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IDENTIFICATION OF THE CONDUCTIVITY COEFFICIENT IN AN ELLIPTIC EQUATION*

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Abstract. Consider an elliptic equation in a two-dimensional domain Ω with conductivity coefficient $a=1+k\chi_D$ $(k\neq 0)$ where D is a subdomain of Ω. From the measurements of a pair of Dirichlet and Neumann data one wishes to identify D. It is proved that this problem is stable in some local sense.

Key words. elliptic equations, conductivity coefficient, identification problem, electrical prospecting

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Introduction. Consider an elliptic equation

$$\operatorname{div}(a\nabla u)=0 \quad \text{in } \Omega \tag{0.1}$$

with Dirichlet data

$$u=f \quad \text{on } \partial\Omega, \quad f\not\equiv \text{const}, \tag{0.2}$$

and with coefficient $a=1+k\chi_D$ $(-1<k<\infty,\ k\neq 0)$, where D is an unknown subdomain of Ω. We seek to determine D by measurements of the Neumann data

$$\frac{\partial u}{\partial \nu}=g \quad \text{on } \partial\Omega. \tag{0.3}$$

This identification problem arises in electrical prospecting, whereby one wishes to discover the location of metals or fluid reservoirs inside the earth.

Let $D(t)$ be a 1-parameter monotone family of domains with $D(0)=D$ such that

$$\left.\frac{d}{dt}\chi_{D(t)}\right|_{t=0}\neq 0 \quad \text{in } \mathscr{D}', \tag{0.4}$$

and denote by $u(t)\equiv u(x,t)$ the solution of (0.1), (0.2) corresponding to $a=1+k\chi_{D(t)}$. Our main result asserts that, in case the $D(t)$ are affine transformations of D, C^1 in t, for all t with $|t|$ small enough, there holds

$$\left\|\frac{\partial}{\partial \nu}[u(t)-u(0)]\right\|_{L^2(\partial\Omega)}\geqq c|t| \tag{0.5}$$

where c is a positive constant.

If we denote by Φ the mapping from a to g (when f is fixed) then (0.5) means formally that $d\Phi/da\neq 0$; thus, if $\Phi(a_1)=g_1$, $\Phi(a_2)=g_2$ and $\|a_2-a_1\|$ is small, then

$$\|a_2-a_1\|\leqq C\|g_2-g_1\| \quad \text{where } \frac{1}{\|d\Phi/da\|}\leqq C<\infty. \tag{0.6}$$

This means that the computation of D among a monotone family of domains is stable with respect to small errors in the measurement of the Neumann data; for more details on the significance of a result of this type see [12] and § 1 below.

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There are other versions of identification problems. In [2], [4]-[6], [11] one measures the quadratic form

$$Q_a = \int_\Omega a|\nabla u|^2$$

for *all* f and shows that this determines $a = a(x)$ in Ω, provided that either $a(x)$ is piecewise analytic [4], [6] or $\|a-1\|$ is small enough [11]. For some special domains the identification problem can be resolved by separation of variables [3], [8] or by explicit representation of u by means of Green's function [9].

In another version (0.1) is replaced by

$$\operatorname{div}(a\nabla u) = l \quad \text{in } \Omega \quad (l \text{ is given})$$

and one wishes to find a, given the knowledge of u throughout *all* of Ω; see [1], [10] and the references given there. This problem is unstable.

References to physical models and numerical computations of identification problems are given in [1], [5].

1. The main result. Let Ω be a bounded simply connected domain in $\mathbb{R}^2$ with $C^{1,\alpha}$ boundary $\partial\Omega$ $(0<\alpha<1)$ and let D be a bounded subdomain of Ω with $C^{2,\alpha}$ boundary ∂D, $\bar{D} \subset \Omega$. We shall designate points in $\mathbb{R}^2$ by $x = (x_1, x_2)$.

Denote by χ_A the characteristic function of a set A.

We assume that D is star-shaped with respect to any point x^* of some nonempty subset D^* of D.

For any $x^* \in D^*$ introduce the 1-parameter family of domains

$$D(t) = \{x^* + (1-t)(x-x^*), x \in D\} \quad (-1<t<1). \tag{1.1}$$

Then $D(t) \subset D(t')$ if $t > t'$. Also

$$\frac{\partial}{\partial t}\chi_{D(t)}\bigg|_{t=0} = \beta \otimes \delta_{\partial D} \quad \text{in } \mathscr{D}', \tag{1.2}$$

that is

$$\frac{\partial}{\partial t}\left[\iint_\Omega \chi_{D(t)}\phi\right]\bigg|_{t=0} = \int_{\partial D} \beta\phi$$

for any $\phi \in C_0^0(\Omega)$, and β is a continuous and strictly negative function on ∂D; $\beta \in C^{1,\alpha}$.

Set $D_e(t) = \Omega \backslash \overline{D(t)}$. We shall use the notation w^e (or w^i) to denote the value of a function w on $\partial D(t)$ taken as a limit from $D_e(t)$ (or $D(t)$).

Let k be a fixed number, $-1<k<0$ or $k>0$, and set

$$a(x,t) = 1 + k\chi_{D(t)}(x), \qquad a(x) = a(x,0). \tag{1.3}$$

Consider the elliptic equation

$$\operatorname{div}(a(x,t)\nabla u) = 0 \quad \text{in } \Omega \tag{1.4}$$

with the Dirichlet condition

$$u = f \quad \text{on} \quad \partial\Omega \tag{1.5}$$

where $f = f(x)$ is in $C^{1,\alpha}(\partial\Omega)$.

It is well known [7] that the solution u of this diffraction problem is in $C^{0,\beta}(\bar{\Omega}) \cap H^1(\Omega)$ for some $0<\beta<1$, as well as in $C^{2,\alpha}(\overline{D(t)})$ and in $C^{2,\alpha}(\overline{D_e(t)}\backslash\partial\Omega)$, and that

$$\frac{\partial u^e}{\partial \nu} = (k+1)\frac{\partial u^i}{\partial \nu} \quad \text{on } \partial D(t) \tag{1.6}$$

where ν is the outward normal to $\partial D(t)$.

Set

$$g(x, t) = \frac{\partial u(x, t)}{\partial \nu}, \qquad x \in \partial\Omega \tag{1.7}$$

where ν is the outward normal to $\partial\Omega$. Then $g \in C^\alpha$.

We would like to determine the conductivity coefficient $a(x)$ from measurements of $g(x) \equiv g(x, 0)$. Since in real terms we can only measure $g(x)$ with some error, we would like to ensure that if the measurements give us a function $g(x, t)$ "close" to $g(x)$ then the corresponding $a(x, t)$ is also "close" to the true coefficient $a(x)$. If that is the case, then by compiling a catalog of various g's corresponding to various a's we can have an effective way of determining the true conductivity: We simply correspond to a function $\tilde{g}$ that we obtained by actual measurements the coefficient a which fits to that g in our catalog that is "nearest" to $\tilde{g}$. This point of view is quite common in inverse problems [12].

If $f \equiv \text{const.}$ then $u \equiv \text{const.}$ for any choice of $a(x, t)$ and thus $g(x, t) \equiv 0$. This means that we cannot gain any information on the coefficient a. Thus we must henceforth assume that

$$f \not\equiv \text{const.} \tag{1.8}$$

THEOREM 1.1. *If* (1.8) *holds then there exists a positive constant c such that*

$$\|g(\cdot, h) - g(\cdot)\|_{L^2(\partial\Omega)} \geqq c|h| \tag{1.9}$$

if $|h|$ *is small enough.*

Theorem 1.1 extends to more general monotone families of domains $D(t)$; see § 3.

Theorem 1.1 means that we can effectively determine D by the procedure outlined in the paragraph following (1.7), provided D is known to be imbedded in a monotone family of domains.

As we shall see in § 3 (Remark 3.2), Theorem 1.1 is generally false if $D(t)$ is not a monotone family (at least in one space dimension, or for Ω an annulus).

The remainder of this paper is devoted to the proof of Theorem 1.1; some generalizations are mentioned at the end of § 3.

2. Proof of Theorem 1.1. Set $g(t) \equiv g(x, t)$. To prove the theorem it suffices to assume that

$$\left\| \frac{g(h) - g(0)}{h} \right\|_{L^2(\partial\Omega)} \to 0 \quad \text{for some sequence } h \to 0 \tag{2.1}$$

and derive a contradiction. From now on h will be restricted to this sequence.

Consider first the case where $0 < h < 1$, so that

$$D(h) \subset D, \tag{2.2}$$

and set

$$a(t) \equiv a(x, t), \qquad u(t) \equiv u(x, t),$$

$$a_h = \frac{a(h) - a(0)}{h}, \qquad v_h = \frac{u(h) - u(0)}{h}.$$

From (1.4) we get

$$\operatorname{div}(a(h)\nabla u(h) - a(0)\nabla u(0)) = 0, \tag{2.3}$$

which implies that there exists a function w^h in $H^1(\Omega)$ such that

$$\frac{1}{h}[a(h)\nabla u(h) - a(0)\nabla u(0)] = \operatorname{curl} w^h; \tag{2.4}$$

here $\operatorname{curl} w = (w_{x_2}, -w_{x_1})$. We normalize w^h so that

$$w^h(x^0) = 0 \quad \text{at some point } x^0 \in \partial\Omega. \tag{2.5}$$

Since

$$\frac{1}{h}[a(h)\nabla u(h) - a(0)\nabla u(0)] = a(0)\nabla u_h + a_h \nabla u(h) \tag{2.6}$$

we can rewrite (2.4) in the form

$$a(0)\frac{\partial}{\partial \bar{z}} v_h + i \frac{\partial}{\partial \bar{z}} w^h = -a_h \frac{\partial u(h)}{\partial \bar{z}}. \tag{2.7}$$

Introduce the function

$$f^h = a(0) v_h + i w^h. \tag{2.8}$$

Then

$$\frac{\partial f^h}{\partial \bar{z}} = \frac{\partial a(0)}{\partial \bar{z}} v_h + a(0) \frac{\partial v_h}{\partial \bar{z}} + i \frac{\partial w^h}{\partial \bar{z}} \quad \text{in } \mathscr{D}' \tag{2.9}$$

and, using (2.7),

$$\frac{\partial f^h}{\partial \bar{z}} = \frac{\partial a(0)}{\partial \bar{z}} v_h - a_h \frac{\partial u(h)}{\partial \bar{z}}: \tag{2.10}$$

the right-hand side is a measure.

It follows by Cauchy's formula that

$$\begin{aligned} f^h(\zeta) = &-\frac{1}{\pi} \iint_\Omega \left(\frac{\partial a(0)}{\partial \bar{z}} v_h - a_h \frac{\partial u(h)}{\partial \bar{z}} \right)(z) \frac{1}{z - \zeta} \, dx \, dy \\ &+ \frac{1}{2\pi i} \int_{\partial\Omega} \frac{f^h(z)}{z - \zeta} \, dz, \qquad \zeta \in \Omega. \end{aligned} \tag{2.11}$$

LEMMA 2.1. *As* $h \to 0$,

$$D_x^\gamma v_h \to 0 \quad \textit{uniformly in compact subsets of } D_e, \quad 0 \leqq |\gamma| \leqq 1, \tag{2.12}$$

and

$$\int_{\partial D} |v_h| \, ds \to 0. \tag{2.13}$$

Proof. Take for simplicity $x^*=0$. Define

$$\Omega_h=\left\{x\in\Omega \text{ and } \frac{x}{1-h}\in\Omega\right\}=\Omega\cap(1-h)\Omega,$$

$$u(x)=u(x,0),$$

$$U^h(x)=\frac{1}{h}\left[u\left(\frac{x}{1-h}\right)-u(x)\right] \quad \text{in } \Omega_h,$$

$$V^h(x)=\frac{1}{h}\left[u(x,h)-u\left(\frac{x}{1-h}\right)\right] \quad \text{in } \Omega_h.$$

Then

$$v_h(x)=U^h(x)+V^h(x) \quad \text{in } \Omega_h.$$

By the $C^{2,\alpha}$ regularity of u in $D_e\cup\partial D$,

$$|U^h|+|\nabla U^h|\leqq C \quad \text{in } D_e\cap\Omega_h. \tag{2.14}$$

Next,

$$\operatorname{div}(a(h)\nabla V^h)=\frac{1}{h}\left\{0-\operatorname{div}_x\left[a(h)\nabla_x u\left(\frac{x}{1-h}\right)\right]\right\}.$$

Since

$$a(h)=1+k\chi_{D(h)}(x)=1+k\chi_{(1-h)D}(x)=1+k\chi_D\left(\frac{x}{1-h}\right),$$

setting $y=x/(1-h)$, $\nabla_x=\nabla_y/(1-h)$, the expression in braces becomes

$$-\frac{1}{(1-h)^2}\operatorname{div}_y[1+k\chi_D(y)\nabla_y u(y)]=0 \quad \text{in } \Omega_h,$$

i.e.,

$$\operatorname{div}(a(h)\nabla V^h)=0 \quad \text{in } \Omega_h. \tag{2.15}$$

Further,

$$V^h(x)=\frac{u(x,h)-f(x/(1-h))}{h} \quad \text{if } x\in\partial\Omega_h\cap\partial((1-h)\Omega),$$

$$V^h(x)=\frac{f(x)-u(x/(1-h))}{h} \quad \text{if } x\in\partial\Omega_h\cap\partial\Omega;$$

since $u(x)$ and $u(x,h)$ are in $C^{1,\alpha}$ in some neighborhood of $\partial\Omega$, it follows that in both cases

$$|V^h|\leqq C,$$

i.e., $|V^h|\leqq C$ on $\partial\Omega_h$. Hence, by the maximum principle,

$$|V^h|\leqq C \quad \text{in } \Omega_h. \tag{2.16}$$

Recalling (2.14) we conclude that

$$|v_h|\leqq C \quad \text{in } D_e\backslash B_\delta(\partial\Omega)$$

for any $\delta > 0$ and h small enough, where $B_\delta(A)$ denotes a δ-neighborhood of a set A. Since v_h is harmonic in D_e and $v_h = 0$ on $\partial\Omega$, we then also have that

$$|v_h| \leqq C \quad \text{in } D_e. \tag{2.17}$$

Hence, for a subsequence,

$$v_h \to v \text{ uniformly on compact subsets of } D_e \cup \partial\Omega \tag{2.18}$$

where v is harmonic in D_e and

$$v = \frac{\partial v}{\partial \nu} = 0 \quad \text{on } \partial\Omega; \tag{2.19}$$

here (2.1) was used. It follows that the zero function is a harmonic extension of v into $\mathbb{R}^2 \backslash \Omega$ and therefore $v \equiv 0$ in D_e. Clearly (2.12) now follows from (2.18) and Harnack's theorem.

To prove (2.13) we multiply (2.15) by V^h and integrate over $\Omega' = \Omega \backslash B_\eta(\partial\Omega)$, where $0 < \eta < \operatorname{dist}(D, \partial\Omega)$. We obtain, for small h,

$$\iint_{\Omega'} a(h)|\nabla V^h|^2 \leqq \int_{\partial\Omega'} a(h)|\nabla V^h|\,|V^h|.$$

Since V^h is harmonic and bounded in $B_{\delta_0}(\partial\Omega')$ (δ_0 is independent of h) the right-hand side is uniformly bounded; hence

$$\int_{\Omega'} |\nabla V^h|^2 \leqq C.$$

Recalling (2.14) we deduce that

$$\int_{\Omega' \cap D_e} |\nabla v_h|^2 \leqq C. \tag{2.20}$$

Now, for any small $\delta > 0$,

$$\int_{\partial D} |v_h| \leqq C \int_{D_e \cap B_\delta(\partial D)} |\nabla v_h| + C \int_{(\Omega' \cap D_e) \backslash B_\delta(\partial D)} |\nabla v_h| + C \int_{\partial\Omega'} |v_h|.$$

The last two integrals on the right-hand side converge to zero as $h \to 0$, whereas the first integral is bounded by $C\delta^{1/2}$ (by (2.20)). It follows that

$$\limsup_{h \to 0} \int_{\partial D} |v_h| \leqq C\delta^{1/2},$$

and, since δ is arbitrary, (2.13) follows.

From (2.12), (2.5), (2.7) we conclude that

$$f^h \to 0, \ \nabla f^h \to 0 \text{ uniformly in closed subsets of } D_e \cup \partial\Omega. \tag{2.21}$$

Taking $h \to 0$ in (2.11) we see that if

$$I = \lim_{h \to 0} \left(-\frac{1}{\pi}\right) \iint_\Omega \frac{\partial a(0)}{\partial \bar{z}} \frac{v_h}{z - \zeta}\, dx\, dy, \tag{2.22}$$

$$J = \lim_{h \to 0} \left(-\frac{1}{\pi}\right) \iint_\Omega a_h \frac{\partial u(h)}{\partial \bar{z}} \frac{1}{z - \zeta}\, dx\, dy \tag{2.23}$$

exist for any $\zeta \in D \cup D_e$, then $f^h(\zeta) \to f^0(\zeta)$, where

$$f^0(\zeta) = I - J \quad \text{if } \zeta \in D \cup D_e; \tag{2.24}$$

from (2.21) we also have

$$f^0(\zeta)=0 \quad \text{if } \zeta \in D_e. \tag{2.25}$$

Now clearly

$$\frac{\partial a(0)}{\partial \bar{z}}=\gamma \otimes \delta_{\partial D} \quad \text{in } \mathscr{D}'$$

where γ is a $C^{1,\alpha}$ function on ∂D. Therefore

$$\lim_{h\to 0}\iint_\Omega \frac{\partial a(0)}{\partial \bar{z}}\frac{v_h}{z-\zeta}\,dx\,dy=\lim_{h\to 0}\int_{\partial D}\frac{\gamma}{z-\zeta}v_h(z)\,ds=0$$

by (2.13), i.e.,

$$I=0. \tag{2.26}$$

Next, by (1.6) and the fact that $u(t)$ is in $C^1(\overline{D(t)})$ and in $C^1(\overline{D_e(t)})$ with moduli of continuity independent of t, it follows that

$$\lim_{h\to 0}\iint_\Omega a_h\frac{\partial u(h)}{\partial \bar{z}}\frac{1}{z-\zeta}\,dx\,dy=\int_{\partial D} k\beta\left(\frac{\partial u(0)}{\partial \bar{z}}\right)^e\frac{1}{z-\zeta}\,ds;$$

here we used (1.2) and (2.2). We conclude that J exists and, by (2.24), (2.26),

$$f^0(\zeta)=\frac{k}{\pi}\int_{\partial D}\beta\left(\frac{\partial u}{\partial \bar{z}}\right)^e\frac{1}{z-\zeta}\,ds \quad \text{if } \zeta \in D\cup D_e; \tag{2.27}$$

here $u=u(x,0)$.

Let $T(z)$ be the positively oriented tangent vector to ∂D at z. Then

$$dz=T(z)\,ds \quad \text{along } \partial D.$$

Using this in (2.27) we get

$$f^0(\zeta)=\frac{k}{\pi}\int_{\partial D}\frac{\beta}{T(z)}\left(\frac{\partial u}{\partial \bar{z}}\right)^e\frac{dz}{z-\zeta} \qquad (\zeta\in D\cup D_e). \tag{2.28}$$

In view of (2.25) and the standard jump relation of the integral in (2.28) across ∂D, we then have

$$f^0(z)=2ik\frac{\beta(z)}{T(z)}\left(\frac{\partial u}{\partial \bar{z}}\right)^e \quad \text{on } \partial D \tag{2.29}$$

where $f^0(z)=(f(z))^i$ is the limit of f^0 from D. Observe also from (2.28) that

$$f^0(z) \quad \text{is holomorphic in } D. \tag{2.30}$$

LEMMA 2.2. *There holds*

$$\beta\left(\frac{\partial u}{\partial \bar{z}}\right)^e=0 \quad \textit{on } \partial D. \tag{2.31}$$

The proof is given in § 3. Assuming its validity, we shall now proceed to complete the proof of Theorem 1.1. Since $\beta\neq 0$ along ∂D,

$$\left(\frac{\partial u}{\partial \bar{z}}\right)^e=0 \quad \text{on } \partial D.$$

Recalling the jump relation (1.6) we deduce that for some constant c, the function $U=u-c$ vanishes on ∂D together with its first derivatives. By the argument following (2.19) it then follows that $U\equiv 0$ in $B_\delta(\partial D)$ and, by analytic continuation, $u\equiv c$ in D_e which contradicts (1.8).

So far we have assumed that (2.2) holds. If $-1<h<0$, so that

$$D(h)\supset D, \tag{2.32}$$

then we replace (2.6) by

$$\frac{1}{h}[a(h)\nabla u(h)-a(0)\nabla u(0)]=a(h)\nabla v_h+a_h\nabla u(0)$$

and proceed as above (with minor changes) to establish (2.24) with the corresponding I vanishing and with J being the same as before.

3. Proof of Lemma 2.2. Set

$$u_1=u|_{D_e},\qquad u_2=u|_D.$$

Then

$$u_1=u_2 \quad \text{on } \partial D, \tag{3.1}$$

$$\frac{\partial u_1}{\partial\nu}=(k+1)\frac{\partial u_2}{\partial\nu} \quad \text{on } \partial D. \tag{3.2}$$

Notice that the function $\partial u_2/\partial z$ is homomorphic in D. Multiplication of both sides of (2.29) by $\partial u_2/\partial z$ gives

$$F'(z)T(z)=2ik\beta(z)\frac{\partial u_1}{\partial\bar z}\frac{\partial u_2}{\partial z} \qquad (z\in\partial D) \tag{3.3}$$

where F is a holomorphic function in D, namely, the primitive of $f^0(z)\partial u_2/\partial z$.

Along ∂D we have

$$2\frac{\partial}{\partial z}=\frac{\partial}{\partial x}-i\frac{\partial}{\partial y}=e^{i\omega}\left(\frac{\partial}{\partial\nu}-i\frac{\partial}{\partial s}\right),$$

$$2\frac{\partial}{\partial\bar z}=\frac{\partial}{\partial x}+i\frac{\partial}{\partial y}=e^{-i\omega}\left(\frac{\partial}{\partial\nu}+i\frac{\partial}{\partial s}\right)$$

where ν is the outward normal, $\partial/\partial s$ is in the tangential direction obtained from $\partial/\partial\nu$ by rotation counterclockwise by $\pi/2$ and ω is a real valued function. Therefore by (3.1), (3.2) we easily obtain

$$4\frac{\partial u_1}{\partial\bar z}\frac{\partial u_2}{\partial z}=(k+1)\left(\frac{\partial u_2}{\partial\nu}\right)^2+\left(\frac{\partial u_2}{\partial s}\right)^2-ik\frac{\partial u_2}{\partial\nu}\frac{\partial u_2}{\partial s}. \tag{3.4}$$

Hence, if $k>0$,

$$4\left|\operatorname{Im}\frac{\partial u_1}{\partial\bar z}\frac{\partial u_2}{\partial z}\right|=k\left|\frac{\partial u_2}{\partial\nu}\frac{\partial u_2}{\partial s}\right|\leqq\frac{k}{2}\left[\left(\frac{\partial u_2}{\partial\nu}\right)^2+\left(\frac{\partial u_2}{\partial s}\right)^2\right]\leqq 2k\left[\operatorname{Re}\frac{\partial u_1}{\partial\bar z}\frac{\partial u_2}{\partial z}\right].$$

Similarly, if $-1<k<0$,

$$4\left|\operatorname{Im}\frac{\partial u_1}{\partial\bar z}\frac{\partial u_2}{\partial z}\right|\leqq\frac{|k|}{2(k+1)}\left[(k+1)\left(\frac{\partial u_2}{\partial\nu}\right)^2+\left(\frac{\partial u_1}{\partial s}\right)^2\right]=\frac{2|k|}{k+1}\left[\operatorname{Re}\frac{\partial u_1}{\partial\bar z}\frac{\partial u_2}{\partial z}\right].$$

Since β is real valued it follows that in both cases, for any $z\in\partial D$,

$$F'(z)T(z)\in G\equiv\{z=x_1+ix_2;\ |x_2|\leqq C|x_1|\} \tag{3.5}$$

where

$$C=\frac{2|k|}{\min\{1,\,k+1\}}.$$

Writing the holomorphic function F in the form $F=V+iW$ we have

$$\frac{dF}{ds}=V_s+iW_s=V_s+iV_\nu \quad \text{along}\,\partial D.$$

Since also

$$\frac{dF}{ds}=\frac{dF}{dz}\frac{dz}{ds}=F'(z)T(z),$$

we conclude from (3.4) that

$$|V_\nu|\leqq C|V_s| \quad \text{along}\,\partial D. \tag{3.6}$$

Suppose $V\not\equiv$ const. in D. Then V must attain its maximum in $\bar{D}$ at a point $x^0\in\partial D$ and $V_\nu(x^0)>0$. Since also $V_s(x^0)=0$, we get a contradiction to (3.6). We have thus proved that $V\equiv$ const. and therefore also $F\equiv$ const. From (3.3) it then follows that

$$\beta\frac{\partial u_1}{\partial \bar{z}}\frac{\partial u_2}{\partial z}=0 \quad \text{on}\,\partial D$$

which, in view of (3.4) and (3.2), implies (2.31).

Remark 3.1. Theorem 1.1 extends (with minor changes in the proof) to the case where the domains $D(t)$ are conformal affine transformations of D varying in C^2 manner and monotonically in t, provided $\beta\not\equiv 0$ on ∂D. The theorem also extends to the case where f depends on t, say $f=f(x,t)$, provided

$$\frac{1}{h}[f(\cdot,h)-f(\cdot,0)]\to 0 \quad \text{in}\ C^{1,\alpha}(\partial D)\text{-norm}$$

as $h\to 0$. If the $D(t)$ do not vary monotonically in t, then Lemma 2.1 is still valid with (2.13) replaced by

$$\int_{\partial(D\cup D(h))}|v_h|\,ds\to 0.$$

But this is not sufficient for proving (2.26); see also next remark.

Remark 3.2. Consider the case where Ω is one-dimensional, say $\Omega=\{0<x<1\}$. The solution of (1.1) with $u(0)=\alpha$, $u(1)=\beta$, $u'(0)=\alpha'$, $u'(1)=\beta'$ is given by

$$u(x)=\alpha+u(0)\alpha'\int_0^x\frac{dy}{a(y)} \tag{3.7}$$

where

$$u(0)\alpha'=(\beta-\alpha)\Big/\int_0^1\frac{dy}{a(y)},\qquad \beta'=\frac{a(0)\alpha'}{a(1)}. \tag{3.8}$$

For any other conductivity $\tilde{a}(x)$ with

$$\tilde{a}(0)=a(0),\quad \tilde{a}(1)=a(1),\quad \int_0^1\frac{dy}{a(y)}=\int_0^1\frac{dy}{\tilde{a}(y)}, \tag{3.9}$$

the Neumann data g corresponding to the Dirchlet data f are the same as for a. Clearly (3.9) is satisfied if $\tilde{a}(t)=1+k\chi_{D(t)}$, $a=\tilde{a}(0)$ whenever $D(t)$ is a translation of D. In this example the mapping $a\to g$ is thus nonunique; furthermore, the assertion of Theorem 1.1 is not valid if $D(t)$ is a translation of D. If however $D(t)$ is monotone in t then the assertion of Theorem 1.1 is valid, as can be verified directly by means of (3.9). Similarly, if Ω is an annulus $\Omega=\{r_1<|x|<r_2\}$ and $f=c_i$ on $\{|x|=r_i\}$, c_i constants, then the assertion of Theorem 1.1 is valid for a family of annuli $D(t)=\{d_1(t)<|x|<d_2(t)\}$ provided the family is monotone in t, but it is generally false if the $D(t)$ do not vary monotonically in t (note however that Ω is not simply connected, as required in Theorem 1.1).

Remark 3.3. Let $\tilde{z}=\phi(z)$ be a conformal mapping of $\bar{\Omega}$ onto the closure of a domain $\tilde{\Omega}$ and set $\tilde{D}(t)=\phi(D(t))$, $a=\tilde{a}\circ\phi$, $f=\tilde{f}\circ\phi$, $u=\tilde{u}\circ\phi$, $\tilde{g}=|\phi'|\tilde{g}\circ\phi$. Then (1.4), (1.5) and (1.7) are equivalent to

$$\operatorname{div}(\tilde{a}\nabla\tilde{u})=0 \quad \text{in } \tilde{\Omega}, \qquad \tilde{u}=\tilde{f} \quad \text{on } \partial\tilde{\Omega},$$

$$\frac{\partial\tilde{u}}{\partial\tilde{\nu}}=\tilde{g} \quad \text{on } \partial\tilde{\Omega}.$$

Since

$$0<c\leqq\frac{\|\tilde{g}(t)-\tilde{g}(0)\|_{L^2}}{\|g(t)-g(0)\|_{L^2}}\leqq C<\infty,$$

Theorem 1.1 extends to the family $\tilde{D}(t)$ of subdomains of $\tilde{\Omega}$.

Remark 3.4. Theorem 1.1 extends to inhomogeneous equations

$$\operatorname{div}(a\nabla u)=l(x) \quad \text{in } \Omega$$

provided $l\in C^{1,\alpha}$ and $S\equiv\operatorname{supp} l$ satisfies: $S\subset D_e$ and $D_e\backslash S$ is connected; if $l\not\equiv 0$ and $l>0$, then the condition (1.8) is not needed. The function l may also be taken to depend on t.

Remark 3.5. The results of this paper extend with minor changes to the case where the Neumann data (1.7) are prescribed, whereas the Dirichlet data $f=f(t,x)$ are measured; here it is assumed that $\int_{\partial\Omega} g=0$ and u is normalized, say, by $\int_{\partial\Omega} u=0$. The assertion (1.9) is replaced by

$$\|f(\cdot,h)-f(\cdot)\|_{L^2(\partial\Omega)}\geqq c|h|$$

where c is a positive constant.

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