{aligned} p(s) &= P\left(s \exp\left[i\left(\frac{\pi}{3} + k\,\frac{2\pi}{3}\right)\right],\ s \exp\left[-i\left(\frac{\pi}{3} + k\,\frac{2\pi}{3}\right)\right]\right) \\ &= s^6 - 3r^2 s^4 + 2s^3 + a_1 s^2 + a_0 \qquad (s \in \mathbf{R}). \end{aligned}$$

The following should be observed. Set

$$z = s \exp\left[i\left(\frac{\pi}{3} + k\,\frac{2\pi}{3}\right)\right] \qquad (s \in \mathbf{R}).$$

Then, by Lemma 1.1, $p(s) < 0$ if and only if $z \in \Omega \setminus E$, $p(s) = 0$ if and only if $z \in \partial\Omega \cup E$. When $z \in \partial\Omega \setminus C$ the tangent direction of $\partial\Omega$ at z is perpendicular to

N_k by the assumed symmetry of Ω (recall also that $D = \varnothing$), hence $p'(s) \neq 0$ (note that $z \notin B$). When $z \in C$ the cusp has the direction of N_k (by symmetry) and it follows that $p'(s) = 0$ and that p changes sign at s. In particular $p''(s) = 0$. Finally, when $z \in E$, $p'(s) = 0$ and $p(t) < 0$ for $t \neq s$ in a neighbourhood of s. In particular $p''(s) \leqq 0$.

We conclude from the above that $p(s) = p'(s) = 0$ if and only if $z \in E \cup C$, and that then $p''(s) \leqq 0$ with p negative on at least one side of s. By the earlier analysis $(E \cup C) \cap N_k$ contains exactly one point. Therefore there exists exactly one $\tau > 0$ with

$$p(\tau) = p'(\tau) = 0. \tag{4.5}$$

Moreover

$$p''(\tau) \leqq 0 \tag{4.6}$$

and p is negative on at least one side of τ.

We shall use this τ as a parameter in the sequel. From (4.4)

$$p'(s) = 2s(3s^4 - 6r^2s^2 + 3s + a_1).$$

Hence (4.5) shows that

$$a_1 = -3\tau(\tau^3 - 2r^2\tau + 1). \tag{4.7}$$

Substituting this into (4.4), (4.5) gives

$$a_0 = \tau^3(2\tau^3 - 3r^2\tau + 1). \tag{4.8}$$

Taking one more derivate gives

$$p''(s) = 2(3s^4 - 6r^2s^2 + 3s + a_1) + 6s(4s^3 - 4r^2s + 1).$$

Thus

$$p''(\tau) = 6\tau(4\tau^3 - 4r^2\tau + 1).$$

Define

$$p_0(\tau) = 4\tau^3 - 4r^2\tau + 1,$$

$$q_0(\tau) = 2\tau^3 - 3r^2\tau + 1,$$

$$q_1(\tau) = \tau^3 - 2r^2\tau + 1.$$

Thus, by (4.6),

$$p_0(\tau) \leqq 0.$$

It also follows that $r > \sqrt{3}/2$, for when $0 < r < \sqrt{3}/2$, $p_0(\tau) > 0$ for all $\tau > 0$, and when $r = \sqrt{3}/2$, $p_0(\tau) \leqq 0$ only for $\tau = \frac{1}{2}$ and then $p(s)$ turns out to be positive on both sides of $s = \tau$.

Finally, since $0 \in E$ or $0 \notin \Omega$ we have $a_0 \geqq 0$, i.e.

$$q_0(\tau) \geqq 0.$$

Now set

$$I_r = \{\tau > 0 : p_0(\tau) \leqq 0,\ q_0(\tau) \geqq 0\},$$

$$Q = \left\{(r, \tau) \in \mathbf{R}^2 : r > \frac{\sqrt{3}}{2},\ \tau \in I_r\right\},$$

and, for $(r, \tau) \in Q$,

$$\begin{aligned} \Omega(r, \tau) &= [\{z \in \mathbf{C} : z^3\bar{z}^3 - z^3 - \bar{z}^3 - 3r^2z^2\bar{z}^2 - 3\tau q_1(\tau) z\bar{z} + \tau^3 q_0(\tau) < 0\}] \\ &= [\{z \in \mathbf{C} : P(z, \bar{z}) < 0\}] \end{aligned} \tag{4.9}$$

with P as in (4.3), (4.7), (4.8). Up to now we have shown that every $\Omega \in Q(\mu_r, \mathrm{AL}^1)$ which is invariant under the same rigid transformations as μ_r and satisfies $\Omega = [\Omega]$ necessarily is of the form $\Omega(r, \tau)$ for some $\tau \in I_r$ and that $r > \sqrt{3}/2$. We shall next establish the converse.

Proposition 4.1. $\Omega(r, \tau) \in Q(\mu_r, \mathrm{AL}^1)$ *for all* $(r, \tau) \in Q$.

Before going to the proof we shall try to get a picture of the set Q. Straightforward analysis shows that the set of $\tau > 0$ satisfying $p_0(\tau) \leqq 0$ is (for $r > \sqrt{3}/2$) an interval $[\tau_1, \tau_2]$, where $0 < \tau_1 < \frac{1}{2} < \tau_2$. $\tau_j = \tau_j(r)$ are monotone functions of r with $\tau_{1,2} \to \frac{1}{2}$ as $r \to \sqrt{3}/2$, $\tau_1 \to 0$, $\tau_2 \to \infty$ as $r \to \infty$. For $r = 2^{-1/6} > \sqrt{3}/2$, $\tau_2 = 2^{-2/3}$ (this to be used later).

For $\sqrt{3}/2 < r \leqq 2^{-1/6}$, $q_0(\tau) \geqq 0$ for all $\tau > 0$. For $r > 2^{-1/6}$ the set of $\tau > 0$ satisfying $q_0(\tau) \geqq 0$ consists of two intervals, $(0, \tau_3]$ and $[\tau_4, \infty)$. Here $0 < \tau_3 < 2^{-2/3} < \tau_4$, τ_j are monotone functions of r with $\tau_{3,4} \to 2^{-2/3}$ as $r \to 2^{-1/6}$, $\tau_3 \to 0$, $\tau_4 \to \infty$ as $r \to \infty$. Observe that in the limiting case $r = 2^{-1/6}$, $\tau_2 = \tau_3 = \tau_4 = 2^{-2/3}$.

Next we observe that for $r > 2^{-1/6}$, $q_0(\tau_1) > 0$ and $q_0(\tau_2) < 0$. In fact, for $\tau = \tau_1$ or τ_2,

$$q_0(\tau) = q_0(\tau) - \tfrac{3}{4}p_0(\tau) = -(\tau^3 - \tfrac{1}{4}) = -(\tau - 2^{-2/3})(\tau^2 + 2^{-2/3}\tau + 2^{-4/3})$$

and $\tau_1 < \frac{1}{2} < 2^{-2/3}$ and (for $r > 2^{-1/6}$) $\tau_2 > 2^{-2/3}$. It follows that $\tau_1 < \tau_3 < \tau_2 < \tau_4$ for $r > 2^{-1/6}$. Hence we have

$$I_r = [\tau_1(r), \tau_2(r)] \qquad \text{for } \sqrt{3}/2 < r \leqq 2^{-1/6},$$

$$I_r = [\tau_1(r), \tau_3(r)] \qquad \text{for } r > 2^{-1/6}.$$

In particular I_r is a closed interval of positive length for every $r > \sqrt{3}/2$.

Thus Q is bounded by the arcs $\tau = \tau_1(r)$ $(r > \sqrt{3}/2)$, $\tau = \tau_2(r)$ $(\sqrt{3}/2 < r < 2^{-1/6})$, $\tau = \tau_3(r)$ $(r > 2^{-1/6})$ (belonging to Q) and the two special points $(\sqrt{3}/2, \frac{1}{2})$ and $(2^{-1/6}, 2^{-2/3})$, of which just the last one belongs to Q. Assuming for a moment that the proposition is already proven we can interpret the different parts of Q as follows. For (r, τ) in the interior of Q, $p_0(\tau) < 0$, $q_0(\tau) > 0$, hence $p''(\tau) < 0$ and $a_0 = p(0) > 0$. This means that $\Omega(r, \tau)$ is doubly-connected $(0 \notin \Omega(r, \tau))$ and that the double zero $p(\tau) = p'(\tau) = 0$ corresponds to (three) points in E. This is the case (iv) in (4.2).

On the arc $\tau = \tau_1(r)$, $p_0(\tau) = 0$ and $q_0(\tau) > 0$. Thus $p(\tau) = p'(\tau) = p''(\tau) = 0$. Moreover $p'''(\tau) \neq 0$ ($p''(\tau) = p'''(\tau) = 0$ holds only for $(r, \tau) = (\sqrt{3}/2, \frac{1}{2}) \notin Q$), hence p changes sign at τ and it follows that τ corresponds to (three) cusps on $\partial\Omega$. Also, $\Omega(r, \tau)$ is doubly-connected. Thus we are in the case (iii) in (4.2). The same applies to the arc $\tau = \tau_2(r)$ (for $\sqrt{3}/2 < r < 2^{-1/6}$). In this case the cusps are located on the exterior component of $\partial\Omega(r, \tau)$, whereas for $\tau = \tau_1(r)$ the cusps are on the interior component.

For $(r, \tau) = (2^{-1/6}, 2^{-2/3})$, $p_0(\tau) = q_0(\tau) = 0$. $p_0(\tau) = 0$ still means that we have three cusps on the exterior component of $\partial\Omega(r, \tau)$ while $q_0(\tau) = 0$ means that the interior component of $\partial\Omega(r, \tau)$ now has degenerated to become a point in E (namely $z = 0$). Thus we are in the case (i) in (4.2).

On the arc $\tau = \tau_3(r)$ finally $p_0(\tau) < 0$, $q_0(\tau) = 0$, which means that we have four points in E ($z = 0$ and one point on each N_k). Hence we are in the case (ii) in (4.2).

Proof of Proposition 4.1. Set

$$J_r = \{\tau \in I_r : \Omega(r, \tau) \in Q(\mu_r, \mathrm{AL}^1)\}$$

and we shall prove that $J_r = I_r$ for all $r > \sqrt{3}/2$. For this it is enough to prove that, for each $r > \sqrt{3}/2$,

(i) J_r is closed,

(ii) $J_r \cap \operatorname{int} I_r$ is open,

(iii) $J_r \cap \operatorname{int} I_r$ is non-empty.

To prove (i), suppose $\tau_n \in J_r$, $\tau_n \to \tau$ $(n \to \infty)$. Thus $\hat{\chi}_{\Omega(r,\tau_n)} = \hat{\mu}_r$ in $\mathbf{C} \setminus \Omega(r, \tau_n)$ by (1.6). Using that $\Omega(r, \tau) = \operatorname{int} \operatorname{clos} \Omega(r, \tau)$ it easily follows that $\hat{\chi}_{\Omega(r,\tau)} = \hat{\mu}_r$ in $\mathbf{C} \setminus \Omega(r, \tau)$. Thus we have $\Omega(r, \tau) \in Q(\mu_r, \mathrm{AL}^1)$, i.e. $\tau \in J_r$, if we merely can prove that $\Omega(r, \tau)$ is connected.

So suppose that $\Omega(r, \tau)$ is not connected. Then it still follows (e.g. by considering the function $S(z) = \bar{z} + (1/\pi)\,(\hat{\chi}_{\Omega(r,\tau)}(z) - \hat{\mu}_r(z))$, which is meromorphic in $\Omega(r, \tau)$ and equals $\bar{z}$ on $\partial\Omega(r, \tau)$) that each component of $\Omega(r, \tau)$ is a quadrature domain. Using also the symmetry properties of $\Omega(r, \tau)$ it more precisely follows that $\Omega(r, \tau)$ has exactly three components, these being quadrature domains for the measures $f \to \pi r^2 f(z_k)$, $k = 1, 2, 3$ respectively. Hence (compare the beginning of this §4) $\Omega(r, \tau)$ is the disjoint union of the discs $D(z_k, r)$, $k = 1, 2, 3$. This however is a contradiction, since $r > \sqrt{3}/2$.

To prove (ii), let $(r, \tau) \in J_r \cap \operatorname{int} I_r$. Then $\Omega(r, \tau) \in Q(\mu_r, \mathrm{AL}^1)$ is doubly-connected with $\partial\Omega(r, \tau)$ smooth algebraic by the earlier analysis. It is enough to prove that there exists a one-parameter family $\Omega(t)$ $(-\varepsilon < t < \varepsilon$, say) of domains in $Q(\mu_r, \mathrm{AL}^1)$ depending smoothly on t, having the symmetry properties of μ_r and satisfying $\Omega(t) = [\Omega(t)]$ and $\Omega(0) = \Omega(r, \tau)$. For then we must have $\Omega(t) = \Omega(r, \tau(t))$ for some function $\tau(t)$ with $\tau(0) = \tau$. If the map $t \to \Omega(t)$ is smooth and one-to-one then $\tau(t)$ must be smooth and invertible (hence also monotone) and therefore produces a neighbourhood of τ in J_r.

Such a family $\Omega(t)$ may be obtained by solving the moving boundary problem in [9]. Unfortunately, the conclusions in [9] are not strong enough to ensure that the (weak) solution constructed in [9] really is in $Q(\mu_r, \mathrm{AL}^1)$ (although the construction easily can be modified to ensure this property). However, in the present case a family $\Omega(t)$ with the desired properties (and being a classical solution of the problem in [9]) can be constructed in a rather elementary way by variation of the mapping function $\phi : W \to \Omega(r, \tau)$ in §1.

By §1 this $\phi : W \to \Omega(r, \tau)$ is a meromorphic function on $\hat{W}$, which is a torus in the present case. $\hat{W}$ is conformally equivalent to $\mathbf{C}/\sigma\mathbf{Z} + 2i\mathbf{Z}$ for some $\sigma > 0$ and ϕ can hence be represented by a doubly periodic (elliptic) function (which we also call ϕ) with periods σ and $2i$. We may assume that the half fundamental domain spanned by 0, σ and i corresponds to W and hence is mapped onto $\Omega(r, \tau)$ by ϕ. The "back-side" $\tilde{W}$ of $\hat{W}$ may then be represented by the half fundamental domain spanned by 0, σ and $-i$.

ϕ shall have three simple poles in each half fundamental domain corresponding to $\tilde{W}$ and elsewhere be regular. It follows that ϕ is of the form

$$\phi(w) = \sum_{k=1}^{3} A_k \zeta(w - w_k; \sigma) + A_0$$

with w_k in the half fundamental domain in $\mathbf{C}$ corresponding to $\tilde{W}$, $A_k \in \mathbf{C}$, $\Sigma_1^3 A_k = 0$ and where $\zeta(w; \sigma)$ denotes minus the anti-derivate of the Weierstrass $\wp$-function with periods σ and $2i$ $(d\zeta(w; \sigma)/dw = -\wp(w))$. $\zeta(w; \sigma)$ may be normalized to be odd and is then uniquely determined. (See e.g. [2] for the above material.) $\zeta(w; \sigma)$ is not elliptic itself, but ϕ becomes elliptic when $\Sigma_1^3 A_k = 0$. Since W admits a one-parameter group of conformal isomorphisms ϕ should also be normalized in some way, let us say by requiring that $\operatorname{Re} w_1 = 0$.

The family $\Omega(t)$ is now obtained as conformal images of the half fundamental domains spanned by 0, $\sigma(t)$ and i under

$$\phi(w, t) = \sum_{k=1}^{3} A_k(t) \zeta(w - w_k(t); \sigma(t)) + A_0(t) \tag{4.10}$$

for suitable functions $\sigma(t)>0$, $w_k(t)$, $A_k(t)$ subject to $\Sigma_1^3 A_k(t)=0$ and the normalization $\operatorname{Re} w_1(t)=0$. To see how these should be chosen let F be a (test) function holomorphic in a neighbourhood of clos $\Omega(0)$ (so that $F\in \mathrm{AL}^1(\Omega(t))$ for all t small enough). Then, as in [8], one obtains

$$
\begin{aligned}
\frac{d}{dt}\int_{\Omega(t)} F(z)dxdy &= \int_{\partial\Omega(t)} F(z)\operatorname{Im}\left[\frac{d\bar z}{dt}\,dz\right] \\
&= \int_{\partial W(t)} F(\phi(w;t))\operatorname{Im}[\overline{\dot\phi(w,t)}\phi'(w,t)dw],
\end{aligned} \tag{4.11}
$$

where $\partial W(t)$ denotes that part of the boundary of the half fundamental domain spanned by 0, $\sigma(t)$ and i for which $\operatorname{Im} w=0$ or 1. (A dot denotes derivative with respect to t.)

We want to have

$$
\frac{d}{dt}\int_{\Omega(t)} F(z)dxdy=0 \tag{4.12}
$$

for F as above, for then $\Omega(0)=\Omega(r,\tau)\in Q(\mu_r,\mathrm{AL}^1)$ implies $\Omega(t)\in Q(\mu_r,\mathrm{AL}^1)$, at least for t small enough. By (4.11), (4.12) holds if

$$
\operatorname{Im}[\overline{\dot\phi(w,t)}\phi'(w,t)]=1 \tag{4.13}
$$

for $\operatorname{Im} w=0$ and 1. ((4.13) is a classical formulation of the moving boundary problem in [9] expressed in terms of conformal mapping functions.) Thus, if we have a solution $\phi(w,t)$ of (4.13) with $\phi(w,0)=\phi(w)$ (mapping onto $\Omega(r,\tau)$) and with $\phi(w,t)$ elliptic of the form (4.10), then $\phi(w,t)$ will be univalent (in w) for t small enough and map onto doubly connected domains $\Omega(t)$ depending smoothly on t (if this is true for $\phi(w,t)$) and satisfying $\Omega(0)=\Omega(r,\tau)$ and, by (4.12), $\Omega(t)\in Q(\mu_r,\mathrm{AL}^1)$. Also $\dot\phi\neq 0$ by (4.13) so $\Omega(t)$ really changes with t.

The treatment of (4.13) essentially parallels that of a similar equation in [8] and we will not give the details here. The result is that the "Ansatz" (4.10) in (4.13) gives a system of ordinary differential equations for $\sigma(t)$, $w_k(t)$, $A_k(t)$ in normal form (i.e. solved for $\dot\sigma$, $\dot w_k$, $\dot A_k$ as smooth functions of σ, w_k, A_k). By standard theory this system has a unique local solution. Hence (4.13) has a unique local solution of the form (4.10) and we obtain a smooth (and non-constant) family of domains $\Omega(t)\in Q(\mu_r,\mathrm{AL}^1)$. By the uniqueness $\Omega(t)$ must have the required symmetry properties and it is also rather immediate that $\Omega(t)=[\Omega(t)]$. This proves (ii).

From (i) and (ii) it follows that for each $r > \sqrt{3}/2$ either $J_r = I_r$ or $J_r \cap \operatorname{int} I_r = \varnothing$. Now J_r is certainly non-empty for all $r > \sqrt{3}/2$ because

$$D(r) := \mathbf{D}(z_1; r) + \mathbf{D}(z_2; r) + \mathbf{D}(z_3; r) \in Q(\mu_r, \mathrm{SL}^1) \subset Q(\mu_r, \mathrm{AL}^1).$$

Clearly $D(r)$ has the required symmetry properties so that

$$[D(r)] = \Omega(r, \tau(r))$$

for a certain function $\tau(r)$.

Thus $\tau(r) \in J_r$ for all $r > \sqrt{3}/2$. Moreover $\tau(r) > \tau_1(r)$ and, for $r \leqq 2^{-1/6}$, $\tau(r) < \tau_2(r)$. In fact, in the contrary case $\partial D(r)$ would have (three) $\frac{3}{2}$-cusps, which is impossible because inside a $\frac{3}{2}$-cusp the logarithmic potential of $\mu_r - \chi_{D(r)}$ becomes negative (cf. [16]) and this contradicts $D(r) \in Q(\mu_r, \mathrm{SL}^1)$ by [13], [14]. However $\tau(r) = \tau_3(r)$ from some r on (at least from $r = 1$ on, since $0 \in \mathbf{D}(z_1; r) \cup \mathbf{D}(z_2; r) \cup \mathbf{D}(z_3; r) \subset D(r)$ for $r > 1$) so $\tau(r) \notin \operatorname{int} I_r$ for large r.

To produce points in $J_r \cap \operatorname{int} I_r$ for larger r ($r > 2^{-1/6}$) we therefore have to modify the construction of $D(r)$ a little. Starting from any $(r_0, \tau_0) \in \operatorname{int} Q$ such that $\Omega(r_0, \tau_0) \in Q(\mu_r, \mathrm{AL}^1)$ (i.e. $\tau_0 \in J_{r_0} \cap \operatorname{int} I_{r_0}$) we set

$$D(r; r_0, \tau_0) = \Omega(r_0, \tau_0) + \sum_{k=1}^{3} \mathbf{D}(z_k; \sqrt{r^2 - r_0^2})$$

for $r > r_0$. Then $D(r; r_0, \tau_0) \in Q(\mu_r, \mathrm{AL}^1)$ and is symmetric. It follows that

$$[D(r; r_0, \tau_0)] = \Omega(r, \tau(r; r_0, \tau_0))$$

for some function $\tau(r; r_0, \tau_0)$. Hence $\tau(r; r_0, \tau_0) \in J_r$ for all $r > r_0$. Moreover $\tau(r; r_0, \tau_0) > \tau_1(r)$ and, if $r \leqq 2^{-1/6}$, $\tau(r; r_0, \tau_0) < \tau_2(r)$ because $\partial D(r; r_0, \tau_0)$ cannot have $\frac{3}{2}$-cusps, by an argument similar to the earlier one

$$(D(r; r_0, \tau_0) \in Q(\mu_{\sqrt{r^2 - r_0^2}} + \chi_{\Omega(r_0, \tau_0)} dx dy, \mathrm{SL}^1)).$$

Notice also that, by construction, $\Omega(r_0, \tau_0) \subset D(r; r_0, \tau_0)$. This implies that $\tau(r; r_0, \tau_0) < \tau_3(r)$ for r in some interval $r_0 < r < r_0 + \varepsilon$ where $\varepsilon = \varepsilon(r_0, \tau_0) > 0$ can be estimated. In fact, if the area of the bounded component of $\Omega(r_0, \tau_0)$ is denoted $A = A(r_0, \tau_0)$ then $\tau(r; r_0, \tau_0) < \tau_3(r)$ whenever $3\pi(r^2 - r_0^2) < A$ (i.e. $r < \sqrt{A(r_0, \tau_0)/3\pi + r_0^2}$) because then the $\mathbf{D}(z_k; \sqrt{r^2 - r^2})$ do not enough area to fill in the hole in $\Omega(r_0, \tau_0)$. The important thing about $\varepsilon(r_0, \tau_0)$ is that it is bounded from below (away from zero) on every compact subset of $\operatorname{int} Q$.

Now we can conclude that $J_r \cap \operatorname{int} I_r \neq \varnothing$ for all $r > \sqrt{3}/2$. For in the contrary case set

$$\rho = \sup\{r > \sqrt{3}/2 : J_r \cap \operatorname{int} I_r \neq \varnothing\}.$$

Then $2^{-1/6} \leq \rho < \infty$, $J_r = I_r$ for $r < \rho$ and we can choose a point $(r_0, \tau_0) \in \operatorname{int} Q$ with $r_0 < \rho$ so close to ρ so that $\rho - r_0 < \varepsilon(r_0, \tau_0)$. Since, by the definition of $\varepsilon(r_0, \tau_0)$, $\tau(r; r_0, \tau_0) \in J_r \cap \operatorname{int} I_r$ for all $r_0 < r < r_0 + \varepsilon(r_0, \tau_0)$ we get a contradiction to the definition of ρ. This proves (iii) and finishes the proof of the proposition.

Remark 4.1. It is worth emphasizing that, with a different parametrization, the family $\{\Omega(r, \tau) : \tau \in \operatorname{int} I_r\}$ for any fixed $t > \sqrt{3}/2$ is a classical solution of the moving boundary problem studied in [9]. In fact, for $\tau \in \operatorname{int} I_r$, $\Omega(r, \tau)$ is doubly connected, $\partial\Omega(r, \tau)$ is smooth algebraic and depends smoothly on t (by the explicit representation (4.9)) and, since $\Omega(r, \tau) \in Q(\mu_r, \mathrm{AL}^1)$,

$$\frac{d}{d\tau} \int_{\Omega(r,\tau)} f dxdy = 0 \tag{4.13}$$

for all f holomorphic in a neighbourhood of clos $\Omega(r, \tau)$. From (4.13) it follows that

$$\frac{d}{d\tau} \int_{\Omega(r,\tau)} u dxdy = \frac{1}{2\pi} \frac{d}{d\tau} \int_{\Omega(r,\tau)} \log|z| dxdy \cdot \int_{\Gamma} \frac{\partial u}{\partial n} ds \tag{4.14}$$

for every u harmonic in a neighbourhood of clos $\Omega(r, \tau)$. Here Γ is any closed oriented curve in $\Omega(r, \tau)$ surrounding, with index $+1$, the hole in $\Omega(r, \tau)$.

(4.14) shows, by [9, Prop. 1.1], that expressed in terms of a parameter $t = t(\tau)$ satisfying

$$\frac{dt}{d\tau} = \frac{1}{2\pi} \frac{d}{d\tau} \cdot \int_{\Omega(r,\tau)} \log|z| dxdy, \tag{4.15}$$

the family $\{\Omega(r, \tau) : \tau \in \operatorname{int} I_r\}$ is a classical solution as stated. The right member of (4.15) turns out to be negative so $t(\tau)$ is monotone decreasing. From [9] it also follows that the family $\{\Omega(r, \tau) : \tau \in \operatorname{int} I_r\}$ is monotone in the sense that if $\Omega(r, \tau)^i$ and $\Omega(r, \tau)^e$ denote the bounded and unbounded components of $\mathbf{C} \setminus \Omega(r, \tau)$ respectively then $\Omega(r, \tau)^i \Supset \Omega(r, \tau')^i$ and $\Omega(r, \tau)^e \Subset \Omega(r, \tau')^e$ for $\tau < \tau'$.

Remark 4.2. The fact that, for $\sqrt{3}/2 < r < 2^{-1/6}$, the domains $\Omega(r, \tau)$ develop cusps on the inner boundary component as $\tau \to \tau_1(r)$ ($I_r = [\tau_1(r), \tau_2(r)]$ now) and on the outer boundary as $\tau \to \tau_2(r)$ has the interesting consequence that within the family $\{\Omega(r, \tau) \in Q(\mu_r, \mathrm{AL}^1) : \tau \in I_r\}$ there must be pairs of (different) domains which are conformally equivalent. Thus there exists different but conformally equivalent quadrature domains (for analytic functions) which admit the same quadrature identity.

To see this let, for any doubly connected domain Ω (with sufficiently regular boundary), $u = u_\Omega$ be the harmonic function in Ω with boundary values

$$u_\Omega = \begin{cases} 1 & \text{on } \gamma_1, \\ 0 & \text{on } \gamma_0, \end{cases}$$

where γ_1, γ_0 denote the inner and outer components of $\partial\Omega$ respectively, oriented as $\partial\Omega$. Set

$$c = c(\Omega) = \int_\Omega |\nabla u|^2 = \int_{\gamma_1} \frac{\partial u}{\partial n}\, ds,$$

the capacity of Ω ($\partial/\partial n$ the outward normal derivative).

We first notice that the quantity $c(\Omega)$ is a measure of the "modulus" of Ω, namely that $c(\Omega_1) = c(\Omega_2)$ if and only if Ω_1 and Ω_2 are conformally equivalent. In fact, if v is the (multiple-valued) harmonic conjungate of u then the function $\exp 2\pi(u + iv)/c$ is univalent on Ω and maps it conformally onto the annulus $\{1 < |z| < \exp 2\pi/c\}$, proving the assertion.

Now the idea is that a cusp on $\partial\Omega$ (pointing inwards) has the effect of creating a peak (top) value for the capacity $c(\Omega)$, so that, within the family $\{\Omega(r, \tau) : \tau \in I_r\}$, $c(\Omega(r, \tau))$ has local maxima at the endpoints of I_r. Therefore $c(\Omega(r, \tau))$ cannot be a monotone function of τ and the assertion follows.

To make the above idea more precise it is convenient to reparametrize $\Omega(r, \tau)$ as in Remark 4.1. Thus set $\Omega(t) = \Omega(r, \tau)$ with t related to τ as in (4.15). Then $\tau \in I_r = [\tau_1(r), \tau_2(r)]$ corresponds to say $t \in [t_1, t_2]$, $t_1 < t_2$ (t_1 (resp. t_2) will correspond to τ_2 (resp. τ_1); that I_r corresponds to a bounded interval follows from (4.15) or from [10]). By Remark 4.1, $\{\Omega(t) : t \in (t_1, t_2)\}$ now is a classical solution of the problem in [9]. This means that $\partial\Omega(t)$ moves with the velocity $-\partial u_{\Omega(t)}/\partial n$, measured in the direction of the outward normal of $\partial\Omega(t)$. From this one easily gets that

$$\frac{\partial u}{\partial t} = \left(\frac{\partial u}{\partial n}\right)^2 \qquad \text{on } \partial\Omega(t). \tag{4.16}$$

Consider now

$$\frac{d}{dt}\, c(\Omega(t)) = \frac{d}{dt} \int_{\Omega(t)} |\nabla u_{\Omega(t)}|^2$$

for $t \in (t_1, t_2)$. The derivate in the right member is the sum of two terms, one in which d/dt acts on the domain of integration and one in which d/dt acts on the integrand. The first term becomes (using that $\partial\Omega(t)$ moves with the velocity $-\partial u/\partial n$)

$$-\int_{\partial\Omega(t)} \frac{\partial u}{\partial n}\,|\nabla u|^2 ds = -\int_{\partial\Omega(t)} \left(\frac{\partial u}{\partial n}\right)^3 ds.$$

For the second term we get, using (4.16),

$$\int_{\Omega(t)} \frac{\partial}{\partial t}\,|\nabla u|^2 = 2\int_{\Omega(t)} \nabla u\cdot\nabla\,\frac{\partial u}{\partial t} = 2\int_{\partial\Omega(t)} \frac{\partial u}{\partial n}\,\frac{\partial u}{\partial t}\,ds = 2\int_{\partial\Omega(t)} \left(\frac{\partial u}{\partial n}\right)^3 ds.$$

Thus we arrive at the formula

$$\frac{d}{dt}\,c(\Omega(t)) = \int_{\partial\Omega(t)} \left(\frac{\partial u}{\partial n}\right)^3 ds$$

for $t\in(t_1, t_2)$ (then we certainly have enough regularity for the derivation to be valid). When $t\to t_1$ (corresponding to $\tau\to\tau_2(r)$) the outer component $\gamma_0(t)$ develops cusps while the inner component $\gamma_1(t)$ remains smooth. If s denotes an arc-length parameter along $\gamma_0(t_1)$ such that $s=0$ corresponds to a cusp point then, in the limiting case $t=t_1$, $\partial u/\partial n$ will have a singularity at $s=0$ at least as bad as $1/\sqrt{s}$. See [13, Lemma 2.4]. Since $\partial u/\partial n$ is negative on γ_0 this shows that

$$\int_{\gamma_0(t_1)} \left(\frac{\partial u}{\partial n}\right)^3 ds = -\infty.$$

Using e.g. Fatou's lemma (and the monotonicity properties of $\Omega(t)$ mentioned in Remark 4.1) one concludes that

$$\int_{\gamma_0(t_1)} \left(\frac{\partial u}{\partial n}\right)^3 ds \to -\infty.$$

as $t\to t_1$ while $\int_{\gamma_1(t)}(\partial u/\partial n)^3 ds$ remains bounded.

Hence $dc(\Omega(t))/dt$ becomes negative for t close enough to t_1. Similarly, it becomes positive for t close enough to t_2. This shows that $c(\Omega(t))$ is not monotone on $[t_1, t_2]$ as claimed.

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