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A SIMPLE PROOF OF THE REGULARITY THEOREM FOR THE VARIATIONAL INEQUALITY OF THE OBSTACLE PROBLEM

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LET Ω BE a bounded open set in $\mathbb{R}^n$ with $\partial\Omega$ smooth and let $a(\cdot,\cdot)$ denote the Dirichlet inner product on $H_0^1(\Omega)$,

$$a(u, v) = \int_\Omega \sum_{i=1}^n \frac{\partial u}{\partial x_i}\frac{\partial v}{\partial x_i}\,dx \qquad (u, v \in H_0^1(\Omega)).$$

The aim of the present paper is to give a relatively short and seemingly new proof of the following well-known result ([6, Chapter IV, theorem 2.3] for example).

THEOREM. Let $n < p < \infty$, $p \geq 2$. Suppose $f \in L^p(\Omega)\ (\subset H^{-1}(\Omega))$, $\psi \in H^{2,p}(\Omega)$ and let $u \in H_0^1(\Omega)$ be the solution of

$$\begin{cases} u \in K & (1)\\ a(u, v-u) \geqq \langle f, v-u\rangle \quad \text{for all } v \in K, & (2)\end{cases}$$

where $K = \{v \in H_0^1(\Omega) : v \geqq \psi\}$. Then

$$f \leqq -\Delta u \leqq \max\{-\Delta\psi, f\}. \tag{3}$$

In particular $u \in H^{2,p}(\Omega)$ and $u \in C^{1,\alpha}(\bar{\Omega})$ with $\alpha = 1 - n/p$.

Notational remark. We are using the notations $H^{m,p}(\Omega)$, $H^m(\Omega) = H^{m,2}(\Omega)$, $H_0^m(\Omega) = H_0^{m,2}(\Omega)$ etc. for Sobolev spaces on Ω (as in [8]) and $\langle\cdot,\cdot\rangle$ for the standard duality pairing between $H^{-1}(\Omega)$ and $H_0^1(\Omega)$. When nothing else is stated, equalities and (nonstrict) inequalities on open sets between elements in function spaces are to be interpreted in the sense of distributions.

The variational inequality (1)–(2) is known as the variational inequality of the obstacle problem in its simplest form ([6, Chapter II, Section 6]) and the existence of a unique solution of it is a widely known fact ([6, Chapter II, theorem 6.2]).

The regularity of the solution of (1)–(2) is also well-known and a number of different proofs of it exist. The first proof (by "penalization") appeared in [7, theorem 3.1]). See also [6, Chapter IV, Section 2]. Two other proofs are in [3, corollaire II.3] and [2, théorème I.1]. The estimate (3) is not mentioned explicitly in these papers but it is more or less implicit in them (it follows e.g. from [7, corollary 3.1] and is in any case well-known).

We are here going to give a proof of the regularity theorem that is somewhat shorter and more elementary than those given in [2, 3, 7]. It should be remarked, however, that in these papers the results are stated under more general hypotheses than here, allowing $a(u, v)$ to have variable, continuously differentiable, coefficients and to be nonsymmetric ($-\Delta$ correspondingly changed), requiring only $\psi \leqq 0$ on $\partial\Omega$ in place of $\psi = 0$ there and, in [3], requiring only $1 < p < \infty$. Compare, however, the remark after our proof. An example of a short variant of the method of [3], specialized to essentially our case, is given in [4, pp. 26–29]. Our proof seems to be new except that I have used the same method in [5, pp. 48–50]. The idea of the proof is to consider a new variational inequality, (4)–(5) below (which appeared naturally in the work on the problem in [5]), for which the estimate (3) *a priori* is fulfilled, and then showing that the solution of that variational inequality also solves the original one. Then the theorem follows by uniqueness of solutions of (1)–(2).

For convenience, before giving the proof, we list a few standard facts that will be used in the proof.

(i) $-\Delta$ is an isomorphism from $H_0^1(\Omega)$ onto $H^{-1}(\Omega)$ [8, theorem 23.1].

(ii) If $-\Delta u \in L^p(\Omega)$ for $u \in H_0^1(\Omega)$ and $1 < p < \infty$, then $u \in H^{2,p}(\Omega)$.

(iii) $a(u, v) = \langle -\Delta u, v\rangle$ for $u, v \in H_0^1(\Omega)$ [8, Section 23].

(iv) If $f \in L^p(\Omega)$ with $p \geqq 2$ and $u \in H_0^1(\Omega)$, then $\langle f, u\rangle = \int_\Omega f \cdot u \, dx$. Here the left-hand side makes sense because $L^p(\Omega) \subset H^{-1}(\Omega)$ and the right-hand side because $H_0^1(\Omega) \subset L^q(\Omega)$, where $(1/p) + (1/q) = 1$.

(v) If $f \geqq 0$, $u \geqq 0$ ($f \in H^{-1}(\Omega)$, $u \in H_0^1(\Omega)$), then $\langle f, u\rangle \geqq 0$.

(vi) If u is continuous in a bounded open set N, $-\Delta u \geqq 0$ in N and $\lim_{x \to y} u(x) = 0$ for any $y \in \partial N$, then $u \geqq 0$ in N (minimum principle for superharmonic functions) [8, statement (30.4) together with proposition 30.6].

(vii) $H^{2,p}(\Omega) \subset C^{1,\alpha}(\bar{\Omega})$ if $n < p < \infty$, $\alpha = 1 - n/p$ and $\partial\Omega$ is smooth [1, theorem 5.4, part IIC′].

Proof of the theorem. We shall prove the theorem in the special case that $\psi = 0$. The general case then follows by applying the special case with $u - \psi$ in place of u and $f + \Delta\psi$ in place of f. Thus we assume from now on that $\psi = 0$.

Put

$$L = \{v \in H_0^1(\Omega) : f \leqq -\Delta v \leqq f^+\}$$

($f^+ = \max\{f, 0\}$). Thus we want to prove that $u \in L$ (where u is the solution of (1)–(2) with $\psi = 0$). Observe that $L \subset H^{2,p}(\Omega) \subset C(\bar{\Omega})$ by (ii) and (vii). In particular, the members of L can (and will) be considered as continuous functions on $\bar{\Omega}$, instead of just equivalence classes of functions or distributions on Ω. Moreover, since $L \subset H_0^1(\Omega)$ and $\partial\Omega$ is smooth, these functions must be identically zero on $\partial\Omega$ [8, Proposition 22.2].

It is easy to see that L is a closed, convex and nonempty set. Therefore, and since $a(\cdot, \cdot)$ is coercive on $H_0^1(\Omega)$, the variational inequality

$$\begin{cases} w \in L & (4) \\ a(v - w, w) \geqq 0 \qquad \text{for all } v \in L & (5) \end{cases}$$

has a (unique) solution w. We shall prove our theorem by showing that w also solves (1)–(2). Since (1)–(2) has only one solution, this will imply that $w = u$ and hence that $u \in L$.

The first, and main, step is to prove that w satisfies

$$\begin{cases} w \geqq 0 & (6) \\ -\Delta w - f \geqq 0 & (7) \\ \langle -\Delta w - f, w \rangle = 0. & (8) \end{cases}$$

By (iii), (5) can be written

$$\langle -\Delta v + \Delta w, w \rangle \geqq 0 \qquad \text{for all } v \in L. \tag{9}$$

First, choose $v \in H_0^1(\Omega)$ with $-\Delta v = f$ in (9). This gives

$$\langle f + \Delta w, w \rangle \geqq 0. \tag{10}$$

Second, choose $v \in H_0^1(\Omega)$ with

$$-\Delta v = \begin{cases} f^+ & \text{in } N \\ -\Delta w & \text{in } \Omega \backslash N, \end{cases} \tag{11}$$

where $N = \{x \in \Omega : w(x) < 0\}$. Clearly, since $w \in H^{2,p}(\Omega) \subset C(\Omega)$, N is a well-defined open set and the right member of (11) defines a function in $L^p(\Omega)$. This gives (using (iv))

$$\int_N (f^+ + \Delta w) \cdot w \, dx \geqq 0. \tag{12}$$

But $(f^+ + \Delta w) \cdot w \leqq 0$ in N. Hence (12) implies $(f^+ + \Delta w) \cdot w = 0$ a.e. in N and so

$$-\Delta w = f^+ \text{ in } N \text{ (a.e. or in the sense of distributions).} \tag{13}$$

Now (13) shows that w is a superharmonic function in N. On the boundary, ∂N, of N in $\mathbb{R}^n$ we have $w = 0$, in view of the definition of N and because $w \in H_0^1(\Omega) \cap C(\bar{\Omega})$. Therefore, the minimum principle for superharmonic harmonic functions (vi) shows that $w \geqq 0$ in N. Comparing with the definition of N it follows that N is empty. Hence

$$w \geqq 0 \quad \text{in } \Omega.$$

This proves (6). (7) is part of (4) and (8) follows by combining (6), (7) and (10), using (v). By this (6)–(8) is proven.

From (6)–(8) we easily deduce (1)–(2) with w in place of u: (1) is the same as (6) and

$$a(w, v - w) - \langle f, v - w \rangle = \langle -\Delta w - f, v - w \rangle = \langle -\Delta w - f, v \rangle - \langle -\Delta w - f, w \rangle \geqq 0$$

for $v \in K$ in view of (7) and (8), proving (2).

Thus the solution w of (4)–(5) is also a solution of (1)–(2), which, as we have remarked earlier, proves the theorem.

Remark. It is easily seen that our proof goes through under more general hypotheses than stated in the theorem. Specifically, what is required of $a(\cdot,\cdot)$ and $-\Delta$ is only that $a(\cdot,\cdot)$ is coercive and that (i), (ii), (iii) and (vi) above hold. This results in about the same hypotheses on $a(\cdot,\cdot)$ and the operator replacing $-\Delta$ as in [2, 3, 7].

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