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ISOPERIMETRIC INEQUALITIES FOR THE STEFAN PROBLEM*

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Abstract. The weak solution (θ, h) of the Stefan problem in some annular domain $\omega \times (0, T)$ is compared with the weak solution (Θ, H) of the "symmetrized" problem, in $\Omega \times (0, T)$, where Ω is a symmetrical annulus having the same measure as ω. For the one-phase Stefan problem—$\theta \geqq 0$, $h \in (-\alpha, 0)$ when $\theta = 0$—it is shown in particular that the "volume of ice" (meas $\{h(t) = -\alpha\}$) remains greatest in spherical symmetry (with initial data decreasing along the radii).

Résumé. Cet article compare la solution faible (θ, h) du problème de Stefan dans un domaine $\omega \times (0, T)$ (où ω est une couronne) avec celle du problème "symétrisé" (Θ, H) dans $\Omega \times (0, T)$, où Ω est une couronne symétrique de même mesure que ω. Pour le problème de Stefan à une phase—$\theta \geqq 0$, $h \in (-\alpha, 0)$ là où $\theta = 0$—on voit en particulier que le "volume de glace" (mes $\{h(t) = -\alpha\}$) est maximum en symétrie de révolution (avec donnée initiale décroissante le long du rayon).

Key words. one-phase Stefan problem, two-phase Stefan problem, solid phase-liquid phase, regularized problem, symmetrized problem, decreasing rearrangement, equimeasurable functions, isoperimetric inequalities

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1. Introduction. We consider the Stefan problem in its simplest form and in an annular space geometry: find a pair (θ, h) of functions defined in $q = \omega \times (0, T)$ such that, in some weak sense,

$$(1.1)\qquad \begin{aligned} &\frac{\partial h}{\partial t} - \Delta\theta = 0 \quad \text{in } q,\\ &\theta = g \quad \text{on } \sigma = \partial\omega \times (0, T),\\ &h|_{t=0} = h_0,\\ &h \in a(\theta) \quad \text{a.e. in } q. \end{aligned}$$

Here we have the following:

- $\omega = \omega_0 \backslash \bar{\omega}_1$, where ω_0, ω_1 are bounded domains in $\mathbb{R}^N$ $(N \geqq 2)$ with smooth boundaries $\gamma_0 = \partial\omega_0$ and $\gamma_1 = \partial\omega_1$, and satisfying $\bar{\omega}_1 \subset \omega_0$.
- g is constant on each of $\sigma_0 = \gamma_0 \times (0, T)$ and $\sigma_1 = \gamma_1 \times (0, T)$, let us say

$$g = \begin{cases} 0 & \text{on } \sigma_0,\\ 1 & \text{on } \sigma_1. \end{cases}$$

- a is a strictly monotone graph in $\mathbb{R}^2$ (regarded as a map from $\mathbb{R}$ into subsets of $\mathbb{R}$). The typical form of a for the Stefan problem is

$$(1.2)\qquad a(\theta) = \begin{cases} \alpha_0(\theta - \lambda) - \alpha & \text{for } \theta < \lambda,\\ [-\alpha, 0] & \text{for } \theta = \lambda,\\ \alpha_1(\theta - \lambda) & \text{for } \theta > \lambda, \end{cases}$$

where α, α_0, α_1 are positive constants, $\lambda \in [0, 1]$. However, our main results are valid for an arbitrary maximal monotone graph a such that $a([0, 1])$ is bounded, and such that the inverse graph $b = a^{-1}$ is a Lipschitz continuous function on $a([0, 1])$.

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• $h_0 \in L^\infty(\omega)$ and satisfies an extra condition (see (1.6), (1.7) below), which essentially means that $\theta_0 = b(h_0)$ belongs to $H^1(\omega)$ and satisfies $0 \leqq \theta_0 \leqq 1$.

The physical interpretation of (1.1) when a is of the form (1.2) is that θ is the temperature and h the enthalpy of some matter that undergoes a phase change (solid–liquid) at temperature λ. The number α is proportional to the latent heat for the phase change, and α_0 and α_1 are proportional to the heat capacities of the solid and liquid states, respectively (α, α_0, α_1 are also inversely proportional to the thermal conductivity coefficients). With more general a (often single-valued), there are many other interpretations of (1.1) (e.g., porous medium equation).

Our boundary and initial data, g and h_0 above, are such that the solution (θ, h) of (1.1), by the maximum principle, will satisfy $0 \leqq \theta \leqq 1$ in all q. In the case of (1.2) with $\lambda = 0$, the temperature in the solid phase (the latter generally defined as the region where $h \leqq -\alpha$ ($h = -\alpha$ in this case)) therefore must be constantly equal to zero. Similarly, in the case $\lambda = 1$, the temperature in the liquid phase $\{h \geqq 0\}$ ($h = 0$ in this case) is constantly equal to 1. Thus, for these extreme cases, in practice we have a one-phase Stefan problem, while for $0 < \lambda < 1$, the problem really is a two-phase problem.

One standard way of making (1.1) precise is to say that (θ, h) is a weak solution of (1.1) if

$$\theta \in L^\infty(q), \quad h \in L^\infty(q), \quad h \in a(\theta) \quad \text{a.e. in } q,$$

$$\iint_q \left(h\frac{\partial \varphi}{\partial t} + \theta \Delta \varphi \right) dx\, dt = \int_0^T \int_\gamma g \frac{\partial \varphi}{\partial \nu}\, d\gamma\, dt - \int_\omega h_0(x)\varphi(x, 0)\, dx \tag{1.3}$$

for every "test function" $\varphi \in \mathscr{C}^1(\bar{q})$ satisfying $(\partial^2\varphi/\partial x_i\, \partial x_j) \in \mathscr{C}(\bar{q})$ and $\varphi = 0$ on $\sigma \cup (\omega \times \{T\})$ (see, e.g., [3] or [4]).

Existence and uniqueness of weak solutions can be proved in several different ways. One method, developed by Oleinik [9] (in one space dimension) and Friedman [3] (see also [4]) is to obtain the weak solution as a limit as $\varepsilon \to 0$ ($\varepsilon > 0$) of the classical solutions $(\theta_\varepsilon, h_\varepsilon)$ of some regularized problems

$$(1.1)_\varepsilon \qquad \begin{aligned} &\frac{\partial h_\varepsilon}{\partial t} - \Delta \theta_\varepsilon = 0 \quad \text{in } q, \\ &\theta_\varepsilon = g \quad \text{on } \sigma, \\ &h_\varepsilon|_{t=0} = h_{\varepsilon_0}, \\ &h_\varepsilon = a_\varepsilon(\theta_\varepsilon) \quad \text{in } q. \end{aligned}$$

Here a_ε (from $[0, 1]$ to $a([0, 1])$) are single-valued smooth functions with

$$a'_\varepsilon \geqq \delta > 0 \qquad (\delta \text{ independent of } \varepsilon) \tag{1.4}$$

such that $a_\varepsilon \to a$ as $\varepsilon \to 0$, in the sense that

(1.5) $\quad b_\varepsilon = a_\varepsilon^{-1}$ (from $a([0, 1])$ to $[0, 1]$) converges uniformly to $b = a^{-1}$.

Moreover, h_{ε_0} are smooth functions such that

$$h_{\varepsilon_0} \to h_0 \quad \text{in } L^1(\omega), \tag{1.6}$$

and that, in terms of $\theta_{\varepsilon_0} = b_\varepsilon(h_{\varepsilon_0})$,

$$0 \leqq \theta_{\varepsilon_0} \leqq 1 \quad \text{in } \omega, \qquad \theta_{\varepsilon_0}|_{\partial\omega} = g,$$

$$\int_\omega |\nabla \theta_{\varepsilon_0}|^2\, dx \text{ is bounded independently of } \varepsilon \quad \text{as } \varepsilon \to 0. \tag{1.7}$$

In this paper, our aim is to give isoperimetric inequalities for the Stefan problem (1.1). They are not standard, since the problem is multivalued, and the domain is doubly connected. The principal one of them, (2.1) below (from which some other inequalities follow as corollaries), is obtained by passing to the limit in the corresponding inequality $(2.1)_\varepsilon$ for $(1.1)_\varepsilon$. The proof of $(2.1)_\varepsilon$ is a direct parabolic one that relies on the techniques developed in [8] for linear parabolic problems. Some of our corollaries confirm the intuitive idea that, among all domains $\omega = \omega_0 \backslash \bar{\omega}_1$, with ω_0, ω_1, of given measures (volumes), and all equimeasurable initial data h_0, the solid "melts slowest" in the symmetrized domain, with symmetrized h_0 ($H_0 = \mathbf{H}_0 = \mathbf{h}_0$). These results were announced in a previous note [5].

One drawback of our method is that it seems to require constant boundary values for θ (on each component of the boundary), and initial data guaranteeing that $0 \leqq \theta \leqq 1$ holds throughout q (where 0 and 1 are the constant boundary values for θ).

Isoperimetric inequalities for a problem similar to $(1.1)_\varepsilon$ have been obtained earlier by Vazquez [10], in the case $\omega = \mathbb{R}^N$, by an elliptic approach (and using semigroup theory). This approach does not seem to work in our geometry.

2. Statements of results. Our isoperimetric inequalities are inequalities between certain quantities for the problems (1.1) and $(1.1)_\varepsilon$, and the corresponding quantities for certain "symmetrized problems" ($(\widetilde{1.1})$ and $(\widetilde{1.1})_\varepsilon$ below). Before describing these symmetrized problems, we must introduce some general notation:

- σ_N denotes the volume of the unit ball in $\mathbb{R}^N$.
- $|E|$ denotes the volume (N-dimensional Lebesgue measure) of a measurable set E in $\mathbb{R}^N$ (also: $|x| = (x_1^2 + \cdots + x_N^2)^{1/2}$ if $x \in \mathbb{R}^N$).
- $f_+ = \max\{f, 0\}$, $f_- = \max\{-f, 0\}$.
- With $\omega = \omega_0 \backslash \bar{\omega}_1$ as in § 1, Ω_j ($j = 0, 1$) denotes the open balls in $\mathbb{R}^N$ centered at the origin and having the same volumes as ω_j. Thus $\bar{\Omega}_1 \subset \Omega_0$. We also set $\Omega = \Omega_0 \backslash \bar{\Omega}_1$, $Q = \Omega \times (0, T)$, $\Gamma_j = \partial \Omega_j$. In general, when a lowercase letter is used for a certain quantity in the original problem, the corresponding capital letter will be used for the same quantity in the symmetrized problem.
- If f is a measurable function defined in ω, f_* denotes the decreasing rearrangement of f:

$$f_*(s) = \operatorname{Inf}\{\xi \in \mathbb{R}: |x: f(x) > \xi| \leqq s\},$$

defined for $s \in \bar{\omega}_* = [0, |\omega|]$ ($\omega_* = (0, |\omega|)$), while $\mathbf{f}$ denotes the rearrangement of f, defined in Ω, that decreases along radii:

$$\mathbf{f}(x) = f_*(\sigma_N |x|^N - m_1),$$

where $m_1 = |\omega_1|$, $x \in \Omega$. If f is defined in q, we consider its rearrangements with respect to the space variable: $f_*(s, t) = (f(\cdot, t))_*(s)$, $\mathbf{f}(x, t) = f_*(\sigma_N |x|^N - m_1, t)$.

Now, the symmetrized problem corresponding to (1.1) is

$$(\widetilde{1.1}) \qquad \begin{aligned} &\frac{\partial H}{\partial t} - \Delta \Theta = 0 \quad \text{in } Q, \\ &\Theta = G \quad \text{on } \Sigma = \partial\Omega \times (0, T), \\ &H|_{t=0} = \mathbf{h}_0, \\ &H \in a(\Theta) \quad \text{a.e. in } Q, \end{aligned}$$

where

$$G = \begin{cases} 0 & \text{on } \Sigma_0 = \Gamma_0 \times (0, T) = \partial\Omega_0 \times (0, T), \\ 1 & \text{on } \Sigma_1 = \Gamma_1 \times (0, T) = \partial\Omega_1 \times (0, T). \end{cases}$$

The problem corresponding to $(1.1)_\varepsilon$ is

$$
(\widetilde{1.1})_\varepsilon \qquad
\begin{aligned}
&\frac{\partial H_\varepsilon}{\partial t} - \Delta\Theta_\varepsilon = 0 \quad \text{in } Q,\\
&\Theta_\varepsilon = G \quad \text{on } \Sigma,\\
&H_\varepsilon|_{t=0} = \mathbf{h}_{\varepsilon_0},\\
&H_\varepsilon = a_\varepsilon(\Theta_\varepsilon) \quad \text{in } Q.
\end{aligned}
$$

Our main technical tool is the following result.

THEOREM 1. *For classical solutions of* $(1.1)_\varepsilon$ *and* $(\widetilde{1.1})_\varepsilon$, *we have the comparison*

$$
(2.1)_\varepsilon \qquad \int_0^s h_{\varepsilon*}(\sigma, t)\, d\sigma - \int_0^t \int_{\gamma_1} \frac{\partial \theta_\varepsilon}{\partial \nu}\, d\gamma\, d\tau \leqq \int_0^s H_{\varepsilon*}(\sigma, t)\, d\sigma - \int_0^t \int_{\Gamma_1} \frac{\partial \Theta_\varepsilon}{\partial \nu}\, d\gamma\, d\tau
$$

for every (s, t) *in* $\bar{q}_* = \bar{\omega}_* \times [0, T] = [0, |\omega|] \times [0, T]$ $(=\bar{Q}_*)$. *Here* $\partial/\partial\nu$ *denotes the outward normal derivative, and* $d\gamma$ *the* $(N-1)$*-dimensional Lebesgue measure of* $\partial\omega$ *and* $\partial\Omega$.

Similarly, for weak solutions of (1.1) *and* $(\widetilde{1.1})$ (*see* (1.3)),

$$
(2.1) \qquad \int_0^s h_*(\sigma, t)\, d\sigma - \int_0^t \int_{\gamma_1} \frac{\partial \theta}{\partial \nu}\, d\gamma\, d\tau \leqq \int_0^s H_*(\sigma, t)\, d\sigma - \int_0^t \int_{\Gamma_1} \frac{\partial \Theta}{\partial \nu}\, d\gamma\, d\tau
$$

for almost every $t \in (0, T)$ *and every* $s \in \bar{\omega}_*$.

Remarks. (1) In (2.1), the terms involving $\partial\theta/\partial\nu$ and $\partial\Theta/\partial\nu$ must be interpreted in a weak sense (see (4.10) below) because we are not guaranteed enough regularity for $\partial\theta/\partial\nu$ and $\partial\Theta/\partial\nu$ to make classical sense.

(2) Some isoperimetric inequalities involving a rearrangement and a "capacity term" (such as $\int_{\gamma_1} (\partial\theta/\partial\nu)\, d\gamma$ in (2.1)) have been previously obtained for elliptic problems in doubly connected domains, with boundary conditions 0 and 1 (respectively) on the two components of the boundary (see [7, p. 62] and [2, p. 168]).

The proof of Theorem 1 appears in §§ 3 and 4. We now give some other isoperimetric inequalities that have more physical significance and therefore can be viewed as the main results of the paper. They are all simple consequences of Theorem 1, and are proved in § 5.

If the solutions (θ, h) and (Θ, H) are "good enough," this meaning in particular that the sets $\{x \in \omega: h(x, \tau) > h_*(s, \tau)\}$ and $\{x \in \Omega: H(x, \tau) > H_*(s, \tau)\}$ have regular boundaries (in ω and Ω): $\gamma(s, \tau) = \{x \in \omega: h(x, \tau) = h_*(s, \tau)\}$ and $\Gamma(s, t) = \{x \in \Omega: H(x, \tau) = H_*(s, \tau)\}$ for almost every $\tau \in (0, t)$, then (2.1) can also be written:

$$
(2.2) \qquad -\int_0^t \int_{\gamma(s,\tau)} \frac{\partial \theta}{\partial \nu}\, d\gamma\, d\tau \geqq -\int_0^t \int_{\Gamma(s,\tau)} \frac{\partial \theta}{\partial \nu}\, d\gamma\, d\tau
$$

(the normal derivatives being directed outward from the sets mentioned above). The members of (2.2) are nonnegative and have the physical interpretation of being the total heat flows during the time interval $(0, t)$ from the warmest parts of volume s of ω and Ω, respectively, into the complementary colder parts. Note that these parts of ω and Ω change continuously with time.

In the particular cases $s = 0$ and $s = |\omega|$, (2.1) reduces to

$$
(2.3) \qquad \int_0^t \int_{\gamma_1} \frac{\partial \theta}{\partial \nu}\, d\gamma\, d\tau \geqq \int_0^t \int_{\Gamma_1} \frac{\partial \Theta}{\partial \nu}\, d\gamma\, d\tau \qquad (\geqq 0),
$$

$$
(2.4) \qquad -\int_0^t \int_{\gamma_0} \frac{\partial \theta}{\partial \nu}\, d\gamma\, d\tau \geqq -\int_0^t \int_{\Gamma_0} \frac{\partial \Theta}{\partial \nu}\, d\gamma\, d\tau \qquad (\geqq 0)
$$

for almost every $t \in (0, T)$. (Equation $(2.1)_\varepsilon$ reduces similarly.)

Another consequence of (2.1) is that, for any measurable set $E(t)\subset\omega$, with $|E(t)|=s$,

$$\int_0^t\int_{\gamma_1}\frac{\partial\theta}{\partial\nu}\,d\gamma\,d\tau-\int_{E(t)}h(x,t)\,dx \geqq \int_0^t\int_{\Gamma_1}\frac{\partial\Theta}{\partial\nu}\,d\gamma\,d\tau-\int_{m_1<\sigma_N|x|^N<m_1+s}H(x,t)\,dx. \tag{2.5}$$

In the case of a one-phase problem, let us say with $\lambda=0$, the quantity

$$t'=\sup\left\{t\in[0,T]:\int_0^\tau\int_{\gamma_0}\frac{\partial\theta}{\partial\nu}\,d\gamma\,d\tau'=0,\text{ a.e. }\tau\in(0,t)\right\} \tag{2.6}$$

can be interpreted as the first instant at which the liquid phase reaches γ_0 (when $\lambda>0$, $t'=0$). With the similar definition for the symmetrized problem, T' also appears to be the time at which all the solid has melted.

We then obtain, for weak solutions of (1.1) and $(\widetilde{1.1})$, some noteworthy comparisons.

THEOREM 2. *With the above definitions*

$$t'\leqq T', \tag{2.7}$$

$$\int_s^{|\omega|}h_*(\sigma,t)\,d\sigma\geqq\int_s^{|\omega|}H_*(\sigma,t)\,d\sigma \tag{2.8}$$

for almost every $t\in(0,t')$, and all $s\in\bar\omega_*$. More generally,

$$\int_s^{|\omega|}\Phi(h_*(\sigma,t))\,d\sigma\geqq\int_s^{|\omega|}\Phi(H_*(\sigma,t))\,d\sigma \tag{2.9}$$

for almost every $t\in(0,t')$, and all $s\in\bar\omega_*$, where Φ is any concave nondecreasing function. This implies

$$\operatorname*{ess\,inf}_{\omega}h(t)\geqq\operatorname*{ess\,inf}_{\Omega}H(t)\quad \textit{a.e. } t\in(0,t'), \tag{2.10}$$

$$\|(h(t)-\beta)_-\|_{L^p(\omega)}\leqq\|(H(t)-\beta)_-\|_{L^p(\Omega)}\quad \textit{a.e. } t\in(0,t') \tag{2.11}$$

for every $\beta\in\mathbb{R}$ and every $p\in[1,\infty]$, and

$$|\{x\in\omega: h(x,t)=-\alpha\}|\leqq|\{x\in\Omega: H(x,t)=-\alpha\}|\quad \textit{a.e. } t\in(0,t'). \tag{2.12}$$

The latter inequality expresses that the volume of the solid remains greater in spherical geometry (up to time t').

3. Proof of $(2.1)_\varepsilon$. Let $(\theta_\varepsilon,h_\varepsilon)$ be the (unique) classical solution of $(1.1)_\varepsilon$ (cf. [6]). By the maximum principle, $0\leqq\theta_\varepsilon\leqq 1$ in all q. Let $t\in(0,T)$ be fixed. Then, for any $\eta\in(0,1)$, we have, by $(1.1)_\varepsilon$,

$$0=\int_\omega\left(\frac{\partial h_\varepsilon}{\partial t}-\Delta\theta_\varepsilon\right)(\theta_\varepsilon-\eta)_+\,dx$$

$$=\int_{\theta_\varepsilon>\eta}\frac{\partial h_\varepsilon}{\partial t}(\theta_\varepsilon-\eta)\,dx+\int_{\theta_\varepsilon>\eta}|\nabla\theta_\varepsilon|^2\,dx-\int_{\gamma_1}\frac{\partial\theta_\varepsilon}{\partial\nu}(\theta_\varepsilon-\eta)\,d\gamma.$$

As (see, e.g., [7, p. 9])

$$\frac{d}{d\eta}\int_{\theta_\varepsilon>\eta}\frac{\partial h_\varepsilon}{\partial t}(\theta_\varepsilon-\eta)\,dx=-\int_{\theta_\varepsilon>\eta}\frac{\partial h_\varepsilon}{\partial t}\,dx,$$

we obtain

$$0=-\int_{\theta_\varepsilon>\eta}\frac{\partial h_\varepsilon}{\partial t}\,dx+\frac{d}{d\eta}\int_{\theta_\varepsilon>\eta}|\nabla\theta_\varepsilon|^2\,dx+\int_{\gamma_1}\frac{\partial\theta_\varepsilon}{\partial\nu}\,d\gamma. \tag{3.1}$$

Let $\mu(\eta)=|\{x\in\omega\colon\theta_\varepsilon(x,t)>\eta\}|$. Using standard rearrangement techniques (see [7]), we get, for almost every $\eta\in(0,1)$,

$$N^2\sigma_N^{2/N}(m_1+\mu(\eta))^{2-(2/N)}\leqq\left[\frac{d}{d\eta}\int_{\theta_\varepsilon>\eta}|\nabla\theta_\varepsilon|\,dx\right]^2\leqq\mu'(\eta)\frac{d}{d\eta}\int_{\theta_\varepsilon>\eta}|\nabla\theta_\varepsilon|^2\,dx. \tag{3.2}$$

Here the first inequality is the isoperimetric inequality relating the volume of the set $\omega_1\cup\{\theta_\varepsilon>\eta\}$ to its perimeter, the latter taken in the sense of De Giorgi (see, e.g., [7] for details). The second inequality is obtained from the Cauchy-Schwarz inequality applied to the difference quotients corresponding to the derivatives after passing to the limit. Combining (3.1) with (3.2) gives

$$N^2\sigma_N^{2/N}(m_1+\mu(\eta))^{2-(2/N)}\leqq-\mu'(\eta)\left[\int_{\gamma_1}\frac{\partial\theta_\varepsilon}{\partial\nu}\,d\gamma-\int_{\theta_\varepsilon>\eta}\frac{\partial h_\varepsilon}{\partial t}\,dx\right] \tag{3.3}$$

for almost every $\eta\in(0,1)$.

Next define

$$k(s,t)=\int_0^s h_{\varepsilon_*}(\sigma,t)\,d\sigma. \tag{3.4}$$

Using results from the theory of relative rearrangement (see [8, Thm. 1.2, p. 60; proof of (2.12), p. 67]), we obtain for almost every η (namely, those for which $|\theta_\varepsilon=\eta|=0$),

$$\begin{aligned}\int_{\theta_\varepsilon>\eta}\frac{\partial h_\varepsilon}{\partial t}\,dx&=\int_{h_\varepsilon>a_\varepsilon(\eta)}\frac{\partial h_\varepsilon}{\partial t}\,dx\\&=\int_0^{\mu(\eta)}\left(\frac{\partial h_\varepsilon}{\partial t}\right)_{*h_\varepsilon}(\sigma,t)\,d\sigma\\&=\int_0^{\mu(\eta)}\frac{\partial h_{\varepsilon_*}}{\partial t}(\sigma,t)\,d\sigma\\&=\frac{\partial k}{\partial t}(\mu(\eta),t).\end{aligned} \tag{3.5}$$

(It should be noted that the rigorous proof of (3.5) is one of the most difficult results in [8]. As it is quite long, we do not repeat it here. We just recall that $(\partial h_\varepsilon/\partial t)_{*h_\varepsilon}$ is called the relative rearrangement of $\partial h_\varepsilon/\partial t$ with respect to h_ε, and may be conceived of as the directional derivative of the map $f\to f_*$, taken at the point h_ε, in the direction $\partial h_\varepsilon/\partial t$.)

Thus (3.3) becomes

$$N^2\sigma_N^{2/N}(m_1+\mu(\eta))^{2-(2/N)}\leqq-\mu'(\eta)\left[\int_{\gamma_1}\frac{\partial\theta_\varepsilon}{\partial\nu}\,d\gamma-\frac{\partial k}{\partial t}(\mu(\eta),t)\right] \tag{3.6}$$

(for almost every $\eta \in (0, 1)$). Set (for $t \in (0, T)$ still fixed)

$$F(s) = \int_{\gamma_1} \frac{\partial \theta_\varepsilon}{\partial \nu} \, d\gamma - \frac{\partial k}{\partial t}(s, t). \tag{3.7}$$

Then F is a continuous function on $\bar{\omega}_*$ because $\partial k / \partial t (s, t) = \int_0^s (\partial h_{\varepsilon *} / \partial t)(\sigma, t) \, d\sigma = \int_0^s (\partial h_\varepsilon / \partial t)_{*h_\varepsilon}(\sigma, t) \, d\sigma$ and $(\partial h_\varepsilon / \partial t)_{*h_\varepsilon}(\sigma, t)$ is integrable as a function of σ (see [8]). We will now prove that F is also nonnegative on $\bar{\omega}_*$.

By (3.6), $(F \circ \mu)(\eta) \geqq 0$ for almost every $\eta \in (0, 1)$. Since μ is continuous from the right, so is $F \circ \mu$. It follows that $F \circ \mu \geqq 0$ on $[0, 1)$, and then on $[0, 1]$, since $(F \circ \mu)(1) = F(0) = \int_{\gamma_1} (\partial \theta_\varepsilon / \partial \nu) \, d\gamma \geqq 0$. It follows next that $F \circ \bar{\mu} \geqq 0$ on $(0, 1]$, where $\bar{\mu}(\eta) = |\theta_\varepsilon \geqq \eta|$, for $\mu(\eta - \delta) \to \bar{\mu}(\eta)$ as $\delta \downarrow 0$. Furthermore,

$$(F \circ \bar{\mu})(0) = F(|\omega|) = \int_{\gamma_1} \frac{\partial \theta_\varepsilon}{\partial \nu} \, d\gamma - \int_0^{|\omega|} \frac{\partial h_{\varepsilon *}}{\partial t}(\sigma, t) \, d\sigma.$$

This last integral also equals $\int_\omega (\partial h_\varepsilon / \partial t) \, dx = \int_\omega \Delta \theta_\varepsilon \, dx = \int_{\partial \omega} (\partial \theta_\varepsilon / \partial \nu) \, d\gamma$ (see [8]; also compare (3.5)), so

$$(F \circ \bar{\mu})(0) = -\int_{\gamma_0} \frac{\partial \theta_\varepsilon}{\partial \nu} \, d\gamma \geqq 0.$$

Thus $F \circ \bar{\mu} \geqq 0$ on all $[0, 1]$. Now let $s \in \bar{\omega}_*$, and set $\eta = \theta_{\varepsilon_*}(s)$, $s' = \mu(\eta)$, $s'' = \bar{\mu}(\eta)$. Then $s' \leqq s \leqq s''$, and we will prove that $F \geqq 0$ on $[s', s'']$. We have $\partial F / \partial s = -(\partial h_{\varepsilon_*} / \partial t)$. Since $\theta_\varepsilon \in H^1(0, T; L^2(\omega))$ (see [4] or § 4 below), $\theta_{\varepsilon_*} \in H^1(0, T; L^2(\omega_*))$ and $\partial h_{\varepsilon_*} / \partial t = a'_\varepsilon(\theta_{\varepsilon_*}) \, \partial \theta_{\varepsilon_*} / \partial t$ is almost everywhere constant on (s', s'') (see [8, pp. 60, 63]). Therefore F is affine on $[s', s'']$, and since $F(s')$ and $F(s'')$ were shown to be nonnegative, we conclude that $F \geqq 0$ on $[s', s'']$. In particular $F(s) \geqq 0$ and since $s \in \bar{\omega}_*$ was arbitrary this proves that $F \geqq 0$ on $\bar{\omega}_*$.

Now (3.3) can be written

$$1 \leqq -N^{-2} \sigma_N^{-(2/N)} (m_1 + \mu(\eta))^{(2/N)-2} F(\mu(\eta)) \mu'(\eta) \quad \text{a.e. } \eta \in (0, 1).$$

Using that $\mu(\eta)$ is a nonincreasing function, integration from η to η', where $0 \leqq \eta < \eta' \leqq 1$, gives

$$\eta' - \eta \leqq N^{-2} \sigma_N^{-(2/N)} \int_{\mu(\eta')}^{\mu(\eta)} (m_1 + s)^{(2/N)-2} F(s) \, ds.$$

As in [7, pp. 24, 31] and [8], this shows that

$$N^{-2} \sigma_N^{-(2/N)} (m_1 + s)^{(2/N)-2} F(s) + \frac{\partial \theta_{\varepsilon *}}{\partial s} \geqq 0 \quad \text{a.e. } s \in \omega_*. \tag{3.8}$$

Now we take the time-dependence into account. Set

$$y(s, t) = \int_0^s h_{\varepsilon_*}(\sigma, t) \, d\sigma - \int_0^t \int_{\gamma_1} \frac{\partial \theta_\varepsilon}{\partial \nu}(x, \tau) \, d\gamma \, d\tau \tag{3.9}$$

for $(s, t) \in \bar{\omega}_* \times [0, T]$. Then

$$\frac{\partial y}{\partial t}(s, t) = -F(s), \qquad \frac{\partial y}{\partial s}(s, t) = h_{\varepsilon_*}(s, t).$$

Since $\theta_\varepsilon = b_\varepsilon(h_\varepsilon)$, where $b_\varepsilon = a_\varepsilon^{-1}$ is strictly increasing, we also have

$$\theta_{\varepsilon_*} = b_\varepsilon(h_{\varepsilon_*}) = b_\varepsilon\left(\frac{\partial y}{\partial s}\right).$$

Therefore (3.8) shows that y satisfies

$$N^{-2} \sigma_N^{-(2/N)} (m_1 + s)^{(2/N)-2} \frac{\partial y}{\partial t} - \frac{\partial}{\partial s}\left(b_\varepsilon\left(\frac{\partial y}{\partial s}\right)\right) \leqq 0 \quad \text{a.e. in } q_* = \omega_* \times (0, T), \tag{3.10}$$

i.e., y is a subsolution of a parabolic equation. Moreover, we get the following boundary and initial conditions for y:

$$(3.11)\qquad \frac{\partial y}{\partial s}(0,t)=a_\varepsilon(1),\qquad \frac{\partial y}{\partial s}(|\omega|,t)=a_\varepsilon(0),\qquad y(s,0)=y_{\varepsilon_0}(s),$$

where $y_{\varepsilon_0}(s)=\int_0^s h_{\varepsilon_0 *}(\sigma)\,d\sigma$.

For the symmetrized problem $\widetilde{(1.1)}_\varepsilon$ we obtain as in [8], for

$$\widetilde{(3.9)}\qquad Y(s,t)=\int_0^s H_{\varepsilon *}(\sigma,t)\,d\sigma-\int_0^t\int_{\Gamma_1}\frac{\partial\Theta_\varepsilon}{\partial\nu}(x,\tau)\,d\gamma\,d\tau,$$

$$\widetilde{(3.10)}\qquad N^{-2}\sigma_N^{-(2/N)}(m_1+s)^{(2/N)-2}\frac{\partial Y}{\partial t}-\frac{\partial}{\partial s}\left(b_\varepsilon\left(\frac{\partial Y}{\partial s}\right)\right)=0\quad\text{in } q_*(=Q_*),$$

$$\widetilde{(3.11)}\qquad \frac{\partial Y}{\partial s}(0,t)=a_\varepsilon(1),\qquad \frac{\partial Y}{\partial s}(|\omega|,t)=a_\varepsilon(0),\qquad Y(s,0)=y_{\varepsilon_0}(s)$$

(observe that $(\mathbf{h}_{\varepsilon_0})_*=h_{\varepsilon_0 *}$). In fact, since Θ_ε, H_ε, are radially symmetric and decreasing, the first line in $\widetilde{(1.1)}_\varepsilon$ can be written:

$$\frac{\partial H_{\varepsilon *}}{\partial t}-\frac{\partial}{\partial s}\left(N^2\sigma_N^{(2/N)}(m_1+s)^{2-(2/N)}\frac{\partial\Theta_{\varepsilon *}}{\partial s}\right)=0\quad\text{in } q_*.$$

Now, we get $\widetilde{(3.10)}$ by integrating between 0 and s, noting that

$$\frac{\partial H_{\varepsilon *}}{\partial t}=\frac{\partial^2 Y}{\partial s\,\partial t}\quad\text{and}\quad\int_{\Gamma_1}\frac{\partial\Theta_\varepsilon}{\partial\nu}\,d\gamma=-N^2\sigma_N^{(2/N)}m_1^{2-(2/N)}\frac{\partial\Theta_{\varepsilon *}}{\partial s}(0).$$

Now $(2.1)_\varepsilon$ of Theorem 1 simply states that $y\leqq Y$ in $\bar q_*$. To prove this inequality, we multiply the difference between (3.10) and $\widetilde{(3.10)}$ by $(y-Y)_+$ and integrate with respect to s for fixed t. Taking the boundary conditions for y and Y into account and using that b_ε is monotone increasing, we get

$$\begin{aligned}
0\geqq&\int_0^{|\omega|}(m_1+s)^{(2/N)-2}\frac{\partial(y-Y)}{\partial t}(y-Y)_+\,ds\\
&\quad-N^2\sigma_N^{2/N}\int_0^{|\omega|}\frac{\partial}{\partial s}\left[b_\varepsilon\left(\frac{\partial y}{\partial s}\right)-b_\varepsilon\left(\frac{\partial Y}{\partial s}\right)\right](y-Y)_+\,ds\\
=&\frac{1}{2}\int_0^{|\omega|}(m_1+s)^{(2/N)-2}\frac{\partial}{\partial t}(y-Y)_+^2\,ds\\
&\quad+N^2\sigma_N^{2/N}\int_0^{|\omega|}\left[b_\varepsilon\left(\frac{\partial y}{\partial s}\right)-b_\varepsilon\left(\frac{\partial Y}{\partial s}\right)\right]\frac{\partial}{\partial s}(y-Y)_+\,ds\\
=&\frac{1}{2}\frac{d}{dt}\int_0^{|\omega|}(m_1+s)^{(2/N)-2}(y-Y)_+^2\,ds\\
&\quad+N^2\sigma_N^{2/N}\int_{y>Y}\left[b_\varepsilon\left(\frac{\partial y}{\partial s}\right)-b_\varepsilon\left(\frac{\partial Y}{\partial s}\right)\right]\left[\frac{\partial y}{\partial s}-\frac{\partial Y}{\partial s}\right]ds\\
\geqq&\frac{1}{2}\frac{d}{dt}\int_0^{|\omega|}(m_1+s)^{(2/N)-2}(y-Y)_+^2\,ds.
\end{aligned}$$

Since $y=Y$ for $t=0$, it follows that

$$\int_0^{|\omega|} (m_1+s)^{(2/N)-2}(y-Y)_+^2\, ds \leqq 0 \quad \text{for } t \geqq 0,$$

and hence that $y \leqq Y$ in $\bar{q}_*$ as desired.

4. Proof of (2.1). The weak solution (θ, h) of (1.1) is obtained as the limit of the solution $(\theta_\varepsilon, h_\varepsilon)$ of $(1.1)_\varepsilon$ as $\varepsilon \to 0$, and we will accordingly obtain (2.1) by letting $\varepsilon \to 0$ in $(2.1)_\varepsilon$. For convenience we review part of the construction of (θ, h).

Let $(\theta_\varepsilon, h_\varepsilon)$ be the solution of $(1.1)_\varepsilon$, and let $q_t = \omega \times (0, t)$ for $t \in (0, T)$. Then we have

$$\begin{aligned} 0 &= \iint_{q_t} \frac{\partial \theta_\varepsilon}{\partial \tau}\left(\frac{\partial h_\varepsilon}{\partial \tau} - \Delta \theta_\varepsilon\right) dx\, d\tau \\ &= \iint_{q_t} \frac{\partial \theta_\varepsilon}{\partial \tau}\frac{\partial h_\varepsilon}{\partial \tau}\, dx\, dt + \int_0^t \left[\iint_\omega \nabla \frac{\partial \theta_\varepsilon}{\partial \tau} \nabla \theta_\varepsilon\, dx - \int_{\partial\omega} \frac{\partial \theta_\varepsilon}{\partial \tau}\frac{\partial \theta_\varepsilon}{\partial \nu}\, d\gamma\right] d\tau \end{aligned}$$

or

$$\iint_{q_t} a_\varepsilon'(\theta_\varepsilon)\left|\frac{\partial \theta_\varepsilon}{\partial \tau}\right|^2 dx\, d\tau + \frac{1}{2}\int_\omega |\nabla \theta_\varepsilon|^2\, dx = \frac{1}{2}\int_\omega |\nabla \theta_{\varepsilon_0}|^2\, dx.$$

By assumptions (1.4) and (1.7), when $\varepsilon \to 0$, $\int_\omega |\nabla \theta_{\varepsilon_0}|^2\, dx$ is bounded, and a_ε' is bounded from below. Hence

$$\iint_{q_t} \left|\frac{\partial \theta_\varepsilon}{\partial \tau}\right|^2 dx\, d\tau \leqq C, \tag{4.1}$$

$$\int_\omega |\nabla \theta_\varepsilon(x, t)|^2\, dx \leqq C, \tag{4.2}$$

for some constant C independent of ε and t. By the maximum principle, the families $\{\theta_\varepsilon\}_{\varepsilon>0}$ and $\{h_\varepsilon\}_{\varepsilon>0}$ are bounded in $L^\infty(q)$, and, by the above estimates, it then follows that $\{\theta_\varepsilon\}_{\varepsilon>0}$ is bounded in $L^\infty(0, T; H^1(\omega))$ and in $H^1(q)$. By repeated extraction of subsequences from $\{\varepsilon\}$ we thus can find a sequence ε_n such that

$$\begin{aligned} &\theta_{\varepsilon_n} \to \theta \text{ weakly* in } L^\infty(q) \text{ and } L^\infty(0, T; H^1(\omega)), \\ &\qquad \text{weakly in } H^1(q), \\ &\qquad \text{strongly in } L^2(q) \text{ (by compactness)}, \\ &h_{\varepsilon_n} \to h \text{ weakly* in } L^\infty(q) \end{aligned} \tag{4.3}$$

for some pair (θ, h) satisfying

$$\theta \in L^\infty(q) \cap H^1(q) \cap L^\infty(0, T; H^1(\omega)), \qquad h \in L^\infty(q). \tag{4.4}$$

In the following, we shall replace ε_n by ε for simplicity.

Now (θ, h) is the required weak solution. In fact, it follows immediately that (1.3) holds, since $(\theta_\varepsilon, h_\varepsilon)$ satisfies the same equation with h_0 replaced by h_{ε_0}. To check that $h \in a(\theta)$ almost everywhere, it is enough (since a is maximally monotone) to check that

$$\langle h' - h, \theta' - \theta \rangle \geqq 0 \tag{4.5}$$

(where $\langle f, g\rangle = \iint_q fg\,dx\,dt$) for all $\theta', h' \in L^\infty(q)$ satisfying $h' \in a(\theta')$ almost everywhere. Writing

$$\langle h'-h, \theta'-\theta\rangle = \langle h_\varepsilon - h, \theta'-\theta\rangle + \langle h'-h_\varepsilon, \theta_\varepsilon-\theta\rangle + \langle h'-h_\varepsilon, \theta'-\theta_\varepsilon\rangle, \tag{4.6}$$

we have (as $\varepsilon \to 0$)

$$\langle h_\varepsilon - h, \theta'-\theta\rangle \to 0 \tag{4.7}$$

since $h_\varepsilon \to h$ weakly* in $L^\infty(q)$ (by (4.3)), and

$$|\langle h'-h_\varepsilon, \theta_\varepsilon - \theta\rangle| \leqq \|h'-h_\varepsilon\|_{L^2(q)} \|\theta_\varepsilon - \theta\|_{L^2(q)} \to 0, \tag{4.8}$$

since $\theta_\varepsilon \to \theta$ in $L^2(q)$, and $\|h'-h_\varepsilon\|_{L^2(q)}$ is bounded (by (4.3)).

Remark. This is the only place where we use the fact that $b = a^{-1}$ is Lipschitz continuous. This property of a makes it possible to choose a_ε satisfying (1.4), from which we get the estimate (4.1) for $\partial\theta_\varepsilon/\partial t$; then weak convergence in $H^1(q)$ and strong convergence in $L^2(q)$ of θ_ε to θ follows. Observe that (4.2) holds independently of (1.4).

For the last term in (4.6), we have

$$\begin{aligned}\langle h'-h_\varepsilon, \theta'-\theta_\varepsilon\rangle &= \langle h'-h_\varepsilon, b(h') - b_\varepsilon(h_\varepsilon)\rangle \\ &= \langle h'-h_\varepsilon, b_\varepsilon(h') - b_\varepsilon(h_\varepsilon)\rangle + \langle h'-h_\varepsilon, b(h') - b_\varepsilon(h')\rangle.\end{aligned}$$

The first bracket in the last member is nonnegative as b_ε is nondecreasing, and the second one tends to zero with ε, as b_ε converges uniformly to b:

$$|\langle h'-h_\varepsilon, b(h') - b_\varepsilon(h')\rangle| \leqq \|h'-h_\varepsilon\|_{L^\infty(q)} \int_q |b(h') - b_\varepsilon(h')|\,dx\,dt.$$

Then

$$\varliminf_{\varepsilon\to 0} \langle h'-h_\varepsilon, \theta'-\theta_\varepsilon\rangle \geqq 0. \tag{4.9}$$

Now (4.5) follows by combining (4.7)-(4.9). Thus (θ, h) is a weak solution.

We now pass to the limit in $(2.1)_\varepsilon$. First, we have to give a weak interpretation of the two members of (2.1), since the regularity of θ that we have is not enough for $\partial\theta/\partial\nu$ and $\partial\Theta/\partial\nu$ to make classical sense.

Let φ be an arbitrary smooth function in ω (say, $\varphi \in \mathscr{C}^2(\bar\omega)$) with boundary values $\varphi = 0$ on γ_0, $\varphi = 1$ on γ_1. Then for (θ, h), a "good enough" solution of $(\partial h/\partial t) - \Delta\theta = 0$,

$$\begin{aligned}\int_0^t \int_{\gamma_1} \frac{\partial\theta}{\partial\nu}\,d\gamma\,d\tau &= \int_0^t \int_{\partial\omega} \varphi \frac{\partial\theta}{\partial\nu}\,d\gamma\,d\tau \\ &= \int_0^t \int_\omega \nabla\varphi\,\nabla\theta\,dx\,d\tau + \int_0^t \int_\omega \varphi\,\Delta\theta\,dx\,d\tau \\ &= \int_0^t \int_\omega \nabla\varphi\,\nabla\theta\,dx\,d\tau + \int_\omega (h(x,t) - h_0(x))\varphi(x)\,dx.\end{aligned} \tag{4.10}$$

Here the last member makes sense for almost every t for any $\theta \in L^\infty(0, T; H^1(\omega))$, $h \in L^\infty(q)$, and defines an (almost everywhere) bounded measurable function of t. Therefore, when (θ, h) is the weak solution of (1.1) satisfying (4.4), we choose the last member of (4.10) to be our definition of $\int_0^t \int_{\gamma_1} (\partial\theta/\partial\nu)\,d\gamma\,d\tau$ for almost every $t \in (0, T)$; we choose similarly for $\int_0^t \int_{\Gamma_1} \partial\Theta/\partial\nu\,d\gamma\,d\tau$. We still have to check that this definition is independent of the choice of φ. This amounts to showing that if $\varphi \in \mathscr{C}^2(\bar\omega)$ has boundary values zero on $\partial\omega$, then

$$\int_0^t \int_\omega \nabla\varphi\,\nabla\theta\,dx\,d\tau + \int_\omega (h(x,t) - h_0(x))\varphi(x)\,dx = 0 \quad \text{a.e. } t \in (0, T). \tag{4.11}$$

Now, (4.11) is an equation of the form $\int_0^t f(\tau)\,d\tau + F(t) = 0$ for almost every $t \in (0, T)$, for two functions f and F in $L^1(0, T)$. The latter equation is equivalent to $\int_0^T f(t)\psi(t)\,dt = \int_0^T F(t)\psi'(t)\,dt$ for every $\psi \in \mathscr{C}^1[0, T]$ with $\psi(T) = 0$. Thus (4.11) is equivalent to

$$\int_0^T \int_\omega \nabla\theta(x, t)\nabla\varphi(x)\psi(t)\,dx\,dt = \int_0^T \int_\omega (h(x, t) - h_0(x))\varphi(x)\psi'(t)\,dx\,dt \tag{4.12}$$

for ψ as above. Now, the truth of (4.12) follows by integration in the definition (1.3) for a weak solution, taking $\varphi(x)\psi(t)$ as a test function (this integration by parts is justified by $\theta \in L^\infty(0, T; H^1(\omega))$ for (θ, h) a weak solution of (1.1)).

For later use we remark that $\int_0^t \int_{\gamma_0} (\partial\theta/\partial\nu)\,d\gamma\,d\tau$ may be defined by the same formula (4.10), taking a test function with the exchanged boundary values, e.g., $1 - \varphi$, with φ as in (4.10). Then, for such φ's,

$$\int_0^t \int_{\gamma_0} \frac{\partial\theta}{\partial\nu}\,d\gamma\,d\tau = -\int_0^t \int_\omega \nabla\varphi\nabla\theta\,dx\,d\tau + \int_\omega (h(x, t) - h_0(x))(1 - \varphi(x))\,dx, \tag{4.10$'$}$$

and from (4.10), (4.10)$'$,

$$\begin{aligned} \int_0^t \int_{\gamma_0} \frac{\partial\theta}{\partial\nu}\,d\gamma\,d\tau + \int_0^t \int_{\gamma_1} \frac{\partial\theta}{\partial\nu}\,d\gamma\,d\tau &= \int_\omega (h(x, t) - h_0(x))\,dx \\ &= \int_0^{|\omega|} (h_*(\sigma, t) - h_{0*}(\sigma))\,d\sigma. \end{aligned} \tag{4.13}$$

Now, with $\varphi \in \mathscr{C}^2(\bar{\omega})$ (respectively, $\Phi \in \mathscr{C}^2(\bar{\Omega})$) in (4.10), (2.1) (to be proven) becomes

$$\begin{aligned} &\int_0^t \int_\omega \nabla\theta\nabla\varphi\,dx\,d\tau + \int_\omega (h(x, t) - h_0(x))\varphi(x)\,dx - \int_0^s h_*(\sigma, t)\,d\sigma \\ &\geqq \int_0^t \int_\Omega \nabla\Theta\nabla\Phi\,dx\,d\tau + \int_\Omega (H(x, t) - \mathbf{h}_0(x))\Phi(x)\,dx - \int_0^s H_*(\sigma, t)\,d\sigma \end{aligned} \tag{4.14}$$

for $s \in \bar{\omega}_*$ and almost every $t \in (0, T)$. For fixed $s \in \bar{\omega}_*$, both members above are integrable functions of t. Therefore (4.14) is equivalent to a statement that, for every $s \in \bar{\omega}_*$, and every nonnegative function $\psi \in \mathscr{C}[0, T]$,

$$\begin{aligned} &\int_0^T \int_0^t \int_\omega \nabla\theta(x, \tau)\nabla\varphi(x)\psi(t)\,dx\,d\tau\,dt + \iint_q (h(x, t) - h_0(x))\varphi(x)\psi(t)\,dx\,dt \\ &\qquad - \int_0^T \int_0^s h_*(\sigma, t)\psi(t)\,d\sigma\,dt \\ &\geqq \int_0^T \int_0^t \int_\Omega \nabla\Theta(x, \tau)\nabla\Phi(x)\psi(t)\,dx\,d\tau\,dt \\ &\qquad + \iint_Q (H(x, t) - \mathbf{h}_0(x))\Phi(x)\psi(t)\,dx\,dt \\ &\qquad - \int_0^T \int_0^s H_*(\sigma, t)\psi(t)\,d\sigma\,dt \end{aligned} \tag{4.15}$$

(we have to make this extra integration because the convergence h_ε to h is not established for fixed t (see (4.3)), as we would need to get (4.14) from the corresponding $(4.14)_\varepsilon$).

By $(2.1)_\varepsilon$, we have $(4.15)_\varepsilon$, that is, (4.15) holds with θ, Θ, h, H, h_0, $\mathbf{h}_0$ replaced by θ_ε, Θ_ε, h_ε, H_ε, h_{ε_0}, $\mathbf{h}_{\varepsilon_0}$, respectively. For the first terms on the left-hand sides of

(4.15) and $(4.15)_\varepsilon$, we have

$$\int_0^T \int_0^t \int_\omega \nabla \theta_\varepsilon(x, \tau) \nabla \varphi(x) \psi(t)\, dx\, d\tau\, dt \to \int_0^T \int_0^t \int_\omega \nabla \theta(x, \tau) \nabla \varphi(x) \psi(t)\, dx\, d\tau\, dt$$

as $\varepsilon \to 0$, because $\theta_\varepsilon \to \theta$ weakly* in $L^\infty(0, T; H^1(\omega))$ (see (4.3)). We have similar results for the first terms on the right-hand sides of (4.15) and $(4.15)_\varepsilon$. As for the second terms on the left-hand sides,

$$\iint_q (h_\varepsilon(x, t) - h_{\varepsilon_0}(x)) \varphi(x) \psi(t)\, dx\, dt \to \iint_q (h(x, t) - h_0(x)) \varphi(x) \psi(t)\, dx\, dt$$

because $h_\varepsilon \to h$ weakly* in $L^\infty(q)$ (see (4.3)), and $h_{\varepsilon_0} \to h_0$ in $L^1(q)$ (see (1.6)). Similarly, for the right-hand sides, ($\mathbf{h}_{\varepsilon_0} \to \mathbf{h}_0$ in $L^1(Q)$ follows from $h_{\varepsilon_0} \to h_0$ because the rearrangement operator is a contraction in L^p-spaces (see [7, pp. 7, 8])). For the last terms on the left-hand sides we have

$$\text{(4.16)} \qquad \varliminf_{\varepsilon \to 0} \int_0^T \int_0^s h_{\varepsilon_*}(\sigma, t) \psi(t)\, d\sigma\, dt \geqq \int_0^T \int_0^s h_*(\sigma, t) \psi(t)\, d\sigma\, dt$$

(the map $h \to \int_0^T \int_0^s h_*(\sigma, t)\psi(t)\, d\sigma\, dt$ is weakly lower semicontinuous $L^1(q) \to \mathbb{R}$). In fact, for every measurable subset $E(t) \subset \omega$, with $|E(t)| = s$, by the Hardy-Littlewood inequality

$$\int_0^s h_{\varepsilon_*}(\sigma, t)\, d\sigma \geqq \int_{E(t)} h_\varepsilon(x, t)\, dx;$$

hence

$$\int_0^T \int_0^s h_{\varepsilon_*}(\sigma, t) \psi(t)\, d\sigma\, dt \geqq \int_0^T \int_{E(t)} h_\varepsilon(x, t) \psi(t)\, dx\, dt,$$

and, if we let $\varepsilon \to 0$,

$$\varliminf_{\varepsilon \to 0} \int_0^T \int_0^s h_{\varepsilon_*}(\sigma, t) \psi(t)\, d\sigma\, dt \geqq \int_0^T \int_{E(t)} h(x, t) \psi(t)\, dx\, dt$$

(since $h_\varepsilon \to h$ weakly in $L^1(q)$ by (4.3)). Now, $E(t)$ can be chosen such that $|E(t) = s|$ and

$$\int_{E(t)} h(x, t)\, dx = \int_0^s h_*(\sigma, t)\, d\sigma,$$

which proves (4.16). Finally we consider the last terms in the right-hand sides of (4.15) and $(4.15)_\varepsilon$. It is clear (by uniqueness of solutions) that Θ_ε, Θ, H_ε, and H are radially symmetric, i.e., are functions of $r = |x|$ and t only. Moreover, Θ_ε and H_ε are nonincreasing as functions of $|x|$ for fixed t (i.e., $\Theta_\varepsilon(x, t) \geqq \Theta_\varepsilon(y, t)$ whenever $|x| \leqq |y|$), as can be seen by applying the parabolic maximum principle to $\partial\Theta_\varepsilon/\partial r$ and $\partial H_\varepsilon/\partial r$ (this pair of functions satisfies a parabolic equation). It also follows that Θ and H are nonincreasing as functions of $|x|$ because the set of such functions (nonincreasing in $|x|$) is closed and convex in, e.g., $L^2(Q)$ and therefore weakly closed in $L^2(Q)$ (use (4.3)). From the above, we get

$$\text{(4.17)} \qquad \int_0^s H_{\varepsilon_*}(\sigma, t)\, d\sigma = \int_{m_1 < \sigma_N |x|^N < m_1 + s} H_\varepsilon(x, t)\, dx,$$

$$\text{(4.18)} \qquad \int_0^s H_*(\sigma, t)\, d\sigma = \int_{m_1 < \sigma_N |x|^N < m_1 + s} H(x, t)\, dx,$$

and since $H_\varepsilon \to H$ weakly* in $L^\infty(Q)$ (see (4.3)) we finally obtain, as $\varepsilon \to 0$,

$$\int_0^T \int_0^s H_{\varepsilon_*}(\sigma, t)\psi(t)\, d\sigma\, dt \to \int_0^T \int_0^s H_*(\sigma, t)\psi(t)\, d\sigma\, dt.$$

Now, all the above shows that (4.15) results from letting $\varepsilon \to 0$ in $(4.15)_\varepsilon$. This completes the proof of Theorem 1.

5. Proofs of (2.2)–(2.5) and (2.7)–(2.12). We first observe that, in view of (4.13), (2.1) also can be written:

$$(5.1)\quad -\int_0^t \int_{\gamma_0} \frac{\partial \theta}{\partial \nu}\, d\gamma\, d\tau + \int_s^{|\omega|} h_*(\sigma, t)\, d\sigma \geqq -\int_0^t \int_{\Gamma_0} \frac{\partial \Theta}{\partial \nu}\, d\gamma\, d\tau + \int_s^{|\omega|} H_*(\sigma, t)\, d\sigma.$$

By taking $s = 0$ in (2.1), and $s = |\omega|$ in (5.1), we obtain (2.3) and (2.4).

More generally, for "classical" solutions (θ, h) and (Θ, H), if $s > 0$ is such that $\gamma(s, \tau) = \{x \in \omega: h(x, \tau) = h_*(s, \tau)\}$ and $\Gamma(s, \tau) = \{x \in \Omega: H(x, \tau) = H_*(s, \tau)\}$ are regular curves (in particular, if they have measure zero) for almost every $\tau \in (0, t)$, using the technique of relative rearrangement [8] gives us

$$\begin{aligned}\int_0^s (h_*(\sigma, t) - h_{0*}(\sigma))\, d\sigma &= \int_0^s \int_0^t \frac{\partial h_*}{\partial \tau}(\sigma, \tau)\, d\tau\, d\sigma \\ &= \int_0^t \int_0^s \frac{\partial h_*}{\partial \tau}(\sigma, \tau)\, d\sigma\, d\tau \\ &= \int_0^t \int_{h(x,\tau) > h_*(s,\tau)} \frac{\partial h}{\partial \tau}\, dx\, d\tau \\ &= \int_0^t \int_{h(x,\tau) > h_*(s,\tau)} \Delta\theta\, dx\, d\tau \\ &= \int_0^t \int_{\gamma_1} \frac{\partial \theta}{\partial \nu}\, d\gamma\, d\tau + \int_0^t \int_{\gamma(s,\tau)} \frac{\partial \theta}{\partial \nu}\, d\gamma\, d\tau\end{aligned}$$

($\partial/\partial\nu$ the outward normal of $\{h(x, \tau) > h_*(s, \tau)\}$); we proceed similarly for (Θ, H). Thus in (2.1),

$$\int_0^t \int_{\gamma_1} \frac{\partial \theta}{\partial \nu}\, d\gamma\, d\tau - \int_0^s h_*(\sigma, t)\, d\tau = -\int_0^t \int_{\gamma(s,\tau)} \frac{\partial \theta}{\partial \nu}\, d\gamma\, d\tau - \int_0^s h_{0*}(\sigma)\, d\sigma,$$

and similarly for (Θ, H), which explains (2.2). (Note also that $(\mathbf{h}_0)_* = h_{0*}$.)

Now, (2.5) holds because, by the Hardy–Littlewood inequality, $\int_{E(t)} h\, dx \leqq \int_0^s h_*\, d\sigma$, when $|E(t)| = s$, and $\int_{m_1 < \sigma_N |x|^N < m_1 + s} H\, dx = \int_0^s H_*\, d\sigma$ (see (4.18)).

Next, suppose that $\int_0^t \int_{\gamma_0} \partial\theta/\partial\nu\, d\gamma\, d\tau = 0$ (the integral being at least defined by (4.10)′), for almost every $t \in (0, t')$ for some $t' > 0$. This occurs in the one-phase problem (with $\lambda = 0$) if the liquid phase does not reach γ_0 initially (because θ is identically zero in the solid phase when $\lambda = 0$). By (2.4), then also

$$\int_0^t \int_{\Gamma_0} \frac{\partial \Theta}{\partial \nu}\, d\gamma\, d\tau = 0 \quad \text{a.e. } t \in (0, t'),$$

that is, we get (2.7), and (5.1) simply reduces to (2.8) for $t \in (0, t')$.

Now (2.9) follows easily as in the convexity result [1, p. 174], and (2.10) follows from (2.8) by dividing by $|\omega| - s$ and letting s tend to $|\omega|$. Choosing $\Phi(h) = -[(h - \beta)_-]^p$ $(p \geqq 1)$ in (2.9) with $s = 0$, we obtain (2.11) for $p < \infty$; letting p tend to ∞ then gives

(2.11) for $p=\infty$. Choosing $\Phi(h)=-(h-\beta)_-/(\alpha+\beta)$, $\beta>-\alpha$, in (2.9) with $s=0$, we finally obtain (2.12) by letting $\beta\to-\alpha$ (observe that $h, H\geqq-\alpha$ almost everywhere when $\lambda=0$).

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